

From real-time tunneling to vacuum decay with conserved charges

DESY Hamburg

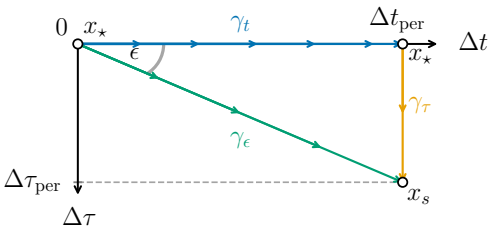
Giulio Barni

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with *T. Steingasser* based on **Phys.Rev.D 114 (2026) 5, 056003**
and *José R. Espinosa* based on **2606.21653**

Outline

1. PART I: Real-time Tunnelling
   - Why, why yes, and why not?
   - A way forward: Steadyons
   - Tunnelling in QM with a conserved charge
2. PART II: Imaginary-time Tunnelling
   - What is a charged state?
   - Tunnelling decay at $Q \neq 0$
   - Charged wall late-time dynamics
   - Generalisation at finite T and μ
3. Conclusion & Outlook



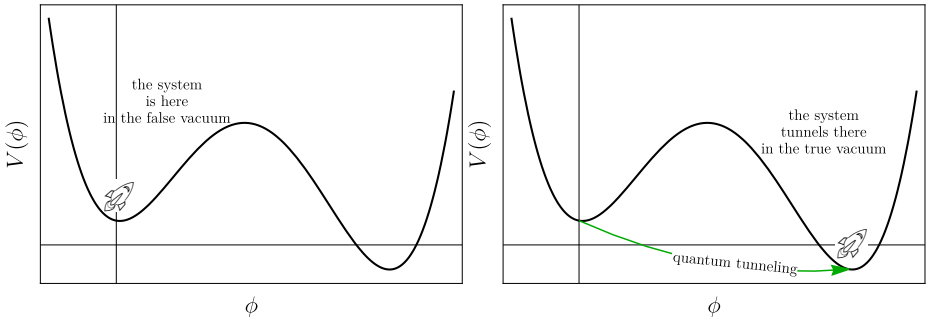


Public code available @ Qubble

PART I: Real-time Tunnelling
Why, why yes, and why not?

Why?

Wick-rotate the PI and saddle point approx. around the Coleman bounce



Tunneling decay rate of the false vacuum

$\Gamma \sim Ae^{-S_E}$, Euclidean action

S_E computed on the sol. of the EOMs

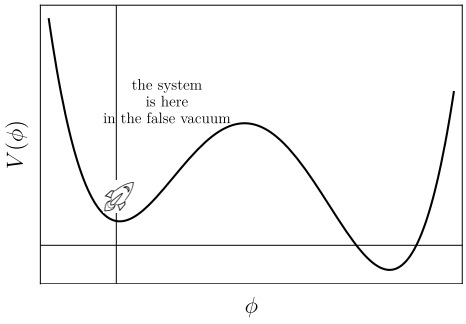


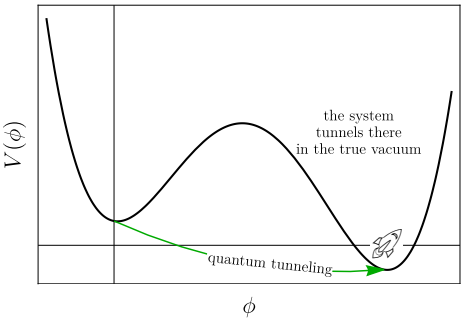
The solutions with the least action are **spherically symmetric**

Coleman ('77); Coleman, Callan ('77)

Why?

But why does this work? And can we always do it?





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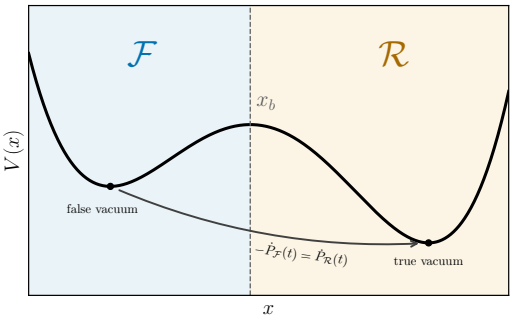


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Why yes? Tunnelling happens in real time

Metastable state in QM with a wave function ψ localized in $\mathcal{F}$ can decay $\mathcal{F} \to \mathcal{R}$ into the escape region.



Survival and escape probabilities

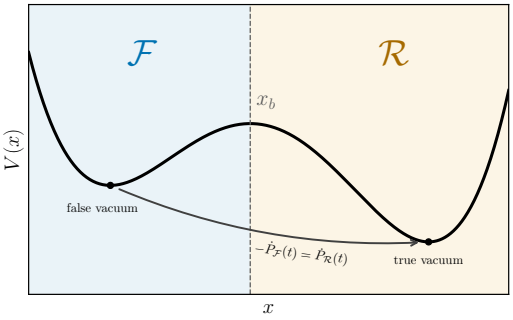
$$P_{\mathcal{F}}(t) = \int_{\mathcal{F}} dx\, |\psi(x,t)|^2, \quad P_{\mathcal{R}}(t) = \int_{\mathcal{R}} dx\, |\psi(x,t)|^2.$$

For $P_{\mathcal{F}}(t) \propto e^{-\Gamma t}$:

$$\boxed{\Gamma = -\frac{\dot{P}_{\mathcal{F}}(t)}{P_{\mathcal{F}}(t)} = \frac{\dot{P}_{\mathcal{R}}(t)}{P_{\mathcal{F}}(t)}}$$

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Direct approach: use the Minkowski path integral

$$D_F(x_f, t; x_i, 0) = \langle x_f | e^{-iHt} | x_i \rangle = \int_{x(0)=x_i}^{x(t)=x_f} \mathcal{D}x\, e^{iS_M[x]}.$$

Andreassen et al. [1602.01102]

How do we actually evaluate the oscillatory Minkowski path integral semiclassically?

Why not?

We would like to evaluate the real-time path integral **semiclassically**:

$$D_F(x_f, t; x_i, 0) = \int_{x(0)=x_i}^{x(t)=x_f} \mathcal{D}x\, e^{iS_M[x]/\hbar}.$$

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1. **Highly oscillatory integral:** no straightforward real saddle-point description of tunnelling.
2. **No real classical tunnelling trajectory:** inside the forbidden region,
$$\frac{m}{2}\dot{x}^2 = E - V(x) < 0.$$
3. **Complex saddles:** we must allow $x(t) \in \mathbb{C}$. How many saddles? Which solutions actually contribute to the original path integral? $\rightarrow$ Picard–Lefschetz thimbles??  *Hic svnt dracones!* 

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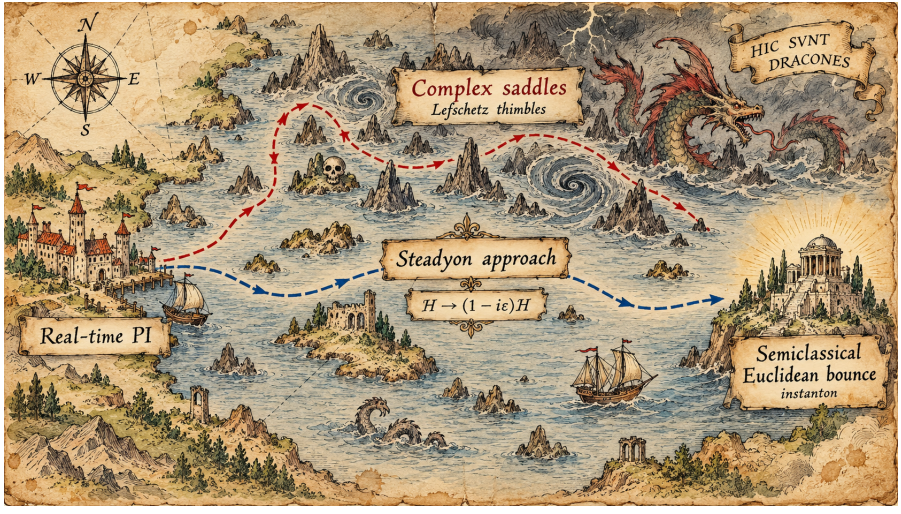
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The challenge

How can we identify the relevant complex saddle directly in real time?

$\longrightarrow$ Andreassen et al. [1602.01102]; Steingasser & Kaiser [2402.00099]

Why not? *Hic sunt dracones!*



A way forward: Steadyons

Steingasser, König & Kaiser [2310.19865] Steingasser & Kaiser [2402.00099] Lin, Scheihing-Hitschfeld & Steingasser [2511.08669, 2511.08670]

A way forward: the steadyon picture in QM

Semiclassically $\rightarrow$ compute the **real-time propagator** around a classical saddle

$$D_F(x_s, \Delta t \mid x_*) = \int_{x_*}^{x_s} \mathcal{D}x \; e^{iS_M[x]}.$$

No real classical trajectory crosses the barrier $\Longrightarrow$ **slightly regularised evolution**

$$e^{-iHt} \longrightarrow e^{-i(1-i\epsilon)Ht}, \qquad 0 < \epsilon \ll 1.$$

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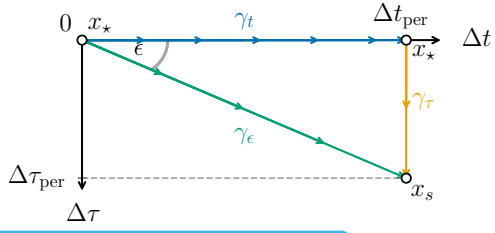
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$$e^{-iHt} \longrightarrow e^{-i(1-i\epsilon)Ht}, \qquad 0 < \epsilon \ll 1.$$

For time-independent H, this is equivalent to a tilted contour in complex time, $z = (1 - i\epsilon)t$.

$$\bar{x}(0) = x_\star, \qquad \bar{x}(\Delta t_{\rm per}) \simeq x_s, \qquad \dot{\bar{x}}(\Delta t_{\rm per}) \simeq 0.$$



The essential limit

$\epsilon \to 0^+$ and $\Delta t_{\rm per} \to \infty$, while $\epsilon \Delta t_{\rm per} \simeq \Delta\tau_{\rm per}$ stays finite.

Steingasser & Kaiser [2402.00099]

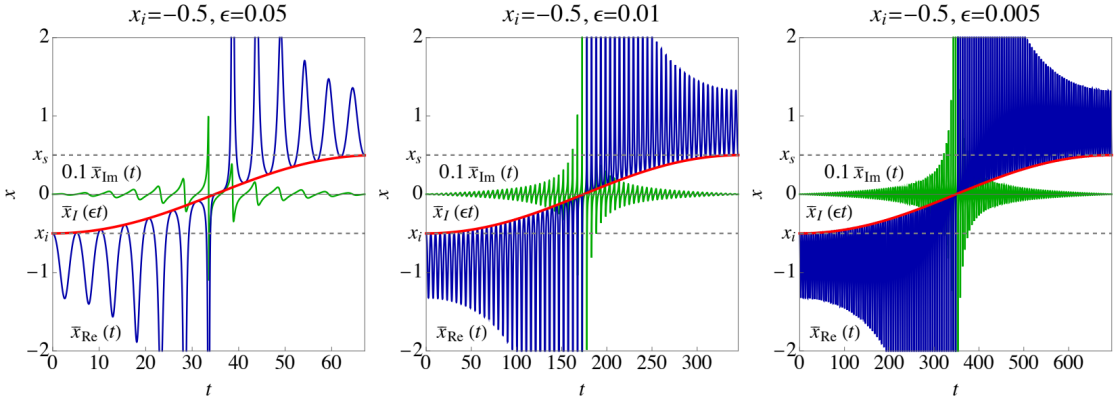
Steadyons: tunnelling at fixed E

Steingasser & Kaiser [2402.00099]

Start with a state displaced from the origin ≡ state with initial energy wrt the false vacuum (e.g. **excited state**)

At fixed energy $E = V(x_i) = V(x_s)$, we solve the **complexified Newton equation**:

$$\ddot{\bar{x}}_\epsilon(t) = -(1 - 2i\epsilon)V'(\bar{x}_\epsilon(t)), \qquad V(x) = \frac{1}{4}(x^2 - 1)^2 \qquad \bar{x}_\epsilon(0) = x_i, \quad \dot{\bar{x}}_\epsilon(0) = 0.$$



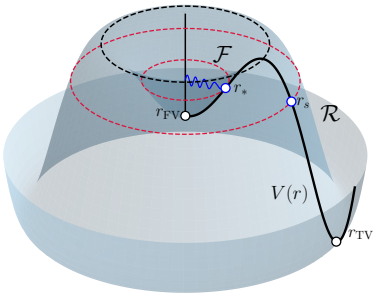
The Euclidean instanton emerges from the complex steadyon in the limit $\epsilon \to 0$.

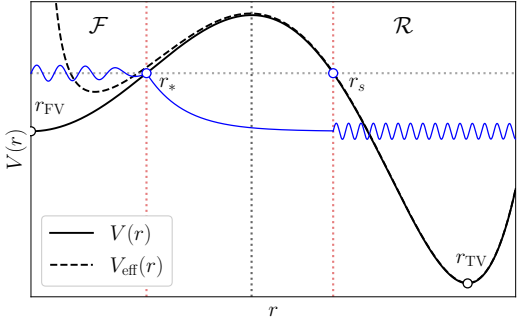
Steadyons: tunnelling at fixed E and J

GB & Steingasser [2604.08660]

Point particle in 2D with some angular momentum J and some energy E:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r), \qquad \Psi_{E,J}(r,\varphi) = e^{iJ\varphi}\,\psi_{E,J}(r).$$





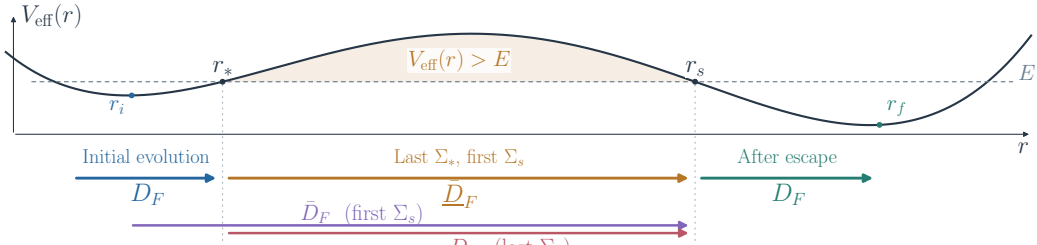
The angular variable is cyclic, so the radial barrier depends on J:

$$J = mr^2\dot{\varphi} = \text{const.}, \qquad V_{\rm eff}(r) = V(r) + \frac{J^2}{2mr^2}, \qquad V_{\rm eff}(r_*) = V_{\rm eff}(r_s) = E.$$

From the propagator to the tunnelling flux

The initial state evolves through the usual propagator:

$$\Psi_{E,J}(\mathbf{x}_f, t) = \int_{\mathcal{F}} d^2x_i \, D_F(\mathbf{x}_f, t|\mathbf{x}_i, 0) \, \Psi_{E,J}(\mathbf{x}_i, 0).$$



Use $\underline{D}_F$ for the **last crossing of** r_*, and $\bar{D}_F$ when the **first crossing of** r_s is also imposed.

$$\boxed{\begin{pmatrix} \mathcal{F}_{E,J} \\ \bar{\mathcal{F}}_{E,J} \end{pmatrix} = \int_0^t d\Delta t \int_0^{2\pi} d\varphi_* \begin{pmatrix} \underline{D}_{F,\epsilon} \\ \bar{\underline{D}}_{F,\epsilon} \end{pmatrix} (\mathbf{x}_s, \Delta t|\mathbf{x}_*) \, \Psi_{E,J}(\mathbf{x}_*, t - \Delta t)}$$

$$\dot{P}_{\mathcal{R}} \simeq 2\,\mathrm{Re} \int_0^{2\pi} d\varphi_s \, \bar{\mathcal{F}}^*_{E,J} \mathcal{F}_{E,J}, \qquad \Gamma = \frac{\dot{P}_{\mathcal{R}}}{P_{\mathcal{F}}}.$$

How does the initial state select the steadyon?

The probability flux involves $\bar{\mathcal{F}}^*_{E,J}\mathcal{F}_{E,J}$. At LO, both are governed by the **same single-traversal saddle**. For an initial state with definite E and J,

$$\Psi_{E,J}(\mathbf{x}_*, t-\Delta t) \propto e^{iJ\varphi_*} e^{-i(1-i\epsilon)E(t-\Delta t)}.$$

The **semiclassical evaluation of** $\mathcal{F}_{E,J}$ gives, up to overall factors,

$$\boxed{\mathcal{F}_{E,J} \sim \int d\Delta t\, d\varphi_*\, \exp\Big[i(S^{\rm cl}_\epsilon + J\varphi_* + (1-i\epsilon)E\Delta t)\Big]}.$$

Stationarity in the crossing angle and time fixes the angular momentum and energy of the steadyon:

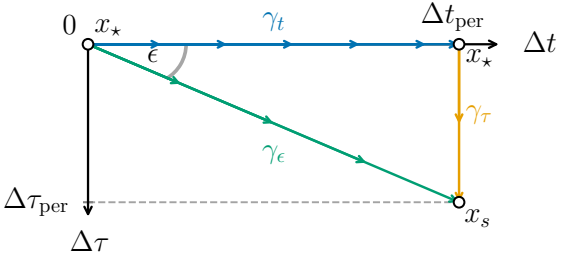
$$\partial_{\varphi_*}: \quad (1+i\epsilon)m\bar{r}^2\dot{\bar{\varphi}} = J,$$

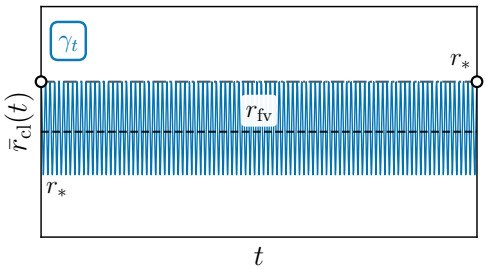
$$\partial_{\Delta t}: \quad \frac{\partial S^{\rm cl}_\epsilon}{\partial \Delta t} = -(1-i\epsilon)E \quad \Rightarrow \quad \dot{\bar{r}}(0) \simeq 0 \quad [V_{\rm eff}(r_*) = E].$$

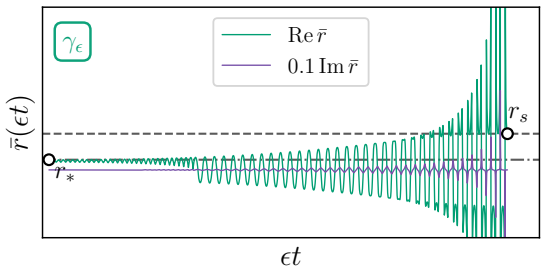
The cyclic angle can be eliminated using the **fixed-J Routhian**:

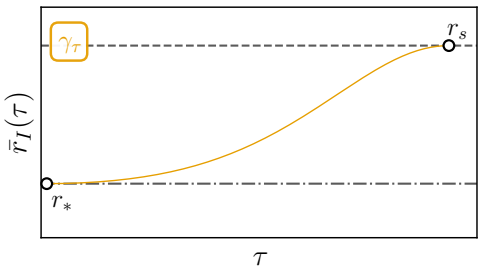
$$S_{R,\epsilon} = S_\epsilon - J\int_0^{\Delta t} dt\, \dot{\bar{\varphi}}.$$

The Steadyon solution

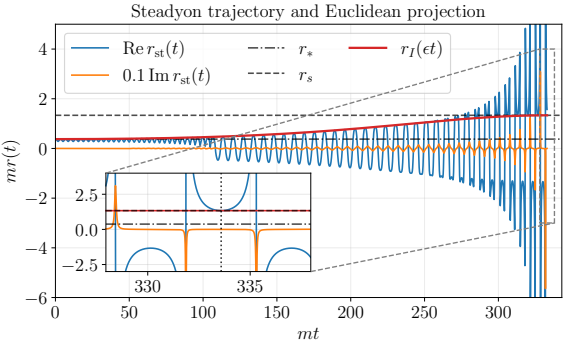


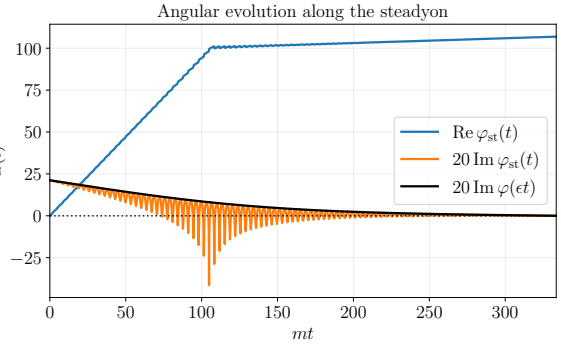




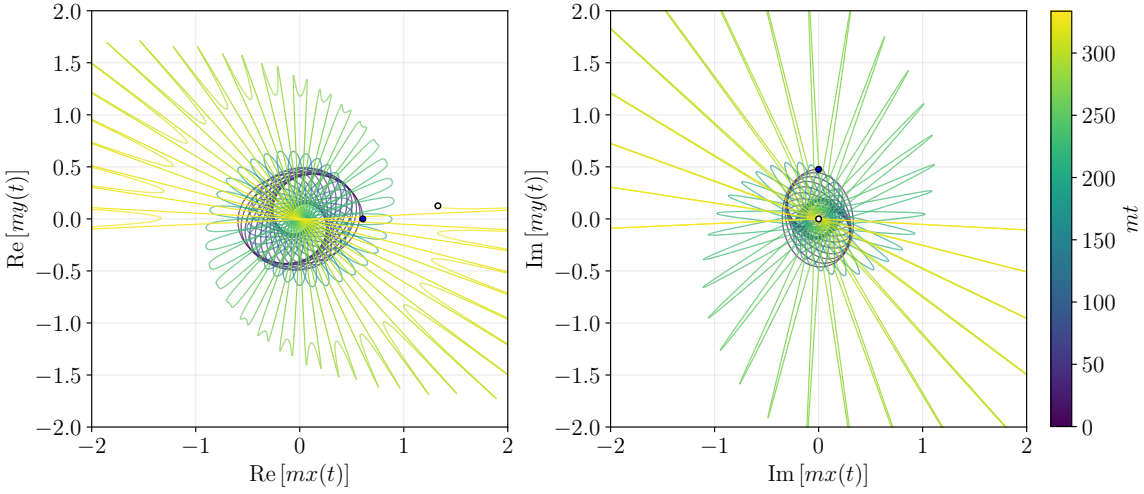


The Steadyon solution





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From Steadyons to Euclidean tunnelling

The steadyon reduces to a Euclidean instanton, but **fixed J requires a complex angular trajectory**

$$mr_I'' = V_{\text{eff}}'(r_I), \qquad J = mr^2\dot{\varphi} \xrightarrow{t=-i\tau} \varphi_I' = -i\frac{J}{mr_I^2}.$$

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The relevant Euclidean action is the **fixed-J Routhian**

$$S_{R,E} = S_E + iJ\Delta\varphi_I = \int_0^{\Delta\tau_{\text{per}}} d\tau \left[\frac{m}{2}(r_I')^2 + V_{\text{eff}}(r_I)\right].$$

The real-time calculation reproduces the **WKB exponent**

$$\begin{aligned}\log\Gamma &\simeq -2\,\text{Im}\,S_{R,\epsilon} + 2E\epsilon\Delta t_{\text{per}} \xrightarrow{\epsilon\to 0} -2\left(S_{R,E} - E\Delta\tau_{\text{per}}\right) \\ &= -2\int_{r_*}^{r_s} dr\,\sqrt{2m\left[V_{\text{eff}}(r) - E\right]}.\end{aligned}$$

The standard Coleman bounce applies only to the **metastable ground state** ($E = 0$). At finite excitation energy, **the tunnelling saddle must be modified** accordingly!

From QM to QFT

The steadyon construction **can be generalized to QFT**, but not today!

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Take home message: steadyons show that tunnelling from the **metastable ground state**, with no initial energy, recovers the Coleman bounce

$$E = 0 \quad \Longrightarrow \quad \Delta\tau_{\rm per} \to \infty \quad \Longrightarrow \quad \text{Coleman-type bounce.}$$

We can therefore **work directly in Euclidean time**, without constructing steadyons in field theory.

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We can therefore **work directly in Euclidean time**, without constructing steadyons in field theory.

Even then, vacuum decay at fixed charge is already a non-trivial problem!

See Part II !!

PART II: Imaginary-time Tunnelling

Tunnelling at fixed $U(1)$ charge Q in QFT?

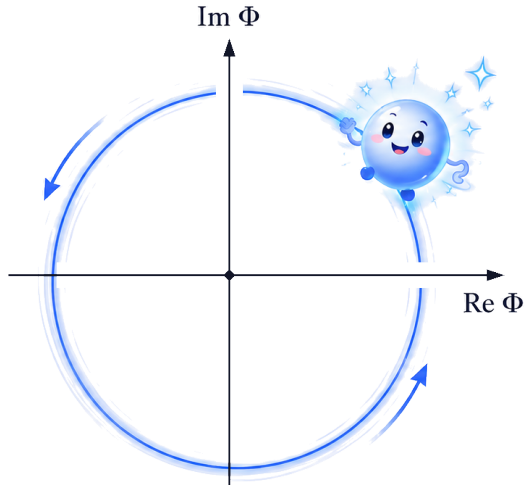
Globally charged vacuum decay: motivations & literature

Globally charged vacuum decay: motivations & literature

- There was a **gap in the tunnelling literature**: what happens when the **tunnelling field itself carries a chemical potential**? The **fixed-Q** problem is the natural building block.

- **Literature:** Fixed-charge tunnelling was already introduced by Lee 1994, but without a **full numerical bounce solution**. Other key aspects were already present in the **wormhole literature** of the 1980s. The work of Levkov, Nugaev, and Popescu 2017 on **Q-ball tunnelling** provides the main basis for the **numerical algorithm**.

What is a charged state?



What is a charged state?

The simplest example is to consider a **complex scalar field charged under a global** $U(1)$

$$\mathcal{L} = |\partial\Phi|^2 - V(\Phi^*\Phi) \qquad \text{where in polar coordinates} \qquad \Phi = \frac{\rho}{\sqrt{2}} e^{i\theta}$$

There is a **conserved Noether charge**

$$Q = \int d^3x\, j^0 = \int d^3x\, \rho^2 \dot{\theta} \qquad j^\mu = i(\Phi \partial^\mu \Phi^* - h.c.)$$

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$$Q = \int d^3x\, j^0 = \int d^3x\, \rho^2 \dot{\theta} \qquad j^\mu = i(\Phi \partial^\mu \Phi^* - h.c.)$$

Then, an **initial homogeneous charged state** is a stationary configuration as

$$\Phi_i = \frac{\rho_i}{\sqrt{2}} e^{i\omega_i t} \quad \text{with} \quad Q = V_3 \omega_i \rho_i^2 \qquad \rightarrow \qquad \begin{cases} E_i = V_3 \left(V(\rho_i) + \frac{1}{2}\omega_i^2 \rho_i^2 \right) = V_3 V(\rho_i) + \frac{1}{2}\omega_i Q \\ S_i = \beta \left(E_i - \omega_i Q \right) \end{cases}$$

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So it is **spinning in field space** with angular velocity ω_i
and displaced from the origin of the potential!

Tunnelling decay at $Q \neq 0$

Tunnelling decay at $Q = 0$

Take a complex scalar field Φ. Start from the **canonical partition function**

$$Z\ (\beta) = \mathrm{Tr}\Big[\quad e^{-\beta \hat{H}}\Big] = \sum_{n \in \mathcal{H}} \langle n | e^{-\beta \hat{H}} | n \rangle$$

In the path integral formulation this gives

$$Z\ (\beta) = \qquad \int \quad \mathcal{D}\Phi\, \mathcal{D}\Phi^*\, e^{-S_E[\Phi, \Phi^*]} \sim e^{-S_E[\Phi_b, \Phi_b^*]} \qquad .$$

with **periodic Euclidean boundary conditions**

$$\Phi(\beta/2, \mathbf{x}) = \quad \Phi(-\beta/2, \mathbf{x}), \qquad \Phi^*(\beta/2, \mathbf{x}) = \qquad \Phi^*(-\beta/2, \mathbf{x}).$$

The decay rate exponent is

$$\boxed{S\ _{,\beta} = S_E^{\mathrm{bounce}} - S_E^i}$$

We can (and we will) study the tunnelling both at finite T (fixed β) or $T = 0$ ($\beta \to \infty$)!

Tunnelling decay at $Q \neq 0$

Take a complex scalar field Φ. Start from the **canonical partition function**

$$Z_Q(\beta) = \mathrm{Tr}\Big[\hat{P}_Q\, e^{-\beta \hat{H}}\Big] = \sum_{n \in \mathcal{H}_Q} \langle n | e^{-\beta \hat{H}} | n \rangle$$

Project onto a fixed charge sector $\hat{P}_Q = \delta(\hat{Q} - Q) = \int \frac{d\theta}{2\pi}\, e^{i\theta(\hat{Q}-Q)}$

In the path integral formulation this gives

$$Z_Q(\beta) = \int \frac{d\theta}{2\pi}\, e^{-i\theta Q} \int_{\text{twisted}} \mathcal{D}\Phi\, \mathcal{D}\Phi^*\, e^{-S_E[\Phi,\Phi^*]} \sim e^{-S_E[\Phi_b,\Phi_b^*]-\eta_b Q}.$$

with **twisted quasi-periodic Euclidean boundary conditions** with complex saddle point $\theta = -i\eta$

$$\Phi(\beta/2, \mathbf{x}) = e^{\eta}\Phi(-\beta/2, \mathbf{x}), \qquad \Phi^*(\beta/2, \mathbf{x}) = e^{-\eta}\Phi^*(-\beta/2, \mathbf{x}).$$

The decay rate exponent is

$$\boxed{S_{Q,\beta} = S_E^{\text{bounce}} - S_E^i + \eta_0 Q} \qquad \eta_0 = \eta_b - \eta_i$$

We can (and we will) study the tunnelling both at finite T (fixed β) or $T = 0$ ($\beta \to \infty$)!

Fixed-Q decay exponent

Initial configuration $\rightarrow$ $\Phi_i = \rho_i/\sqrt{2}e^{i\omega_i t}$ with $Q = \omega_i \rho_i^2 V_3$

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The **action** of the theory is defined as $S_E[\phi, \bar{\phi}] = \int_{-\beta/2}^{\beta/2} d\tau \int d^3x \, \left[\partial_\tau \phi \, \partial_\tau \bar{\phi} + \nabla\phi \cdot \nabla\bar{\phi} + V(\phi\bar{\phi})\right].$

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Extremizing it, we get the EOMs

$$(\partial_\tau^2 + \nabla^2)\phi - V'(\phi\bar{\phi})\,\phi = 0, \qquad (\partial_\tau^2 + \nabla^2)\bar{\phi} - V'(\phi\bar{\phi})\,\bar{\phi} = 0$$

that is a 2d BVP in (r, τ) with $\rho \rightarrow \rho_i$ and $\alpha \rightarrow \omega_i \tau$.

Example at $Q \neq 0$

$U(1)$ Complex scalar field: $\mathcal{L} = \partial_\mu \Phi^* \partial_\mu \Phi - V(\Phi^* \Phi)$ where in polar coordinates $\Phi = (\rho/\sqrt{2}) \exp i\vartheta$

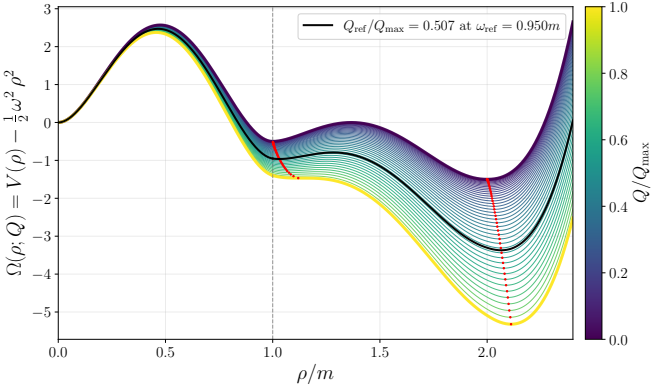
charged configuration
with $Q = \omega_i \rho_i^2 V_3$

$\Updownarrow$

initial state
displaced from the origin

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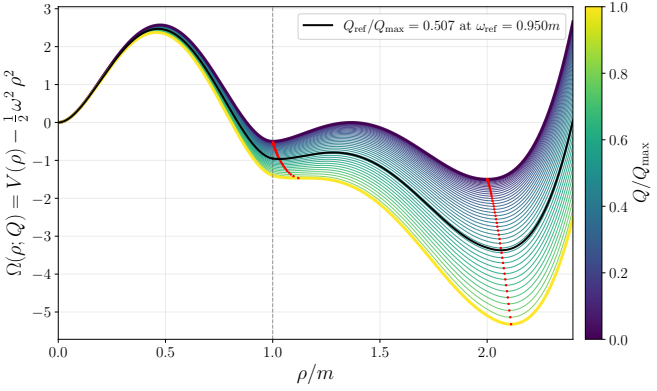
initial state
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$$V(\rho) = -\frac{1}{2} + (\rho - 1)^2 \left[2\rho - 5 + (2 - \rho)^2 \log \frac{(\rho - 2)^2 (\rho_0 - 1)^2}{(\rho - 1)^2 (\rho_0 - 2)^2} \right] , \qquad \rho \geq 1 ,$$

The $d = 4$ bounce action is known, $S_4^{(0)} = -\frac{\pi^2}{3} \left[\rho_0 - 1 + \mathrm{Li}_2 \left(\frac{\rho_0 - 1}{\rho_0 - 2} \right) \right]$.

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Reduced potential:

$$\partial_\mu^2 \rho = \frac{\partial}{\partial \rho}\left[V(\rho) - \frac{1}{2}\rho^2 (\partial_\mu \alpha)^2\right]$$

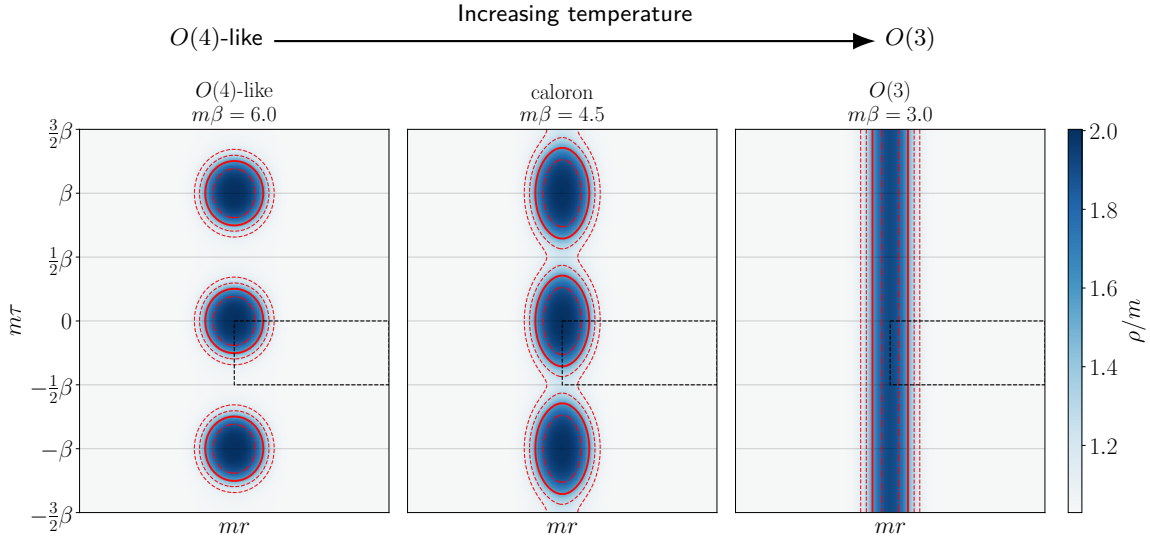
when $\alpha = \omega\tau + \tilde{\alpha}$

$$\Omega(\rho;Q) = V(\rho) - \frac{1}{2}\omega^2\rho^2$$

$$V(\rho) = -\frac{1}{2} + (\rho-1)^2\left[2\rho - 5 + (2-\rho)^2 \log\frac{(\rho-2)^2(\rho_0-1)^2}{(\rho-1)^2(\rho_0-2)^2}\right], \qquad \rho \geq 1\,,$$

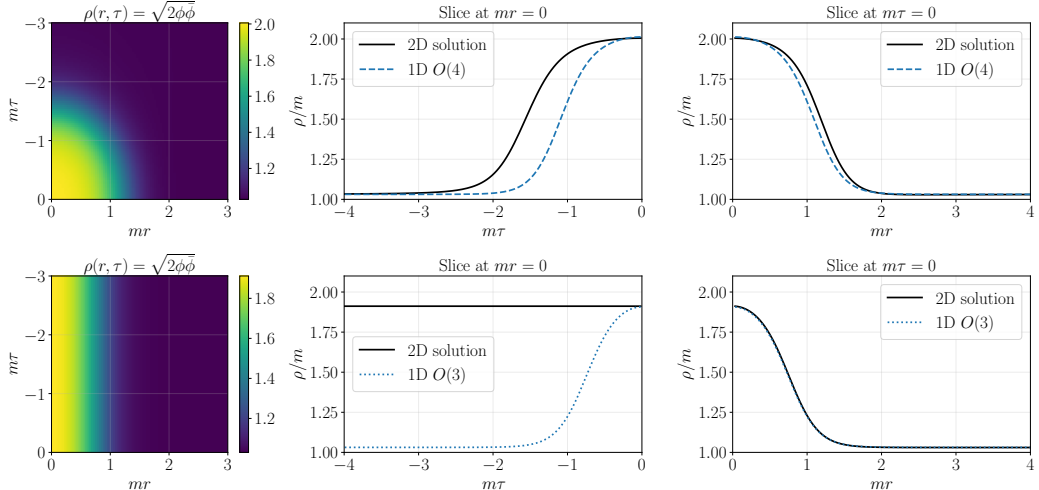
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Solutions of EOMs at $Q \neq 0$: from $O(4)$ to "calorons" to $O(3)$



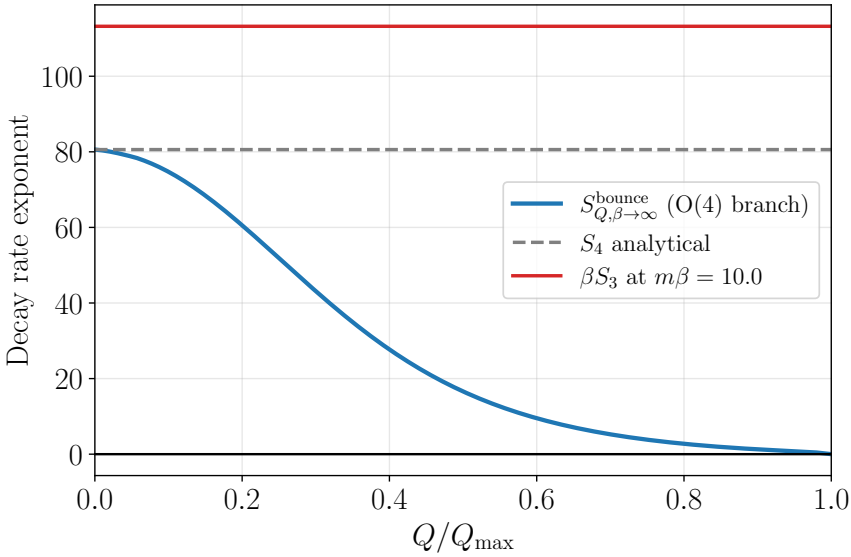
Solutions of EOMs at $Q \neq 0$

Two branches: $O(4)-$like tunnelling at $T = 0$ $(\beta \to \infty)$ and $O(3)$ tunnelling (finite β)



Note: η_0 is tuned to match the charge Q of the initial configuration!

Decay rate exponent: $S_{Q,\beta} = S_E^{\text{bounce}} - S_E^i + \eta_0 Q$



Thin-wall estimates

Thin-wall estimates at fixed Q

At small charge, the modulus retains its neutral thin-wall profile, while the phase solves a **4D dielectric problem**:

$$\rho_0 = \begin{cases} \rho_f, & r_E < R, \\ \rho_i, & r_E > R, \end{cases} \qquad \partial_\mu(\rho_0^2 \partial^\mu \alpha_1) = 0, \quad \rightarrow \quad \nabla_4^2 \alpha_1 = 0 \quad \text{in each region, with} \quad \delta_Q = \frac{1}{2} C_{\text{diel}} q^2 > 0.$$

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$$S_E^{\text{bounce}} - S_E^i \simeq 2\pi^2 \sigma_0 R^3 - \frac{\pi^2 R^4}{2}(\Delta V - \delta_Q).$$

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giving the **physical tunnelling exponent**:

$$\boxed{S_{Q,\rm tw}(R) \simeq 2\pi^2 \sigma_0 R^3 - \frac{\pi^2 R^4}{2}(\Delta V + \delta_Q).}$$

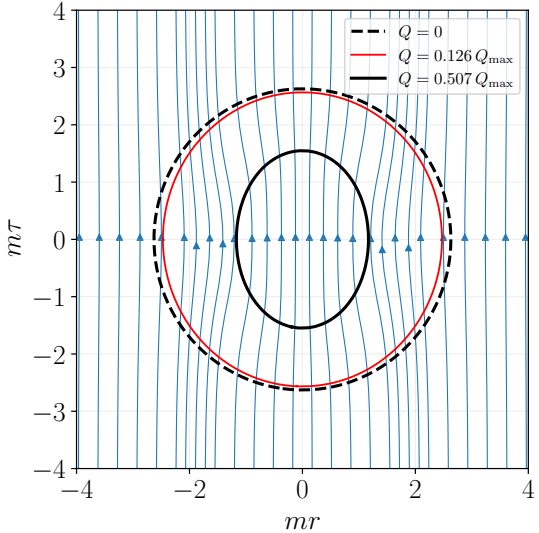
Extremizing:

$$\boxed{R_Q = \frac{3\sigma_0}{\Delta V + \delta_Q} < R_0, \qquad S_{Q,\rm tw}(R_Q) = \frac{S_4^{(0)}}{(1 + \delta_Q/\Delta V)^3}.}$$

Finite charge enhances nucleation and reduces the bubble size!

Charge flux lines

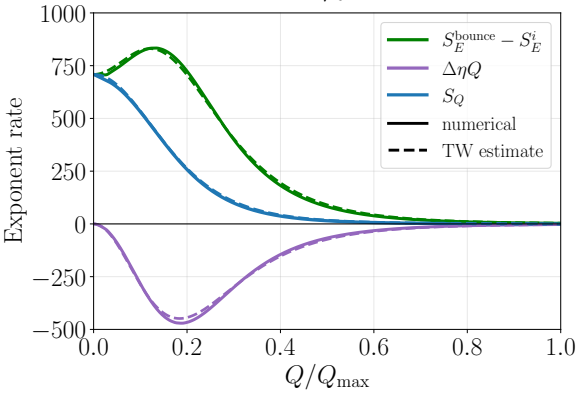
$$j_\tau = \bar{\phi}\,\partial_\tau \phi - \phi\,\partial_\tau \bar{\phi}, \qquad j_r = \bar{\phi}\,\partial_r \phi - \phi\,\partial_r \bar{\phi}.$$



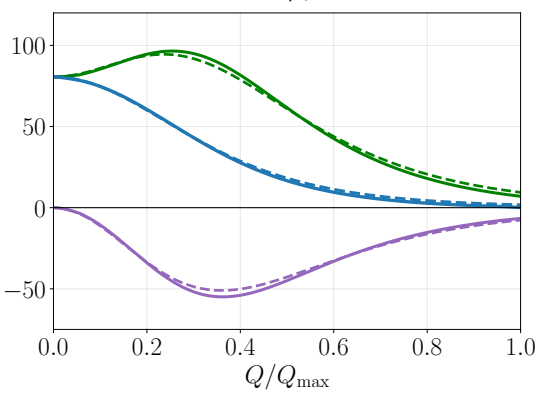
Decay rate exponent in the TW limit

$$S_E^{\text{bounce}} - S_E^i \simeq S_4^{(0)} \frac{1 + 7B_{\text{diel}}x^2}{(1 + B_{\text{diel}}x^2)^4}, \qquad \eta_0 Q \simeq -6S_4^{(0)} \frac{B_{\text{diel}}x^2}{(1 + B_{\text{diel}}x^2)^4}, \qquad \boxed{S_{Q,\text{tw}} \simeq S_4^{(0)} \frac{1}{(1 + B_{\text{diel}}x^2)^3}.}$$

Thin wall: $\rho_0 = 2 - 10^{-9}$



Thick wall: $\rho_0 = 2 - 10^{-3}$



Charged wall late-time dynamics

Lorentzian dynamics of the charged bubble

1 Before nucleation

outer phase
homogeneous rotation

$\omega_+,\ q = \rho^2 \dot{\theta}$

2 Bubble nucleates

outer phase

inner phase
$\omega_- < \omega_+$
$R_M(t)$
wall

3 Expansion in a charged medium

outer phase

$\omega_- < \omega_+$

causal front $r = t$
$R_M(t) = vt$
(wall)
phase shell / radial gradient
the phase interpolates between ω_- and ω_+

4 Late-time self-similar regime

$E_{\text{shell}} \sim E_{\text{vac}}$
$v = v_* < 1$
terminal velocity

$\omega_- < \omega_+$

$\xi = r/t$

0 — $\xi = v$ (wall) — $\xi = 1$ (causal front) $r = t$ — ξ

Late-time dynamics: Lorentzian TW estimate

After nucleation, **charge conservation** induces phase gradients around the expanding bubble.

Assume a **self-similar thin-wall solution**:

$$R_M(t) = vt, \qquad \xi = \frac{r}{t}, \qquad \alpha(t,r) = \omega t\, f(\xi), \qquad \rho = \begin{cases} \rho_f, & \xi < v, \\ \rho_i, & \xi > v. \end{cases}$$

Solving the Lorentzian phase equation gives a shell energy:

$$\partial_\mu(\rho^2 \partial^\mu \alpha) = 0, \qquad E_\alpha^M = 4\pi t^3\, \mathcal{E}_\alpha(v).$$

Balancing it against the released vacuum energy,

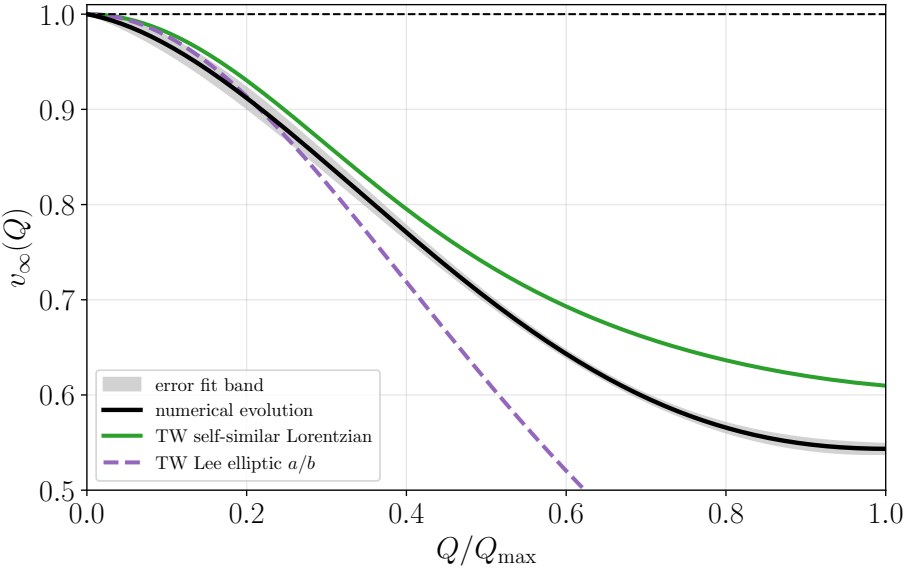
$$E_{\rm vac} = \frac{4\pi}{3} v^3 t^3 \Delta V,$$

determines the **terminal velocity**:

$$\boxed{\mathcal{E}_\alpha(v_\star) = \frac{\Delta V}{3} v_\star^3.}$$

Charge conservation can produce subluminal terminal wall velocities, even in vacuum!

Subluminal velocity



Generalisation at finite T and μ

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   - If charge exchange is slow during tunnelling, so the **charge is exactly conserved** and μ prepares the initial state
   $$B(\mu) = S_{Q_i(\mu),\beta}.$$
   - If the system stays coupled to a reservoir, one needs a genuinely **fixed-μ** bounce
   $$B_{\mu,\beta} = S^{\text{bounce}}_{E,\mu} - S^{i}_{E,\mu}, \qquad \text{where now } Q \textbf{ is an output}.$$

Conclusions and outlook

Conclusions I: Real-time tunnelling

- The real-time path integral can be evaluated semiclassically using **complex saddle points**: the steadyons.

$$H \to (1 - i\epsilon)H \quad \Longrightarrow \quad \bar{x}_\epsilon(t) \in \mathbb{C}.$$

- The **initial quantum state selects the saddle**:

$$\partial_{\Delta t} \longrightarrow E, \qquad \partial_{\varphi_*} \longrightarrow J.$$

- The **Euclidean instanton emerges directly** from the steadyon construction:

$$\boxed{\epsilon \Delta t_{\rm per} \to \Delta \tau_{\rm per}, \qquad \Gamma \sim e^{-B(E,J)}.}$$

- For the **metastable ground state**, we recover the familiar Coleman prescription:

$$\boxed{E = 0 \quad \Longrightarrow \quad \Delta \tau_{\rm per} \to \infty.}$$

This can be generalized to QFT, allowing us to **work directly in Euclidean time**.

Conclusions II: Globally charged vacuum decay

- Fixed$-Q$ changes the tunnelling problem at a **structural level**:

$$\text{charge projection} \quad \Longrightarrow \quad \text{twisted Euclidean boundary conditions.}$$

- The dominant saddle is a **complex saddle point** of the PI.

- The decay exponent is

$$\boxed{S_{Q,\beta} = S_E^{\text{bounce}} - S_E^i + \eta_0 Q, \qquad \text{with } \eta_0 Q < 0}$$

  with $\eta_0 Q$ essential for the fixed-Q projection.

- Finite charge **lowers the barrier**, **reduces the critical radius**, **enhances nucleation** and generate **subluminal terminal wall velocities** even in vacuum.

Outlook

- **Gauged tunnelling:** extend both the real-time **steadyon framework** and the **Euclidean field-theory treatment** to gauged $U(1)$ theories, consistently including Gauss's law.
- **Beyond fixed Q:** fixed-μ bounces, finite-temperature effects and realistic equations of state for dense matter.
- **Beyond the leading exponent:** fluctuation determinants, prefactors and stability of charged configurations.
- **Cosmological and astrophysical applications:** phase transitions in dense matter, bubble dynamics and charge transport.
- **Gravitational waves from charged bubble collisions:** how do charge conservation and modified wall dynamics affect the resulting GW spectrum?


Thank you!
Q

References I

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Backups

Steadyons for fixed E and J

1. Physical setup: a metastable state at fixed E, J

A particle in a rotationally invariant potential, with a metastable annular region $\mathcal{F}$ and escape region $\mathcal{R}$:

$$L = \frac{m}{2}\left(\dot{r}^{\,2} + r^2\dot{\varphi}^{\,2}\right) - V(r), \qquad \Psi_{E,J}(r,\varphi) = e^{iJ\varphi}\psi_{E,J}(r).$$

Narrow resonance (the decay is much slower than local motion):

$$\widetilde{E} = E - \frac{i\Gamma}{2}, \qquad \Gamma \ll E, \qquad P_{\mathcal{F}}(t) \simeq P_{\mathcal{F}}(0)e^{-\Gamma t}.$$

At fixed J, the radial turning points satisfy

$$\boxed{V_{\text{eff}}(r) = V(r) + \frac{J^2}{2mr^2}, \qquad V_{\text{eff}}(r_*) = V_{\text{eff}}(r_s) = E.}$$

$\Sigma_*:\ r = r_*$ (entrance) $\qquad \Sigma_s:\ r = r_s$ (exit).

GB & Steingasser [2604.08660], Sec. II

2. The decay rate is a probability flux

Define the survival and escape probabilities:

$$P_{\mathcal{F}}(t) = \int_{\mathcal{F}} d^2x\, |\Psi(\mathbf{x},t)|^2, \qquad P_{\mathcal{R}}(t) = \int_{\mathcal{R}} d^2x\, |\Psi(\mathbf{x},t)|^2.$$

Assuming negligible back-tunnelling, conservation of probability gives

$$\boxed{\Gamma = -\frac{\dot{P}_{\mathcal{F}}}{P_{\mathcal{F}}} \simeq \frac{\dot{P}_{\mathcal{R}}}{P_{\mathcal{F}}}.}$$

The initial wave function is evolved by the **ordinary propagator**:

$$\Psi(\mathbf{x}_f,t) = \int d^2x_i\, D_F(\mathbf{x}_f,t \mid \mathbf{x}_i,0)\Psi_{E,J}(\mathbf{x}_i),$$

$$D_F(\mathbf{x}_f,t \mid \mathbf{x}_i,0) = \int_{\mathbf{x}_i}^{\mathbf{x}_f} \mathcal{D}\mathbf{x}\, e^{iS_M[\mathbf{x}]}.$$

Hence $P_{\mathcal{R}}$ contains **one** D_F **and one** D_F^*.

Eqs. (2)–(6)

3. Why three restricted propagators?

For a path $\mathbf{x}(t)$ define the crossing-time functionals

$$F_{\Sigma_s}[\mathbf{x}] = \text{first crossing time of } \Sigma_s, \qquad G_{\Sigma_*}[\mathbf{x}] = \text{last crossing time of } \Sigma_*.$$

The crossing conditions restrict the path integral:

$$\overline{D}_F(\mathbf{x}_s, t_s \mid \mathbf{x}_i, 0) \equiv \int \mathcal{D}\mathbf{x}\, e^{iS_M}\, \delta(F_{\Sigma_s} - t_s)\,,$$

$$\underline{D}_F(\mathbf{x}_f, t_f \mid \mathbf{x}_*, t_*) \equiv \int \mathcal{D}\mathbf{x}\, e^{iS_M}\, \delta(G_{\Sigma_*} - t_*)\,,$$

$$\overline{\underline{D}}_F(\mathbf{x}_s, t_s \mid \mathbf{x}_*, t_*) \equiv \int \mathcal{D}\mathbf{x}\, e^{iS_M}\, \delta(G_{\Sigma_*} - t_*)\, \delta(F_{\Sigma_s} - t_s)\,.$$

First crossing identifies the escape event; **last crossing** isolates the final barrier traversal.

Eqs. (7)–(9); endpoint conditions are implicit

4. Decomposing the original propagator

Every path reaching $\mathcal{R}$ must first cross Σ_*, then cross Σ_s. Classify the paths by their **first exit time** t_s:

$$\boxed{D_F(\mathbf{x}_f, t \mid \mathbf{x}_i, 0) = \int_0^t dt_s \int_0^{2\pi} d\varphi_s \, D_F(\mathbf{x}_f, t \mid \mathbf{x}_s, t_s) \overline{D}_F(\mathbf{x}_s, t_s \mid \mathbf{x}_i, 0).}$$

Or classify them by their **last entrance time** t_*:

$$D_F(\mathbf{x}_s, t \mid \mathbf{x}_i, 0) = \int_0^t dt_* \int d\varphi_* \, \underline{D}_F(\mathbf{x}_s, t \mid \mathbf{x}_*, t_*) D_F(\mathbf{x}_*, t_* \mid \mathbf{x}_i, 0),$$

$$\overline{D}_F(\mathbf{x}_s, t \mid \mathbf{x}_i, 0) = \int_0^t dt_* \int d\varphi_* \, \overline{\underline{D}}_F(\mathbf{x}_s, t \mid \mathbf{x}_*, t_*) D_F(\mathbf{x}_*, t_* \mid \mathbf{x}_i, 0).$$

These are **path decompositions**, not saddle-point approximations.

Eqs. (10)–(12); crossing measures absorbed into D_F

5. From first crossings to the outgoing flux

Start from the propagation kernel in $P_{\mathcal{R}}$:

$$p_{\mathcal{R}}(\mathbf{x}_1, \mathbf{x}_2; t) = \int_{\mathcal{R}} d^2x_f \, D_F(\mathbf{x}_f, t \mid \mathbf{x}_1, 0) D_F^*(\mathbf{x}_f, t \mid \mathbf{x}_2, 0).$$

No back-tunnelling: after crossing Σ_s, propagation stays predominantly in $\mathcal{R}$. Consequently

$$\begin{aligned} &\int_{\mathcal{R}} d^2x_f \, D_F(\mathbf{x}_f, t \mid \mathbf{x}_s, t_s) D_F^*(\mathbf{x}_f, t \mid \mathbf{x}'_s, t'_s) \\ &\qquad \simeq D_F(\mathbf{x}'_s, t'_s \mid \mathbf{x}_s, t_s) \qquad (t'_s > t_s), \end{aligned}$$

using completeness of the final states in the escape region. After separating time orderings and differentiating:

$$\boxed{\dot{p}_{\mathcal{R}} \simeq \int_0^{2\pi} d\varphi_s \, \overline{D}_F(\mathbf{x}_s, t \mid \mathbf{x}_1, 0) D_F^*(\mathbf{x}_s, t \mid \mathbf{x}_2, 0) + \text{c.c.}}$$

Eqs. (13)–(19); the approximation is the no-return assumption

6. Insert the $|E, J\rangle$ initial state

Convolve the unrestricted early evolution with the wave function:

$$\boxed{\int d^2x_i \, D_F(\mathbf{x}_*, t_* \mid \mathbf{x}_i, 0)\Psi_{E,J}(\mathbf{x}_i) = \Psi_{E,J}(\mathbf{x}_*, t_*).}$$

Define the two angular-integrated contributions ($\Delta t \equiv t - t_*$):

$$\begin{pmatrix} \mathcal{F}_{E,J} \\ \overline{\mathcal{F}}_{E,J} \end{pmatrix} \equiv \int_0^t d\Delta t \int_0^{2\pi} d\varphi_* \begin{pmatrix} D_F \\ \overline{D}_F \end{pmatrix} (\mathbf{x}_s, \Delta t \mid \mathbf{x}_*) \, \Psi_{E,J}(\mathbf{x}_*, t - \Delta t).$$

The **probability flux** now takes the compact form

$$\boxed{\dot{P}_{\mathcal{R}}(t) \simeq 2 \, \mathrm{Re} \int_0^{2\pi} d\varphi_s \, \overline{\mathcal{F}}^*_{E,J} \mathcal{F}_{E,J}.}$$

The stationary resonance state contains $e^{iJ\varphi_*}$ and e^{-iEt_*}, which will select the saddle.

Eqs. (18)–(24)

7. The real-time path integral needs a regulator

The relevant complex saddles are selected through **slightly nonunitary evolution**:

$$\boxed{H \longrightarrow H_\epsilon = (1 - i\epsilon)H, \qquad 0 < \epsilon \ll 1.}$$

For $H = p_r^2/(2m) + p_\varphi^2/(2mr^2) + V(r)$, its Legendre transform gives the regulated action:

$$\boxed{S_\epsilon = \int_0^{\Delta t} dt \left[\frac{m}{2(1 - i\epsilon)} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) - (1 - i\epsilon) V(r) \right].}$$

This also corresponds to complex time,

$$\boxed{z = (1 - i\epsilon)t.}$$

The approximation is **semiclassical**: restricted path integrals $\sim e^{iS_\epsilon[\bar{x}]}$ on the relevant complex saddle $\bar{x}$.

Eqs. (25)–(26), (50); Hamiltonian-consistent regulator form

8. Where the E- and J-dependent phases enter

Inside the metastable region, neglecting the small physical width in the initial state label,

$$\Psi_{E,J}(r_*, \varphi_*; t - \Delta t) \simeq \psi_{E,J}(r_*; 0)\, e^{iJ\varphi_*} e^{-i(1-i\epsilon)E(t-\Delta t)}.$$

For either restricted path integral, suppressing its crossing conditions and the overall $e^{-i(1-i\epsilon)Et}$, we obtain

$$\boxed{\int_0^t d\Delta t \int_0^{2\pi} d\varphi_* \int_{\text{cross}} \mathcal{D}\mathbf{x}\, \exp\{i\,[S_\epsilon[\mathbf{x}] + J\varphi_* + (1-i\epsilon)E\Delta t]\}\,.}$$

Three independent saddle conditions:

$$\boxed{\delta_{\mathbf{x}} S_\epsilon = 0, \qquad \partial_{\varphi_*}(S_\epsilon + J\varphi_*) = 0, \qquad \partial_{\Delta t}(S_\epsilon + (1-i\epsilon)E\Delta t) = 0.}$$

Eqs. (27)–(32); fixed endpoints and crossing conditions implicit

9. Stationarity of φ_* fixes J

The cyclic equation of motion implies

$$\frac{d}{dt}\left(\frac{mr^2\dot{\varphi}}{1-i\epsilon}\right)=0 \quad\Longrightarrow\quad p^{\epsilon}_{\varphi}=\frac{m\bar{r}^2\dot{\bar{\varphi}}}{1-i\epsilon}=\text{constant}.$$

An initial-angle variation changes the on-shell action by an **endpoint term**:

$$\delta S_\epsilon\big|_{\rm EOM}=\left[p^{\epsilon}_{\varphi}\delta\varphi\right]_0^{\Delta t}=-p^{\epsilon}_{\varphi}\,\delta\varphi_*,$$

where φ_s is fixed. Stationarity of the total exponent then gives

$$0=\frac{\partial S_\epsilon}{\partial\varphi_*}+J=-p^{\epsilon}_{\varphi}+J,$$

$$\boxed{\frac{m\bar{r}^{\,2}\dot{\bar{\varphi}}}{1-i\epsilon}=J \quad\Longrightarrow\quad \dot{\bar{\varphi}}=(1-i\epsilon)\frac{J}{m\bar{r}^2}.}$$

Eqs. (30)–(38); the charge of the saddle is fixed by the state

10. Stationarity of Δt fixes the energy

The propagation time is **integrated over**, not chosen by hand:

$$0 = \frac{\partial S_\epsilon}{\partial \Delta t} + (1 - i\epsilon)E.$$

On shell, a variation of the initial traversal time yields $\partial_{\Delta t} S_\epsilon = -H_\epsilon^{\mathrm{cl}}$, so that the **unregulated classical energy** is E.

$$\frac{m}{2}\left(\frac{\dot{\bar{r}}}{1 - i\epsilon}\right)^2 + V(\bar{r}) + \frac{J^2}{2m\bar{r}^2} = E.$$

At the entrance turning point, $V_{\mathrm{eff}}(r_*) = E$, giving

$$\boxed{\bar{r}(0) = r_*, \qquad \dot{\bar{r}}(0) = 0.}$$

The final crossing is at r_s, and the regulated solution only satisfies the exit condition approximately at finite ϵ:

$$\bar{r}(\Delta t_{\mathrm{per}}) \simeq r_s, \qquad \dot{\bar{r}}(\Delta t_{\mathrm{per}}) \simeq 0.$$

Eqs. (32), (40)–(41); $O(\epsilon)$ endpoint mismatch

11. Integrating out the angle: the fixed-J Routhian

Do not merely substitute $\dot\varphi$ into L. At fixed momentum, first make the partial Legendre transform:

$$\boxed{R_\epsilon(r,\dot r;J) = L_\epsilon - J\dot\varphi, \qquad \frac{\partial L_\epsilon}{\partial \dot\varphi} = J.}$$

Using $\dot\varphi = (1-i\epsilon)J/(mr^2)$ gives

$$\begin{aligned} R_\epsilon &= \frac{m\dot r^2}{2(1-i\epsilon)} + \frac{(1-i\epsilon)J^2}{2mr^2} - (1-i\epsilon)V(r) - (1-i\epsilon)\frac{J^2}{mr^2} \\ &= \boxed{\frac{m\dot r^2}{2(1-i\epsilon)} - (1-i\epsilon)V_{\text{eff}}(r)}. \end{aligned}$$

Thus

$$S_{R,\epsilon} = \int_0^{\Delta t} dt\, R_\epsilon, \qquad \boxed{m\ddot{\bar r} = -(1-i\epsilon)^2 V'_{\text{eff}}(\bar r)}.$$

Eqs. (44)–(46), written directly from H_ϵ

12. The rate: why the normalization matters

At the semiclassical saddle, the initial-angle and final-angle phases cancel between the two factors of $\dot{P}_{\mathcal{R}}$. To leading exponential accuracy,

$$\boxed{\dot{P}_{\mathcal{R}}(t) \sim e^{-2\epsilon E t} \exp[-2\,\mathrm{Im}\, S_{R,\epsilon}[\bar{r}; J] + 2\epsilon E \Delta t_{\mathrm{per}}]\, |\psi_{E,J}(r_*; 0)|^2.}$$

The same regulated evolution gives

$$P_{\mathcal{F}}(t) = \int_{\mathcal{F}} d^2x\, |\Psi_{E,J}(\mathbf{x}, t)|^2 \sim e^{-2\epsilon E t} P_{\mathcal{F}}(0).$$

The unphysical damping cancels in $\Gamma = \dot{P}_{\mathcal{R}}/P_{\mathcal{F}}$:

$$\boxed{\Gamma \sim \exp[-2\,\mathrm{Im}\, S_{R,\epsilon}[\bar{r}; J] + 2\epsilon E \Delta t_{\mathrm{per}}]\, \frac{|\psi_{E,J}(r_*; 0)|^2}{P_{\mathcal{F}}(0)}.}$$

Overall Jacobians and fluctuation determinants are omitted.

Eqs. (47)–(49)

13. The steadyon becomes an Euclidean instanton

For a time-independent Hamiltonian, the regulator amounts to the complex-time contour $z = (1 - i\epsilon)t$.

$$\boxed{\epsilon \to 0^+, \qquad \Delta t_{\rm per} \to \infty, \qquad \epsilon \Delta t_{\rm per} \simeq \Delta \tau_{\rm per}.}$$

Deform the tilted contour into a real segment γ_t and a vertical Euclidean segment γ_τ:

$$\gamma_\tau : \qquad z = \Delta t_{\rm per} - i\tau, \qquad 0 \le \tau \le \Delta \tau_{\rm per}.$$

The real-time segment contributes a **real action**; on γ_τ we obtain

$$\boxed{m r_I''(\tau) = V_{\rm eff}'(r_I), \qquad S_{R,E}[r_I; J] = \int_0^{\Delta \tau_{\rm per}} d\tau \left[\frac{m}{2} (r_I')^2 + V_{\rm eff}(r_I) \right].}$$

$\Delta \tau_{\rm per}$ is one **single** under-barrier traversal; the full bounce traverses the barrier twice.

Eqs. (50)–(59), Fig. 2

Thin Wall estimates: more in details

Thin-wall solution at small charge

We expand around the neutral thin-wall bounce

$$\rho = \rho_0 + \epsilon^2 \rho_2 + \mathcal{O}(\epsilon^4), \qquad \alpha = \epsilon \alpha_1 + \mathcal{O}(\epsilon^3).$$

At leading order, the modulus is the neutral step-wall profile

$$\rho_0(r_E) = \begin{cases} \rho_f, & r_E < R, \\ \rho_i, & r_E > R, \end{cases} \qquad r_E^2 = r^2 + \tau^2, \qquad \tau = r_E \cos\vartheta.$$

The phase is not a step function. It solves the a **dielectric problem in** $4D$

$$\partial_\mu \left[\rho_0^2 \partial^\mu \alpha_1\right] = 0, \qquad \alpha_1 \to \omega_b \tau = (\omega_i + \eta_0/\beta)\tau \quad \text{far away} \qquad \to \qquad \nabla_4^2 \alpha_1 = 0 \qquad \text{in each region.}$$

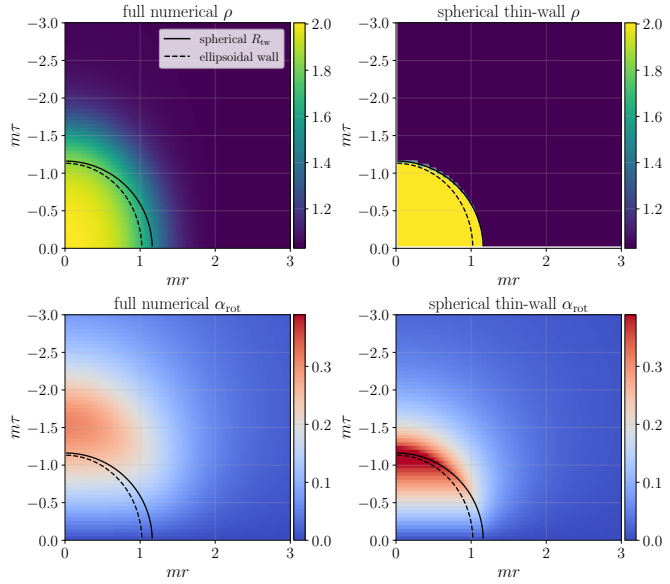
The far-field condition selects only the dipole harmonic

$$\alpha_{\rm in} = A\, \omega_b \tau, \qquad \alpha_{\rm out} = \omega_b \left[1 + B \frac{R^4}{r_E^4}\right] \tau.$$

Matching the phase and the normal charge flux at $r_E = R$ we find A and B. Therefore

$$\boxed{\alpha_{\rm in} = \omega_b \frac{4\rho_i^2}{\rho_f^2 + 3\rho_i^2} \tau, \qquad \alpha_{\rm out} = \omega_b \left[1 + \frac{\rho_i^2 - \rho_f^2}{\rho_f^2 + 3\rho_i^2} \frac{R^4}{r_E^4}\right] \tau.}$$

Thin-wall solution vs numerical bounce



Full numerical solution

$$\rho = \sqrt{2\phi\bar{\phi}}, \quad \alpha_{\rm rot} = \alpha - \omega_i \tau,$$

$$\alpha = \frac{1}{2}\log\frac{\phi}{\bar{\phi}}.$$

Thin-wall solution

$$\rho = \begin{cases} \rho_f, & r_E < R, \\ \rho_i, & r_E > R, \end{cases}$$

$$\alpha = \begin{cases} \omega_i \dfrac{4\rho_i^2}{3\rho_i^2 + \rho_f^2}\,\tau, & r_E < R, \\ \omega_i \left[1 + \dfrac{\rho_i^2 - \rho_f^2}{3\rho_i^2 + \rho_f^2}\dfrac{R^4}{r_E^4}\right]\tau, & r_E > R. \end{cases}$$

Fixed-charge shift δ_Q

The fixed-Q Lee solution gives an effective bulk shift in the thin-wall action:

$$S_{\rm TW}(R) = S_{\rm TW}^{Q=0} + V_4(R)\,\delta_Q$$

Where

$$S_{\rm TW}^{Q=0} \simeq \int d^4x \left[\frac{1}{2}(\rho_0')^2 + V(\rho_0)\right] + O(\epsilon^4)$$

$$S_\alpha^{\rm TW} \simeq \int d\tau \int d^3x\ \frac{1}{2}\rho_0^2(\partial_\mu\alpha_1\partial^\mu\alpha_1)^2 + O(\epsilon^4) \simeq V_4(R)\,\delta_Q$$

The shift is the **dielectric energy gain** of the phase profile,

$$\boxed{\delta_Q = \frac{1}{2}C_{\rm diel}q^2 > 0\ , \qquad q = \omega_i\rho_i^2\ , \qquad C_{\rm diel} = \frac{4\Delta\rho^2}{\rho_i(\rho_f + 3\rho_i)}\ .}$$

Thin-wall estimate: projected exponent

Evaluating the **naive** Euclidean action on the thin-wall solution gives

$$S_E^{\rm bounce} - S_E^i \simeq 2\pi^2\sigma_0 R^3 + \frac{\pi^2 R^4}{2}\left[-\Delta V + \delta_Q\right], \qquad (> S_{Q=0}?)$$

But this is **not** the fixed-Q tunnelling exponent. The fixed-charge projection gives

$$S_Q = S_E^{\rm bounce} - S_E^i + \eta_0 Q.$$

The residual twist is fixed by the **saddle condition** (in the limit $\beta \to \infty$)

$$\frac{\partial S_Q}{\partial \eta_0} = 0 \qquad \Longleftrightarrow \qquad Q_{\rm bounce}(\eta_0) = Q_{\rm target} \quad \to \quad \eta_0 Q \simeq -\pi^2 R^4 \delta_Q < 0.$$

Therefore the **physical projected exponent** is

$$\boxed{S_{Q,\rm tw}(R) \simeq 2\pi^2\sigma_0 R^3 - \frac{\pi^2 R^4}{2}\left[\Delta V + \delta_Q\right].}$$

Extremizing with respect to the radius,

$$\frac{\partial S_{Q,\rm tw}}{\partial R} = 0, \quad \to \quad \boxed{R_Q = \frac{3\sigma_0}{\Delta V + \delta_Q} = \frac{R_0}{1+z_Q}, \qquad z_Q \equiv \frac{\delta_Q}{\Delta V}.}$$

The charge lowers the decay rate exponent and the bubble shrinks!

Warning: energy conservation

The spherical TW configuration will be sufficient for the **action** at order ϵ^2!
The reason is **variational**: the neutral sphere already extremizes the action, so an order-ϵ^2 shape deformation changes S only at higher order.

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However, checking **energy conservation slice by slice** is more restrictive. The explicit Lee phase contribution has a non-trivial τ-dependence

$$E_\alpha^{(2)}(z) \propto (1-z^2)^{3/2}\left(C_0 + C_2 z^2\right), \qquad z = \tau/R\ ,$$

This, with ρ_0, alone do not conserve energy!

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The missing piece is the **modulus response**. In TW language it can be represented by an order-ϵ^2 ellipsoidal deformation,

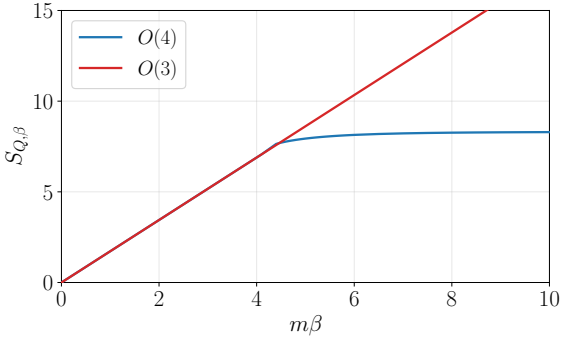
$$a = R_Q(1+\varepsilon^2 A), \qquad b = R_Q(1+\varepsilon^2 B),$$

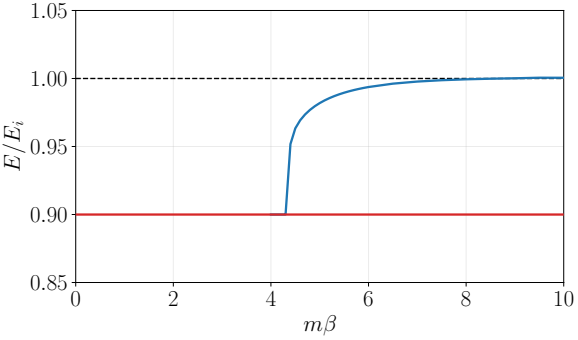
which contributes

$$E_{0,\mathrm{ell}}^{(2)}(z) \propto (1-z^2)^{3/2}\left[A + (A-B)z^2\right].$$

Choosing A and B cancels the two Lee coefficients. Thus the energy is conserved at order ε^2, while the phase solution itself can still be taken spherical: the non-sphericity affects E_α only at order ϵ^4.

Decay rate exponent at fixed Q





Backup: fixed-charge projector

The fixed-charge partition function is obtained by projecting the trace:

$$Z_Q[\beta] = \mathrm{Tr}_Q\, e^{-\beta \hat{H}} = \mathrm{Tr}\left[\hat{P}_Q e^{-\beta \hat{H}}\right].$$

The projector is

$$\hat{P}_Q = \delta[\hat{Q} - Q] = \int \frac{d\theta}{2\pi}\, e^{i\theta[\hat{Q}-Q]}.$$

Therefore

$$Z_Q[\beta] = \int \frac{d\theta}{2\pi}\, e^{-i\theta Q}\, \mathrm{Tr}\left[e^{-\beta \hat{H}} e^{i\theta \hat{Q}}\right].$$

The operator $e^{i\theta\hat{Q}}$ generates a global $U[1]$ rotation:

$$e^{i\theta\hat{Q}}\hat{\Phi}[\mathbf{x}]e^{-i\theta\hat{Q}} = e^{i\theta}\hat{\Phi}[\mathbf{x}], \qquad e^{i\theta\hat{Q}}\hat{\Phi}^\dagger[\mathbf{x}]e^{-i\theta\hat{Q}} = e^{-i\theta}\hat{\Phi}^\dagger[\mathbf{x}].$$

Backup: from the trace to twisted boundary conditions

Let $|\phi\rangle$ be an eigenstate of the field operator,

$$\hat{\Phi}[\mathbf{x}]|\phi\rangle = \phi[\mathbf{x}]|\phi\rangle.$$

Then

$$e^{i\theta\hat{Q}}|\phi\rangle = |e^{i\theta}\phi\rangle,$$

because

$$\hat{\Phi}[\mathbf{x}]\, e^{i\theta\hat{Q}}|\phi\rangle = e^{i\theta\hat{Q}}\left[e^{-i\theta\hat{Q}}\hat{\Phi}[\mathbf{x}]e^{i\theta\hat{Q}}\right]|\phi\rangle = e^{i\theta}\phi[\mathbf{x}]\, e^{i\theta\hat{Q}}|\phi\rangle.$$

Inserting complete sets of field eigenstates:

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}e^{i\theta\hat{Q}}\right] = \int \mathcal{D}\phi\,\mathcal{D}\phi^*\,\langle\phi|e^{-\beta\hat{H}}e^{i\theta\hat{Q}}|\phi\rangle = \int \mathcal{D}\phi\,\mathcal{D}\phi^*\,\langle\phi|e^{-\beta\hat{H}}|e^{i\theta}\phi\rangle.$$

Hence the path integral has twisted closure:

$$\Phi[\beta/2,\mathbf{x}] = e^{i\theta}\Phi[-\beta/2,\mathbf{x}], \qquad \Phi^*[\beta/2,\mathbf{x}] = e^{-i\theta}\Phi^*[-\beta/2,\mathbf{x}].$$

Backup: imaginary twist and real Euclidean fields

The projected path integral is

$$Z_Q[\beta] = \int \frac{d\theta}{2\pi}\, e^{-i\theta Q} \int_{\text{twisted}} \mathcal{D}\Phi\, \mathcal{D}\Phi^*\, e^{-S_E[\Phi,\Phi^*]}.$$

For real charge, the saddle lies at imaginary twist:

$$\theta = -i\eta, \qquad \eta \in \mathbb{R}.$$

Thus

$$\Phi[\beta/2, \mathbf{x}] = e^{\eta}\Phi[-\beta/2, \mathbf{x}], \qquad \Phi^*[\beta/2, \mathbf{x}] = e^{-\eta}\Phi^*[-\beta/2, \mathbf{x}].$$

The Euclidean saddle is complex in the original variables, but can be written as a real problem with

$$\phi[\tau, \mathbf{x}], \qquad \bar{\phi}[\tau, \mathbf{x}]$$

treated as independent fields.
Polar form:

$$\phi = \frac{\rho}{\sqrt{2}} e^{\alpha}, \qquad \bar{\phi} = \frac{\rho}{\sqrt{2}} e^{-\alpha}.$$

$$S_E[\phi, \bar{\phi}] = \int d\tau\, d^3x \left[\partial_\tau \phi\, \partial_\tau \bar{\phi} + \nabla\phi \cdot \nabla\bar{\phi} + V[\phi\bar{\phi}]\right].$$

Backup: fixed-Q saddle over the twist

At fixed η, let Φ_η be the Euclidean saddle satisfying the twisted closure. Then

$$Z_Q[\beta] \sim \int d\eta \, \exp\left[-S_E[\Phi_\eta, \Phi_\eta^*] - \eta Q\right].$$

The saddle condition in η fixes the charge:

$$Q = -\left.\frac{\partial S_E[\Phi_\eta, \Phi_\eta^*]}{\partial \eta}\right|_{\eta=\eta_{\rm cl}}.$$

At the dominant saddle:

$$S_{Q,\beta}^{\rm bounce} = S_E[\Phi_{\eta_{\rm cl}}, \Phi_{\eta_{\rm cl}}^*] + \eta_{\rm cl} Q.$$

The decay rate is extracted from the bounce contribution divided by the metastable homogeneous contribution:

$$\Gamma \propto \frac{\mathrm{Im}\, Z_Q^{\rm bounce}}{Z_Q^{\rm fv}} \sim e^{-S_{Q,\beta}}.$$

Thus

$$S_{Q,\beta} = S_E[\phi_b, \bar{\phi}_b] + \eta_{\rm cl} Q - \beta E_i[Q].$$

Backup: reference subtraction and residual twist

The homogeneous charged reference state is

$$\phi_i[\tau] = \frac{\rho_i}{\sqrt{2}} e^{+\omega\tau}, \qquad \bar{\phi}_i[\tau] = \frac{\rho_i}{\sqrt{2}} e^{-\omega\tau}.$$

Its charge and energy are

$$Q = V_3 \omega \rho_i^2, \qquad E_i[Q] = V_3 \left[V[\rho_i^2/2] + \frac{1}{2}\omega^2 \rho_i^2 \right].$$

The Euclidean action of the homogeneous reference is

$$S_E[\phi_i, \bar{\phi}_i] = \beta V_3 \left[V[\rho_i^2/2] - \frac{1}{2}\omega^2 \rho_i^2 \right] = \beta \left[E_i[Q] - \omega Q \right].$$

The total twist is separated as

$$\eta_{\rm cl} = \omega\beta + \eta_0, \qquad \eta_0 \equiv \eta_{\rm cl} - \omega\beta.$$

Therefore

$$S_{Q,\beta} = S_E[\phi_b, \bar{\phi}_b] - S_E[\phi_i, \bar{\phi}_i] + \eta_0 Q.$$

$$\boxed{S_{Q,\beta} = S_E^{\rm b} - S_E^i + \eta_0 Q}$$

Backup: BVP equations and boundary conditions

Assuming spherical symmetry, $\phi = \phi[\tau, r]$, $\bar{\phi} = \bar{\phi}[\tau, r]$:

$$\partial_\tau^2 \phi + \partial_r^2 \phi + \frac{2}{r} \partial_r \phi - V'[\phi\bar{\phi}]\phi = 0,$$

$$\partial_\tau^2 \bar{\phi} + \partial_r^2 \bar{\phi} + \frac{2}{r} \partial_r \bar{\phi} - V'[\phi\bar{\phi}]\bar{\phi} = 0.$$

Boundary conditions:

$$\partial_r \phi[\tau, 0] = 0, \qquad \partial_r \bar{\phi}[\tau, 0] = 0,$$
$$\phi[\tau, L_r] \to \phi_i[\tau], \qquad \bar{\phi}[\tau, L_r] \to \bar{\phi}_i[\tau].$$

Turning slice:

$$\phi[0, r] = \bar{\phi}[0, r], \qquad \partial_\tau \phi[0, r] = -\partial_\tau \bar{\phi}[0, r].$$

Twisted reflected closure:

$$\phi[-\beta/2, r] = e^{-\eta_{\mathrm{cl}}} \bar{\phi}[-\beta/2, r].$$

Backup: Euclidean energy conservation

Define

$$C[\tau] = \int d^3x \left[-\partial_\tau \phi \, \partial_\tau \bar{\phi} + \nabla\phi \cdot \nabla\bar{\phi} + V[\phi\bar{\phi}] \right].$$

Using the equations of motion,

$$\frac{dC}{d\tau} = 0.$$

In polar variables:

$$C[\tau] = 4\pi \int dr \, r^2 \left[\frac{1}{2}\rho_r^2 - \frac{1}{2}\rho^2\alpha_r^2 + \frac{1}{2}\rho^2\alpha_\tau^2 - \frac{1}{2}\rho_\tau^2 + V[\rho] \right].$$

At the turning slice, $\rho_\tau[0, r] = 0$, hence

$$C[0] = E_{\text{bubble}}.$$

Asymptotically,

$$C[-\beta/2] \simeq E_{\text{homogeneous}}.$$

Therefore the nucleated bubble has the same conserved energy as the initial charged state.

Backup: effective potential Ω

For a homogeneous rotating state,

$$\Phi[t] = \frac{\rho}{\sqrt{2}} e^{i\omega t}.$$

The modulus equation is controlled by

$$\Omega[\rho;\omega] = V[\rho] - \frac{1}{2}\omega^2\rho^2.$$

Stationary homogeneous branches satisfy

$$\frac{d\Omega}{d\rho} = 0.$$

Interpretation:

$$\Omega = \text{potential felt by } \rho \text{ at fixed rotation frequency.}$$

Important caveat:

$$\Omega[\rho;\omega_i] \text{ is useful for intuition and seeds,}$$

but the full fixed-Q bounce is not obtained by holding ω fixed everywhere.

$$Q = \int d^3x\, \rho^2 \partial_\tau \alpha$$

must be conserved globally.

Backup: charged Bitar–Chang barrier

The fixed-Q exponent can be rewritten using a modified Euclidean action:

$$S_{Q,\beta} = S_E^{\text{bounce}} - S_E^i + \eta_{\text{cl}} Q = \Delta \widetilde{S}_E.$$

Using the phase EOM and the boundary twist,

$$\eta_{\text{cl}} Q = \int d^4x \, \rho^2 \left[(\partial_\tau \alpha)^2 + (\nabla \alpha)^2 \right],$$

so that

$$\widetilde{S}_E = \int d^4x \left[\frac{1}{2}(\partial_\tau \rho)^2 + \frac{1}{2}\rho^2 (\partial_\tau \alpha)^2 + \frac{1}{2}(\nabla \rho)^2 + \frac{1}{2}\rho^2 (\nabla \alpha)^2 + V(\rho) \right].$$

Thus the projection term turns the phase contributions in S_E into positive terms in $\widetilde{S}_E$.

Backup: Bitar–Chang energy interpretation

The conserved Euclidean energy is

$$\mathcal{E} = \int d^3x \left[-\frac{1}{2}(\partial_\tau \rho)^2 + \frac{1}{2}\rho^2 (\partial_\tau \alpha)^2 + \frac{1}{2}(\nabla \rho)^2 - \frac{1}{2}\rho^2 (\nabla \alpha)^2 + V(\rho) \right].$$

Using $\Delta \mathcal{E} = 0$, the modified action becomes

$$\boxed{\Delta \widetilde{S}_E = 2 \int_{-\beta/2}^{0} d\tau \, \Delta \widetilde{E}(\tau)}$$

with the charged Bitar–Chang barrier

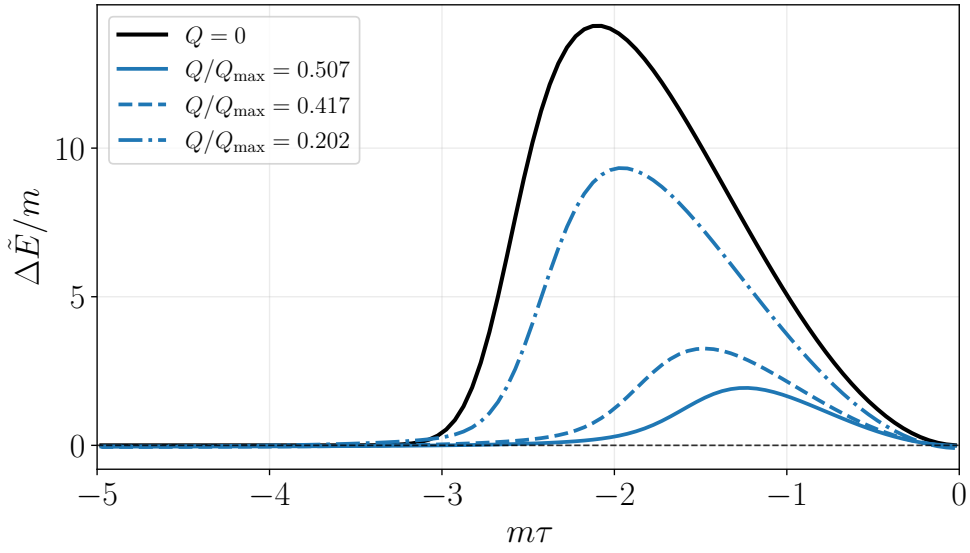
$$\boxed{\widetilde{E}(\tau) = \int d^3x \left[\frac{1}{2}(\nabla \rho)^2 + \frac{1}{2}\rho^2 (\partial_\tau \alpha)^2 + V(\rho) \right].}$$

At the endpoints,

$$\partial_\tau \rho = 0, \qquad \nabla \alpha = 0, \qquad \widetilde{E} = E,$$

so the barrier vanishes at the initial state and at the turning slice.

BC energy barrier at $Q \neq 0$



Backup: thin-wall phase solution I

Thin-wall modulus:

$$\rho[\xi] = \begin{cases} \rho_f, & \xi < R, \\ \rho_i, & \xi > R, \end{cases} \qquad \xi^2 = r^2 + \tau^2, \qquad \tau = \xi \cos\vartheta.$$

The phase solves

$$\partial_\mu \left[\rho^2 \partial^\mu \alpha\right] = 0.$$

Since ρ is constant in each region:

$$\nabla_4^2 \alpha = 0 \qquad \text{inside and outside.}$$

Far away:

$$\alpha_{\text{out}} \to \omega_i \tau = \omega_i \xi \cos\vartheta.$$

Therefore only the dipole harmonic is needed:

$$\alpha_{\text{in}} = A\, \omega_i \xi \cos\vartheta, \qquad \alpha_{\text{out}} = \omega_i \left[\xi + B \frac{R^4}{\xi^3}\right] \cos\vartheta.$$

Equivalently,

$$\alpha_{\text{in}} = A \omega_i \tau, \qquad \alpha_{\text{out}} = \omega_i \left[1 + B \frac{R^4}{\xi^4}\right] \tau.$$

Backup: thin-wall phase solution II

At the wall, impose continuity of the phase:

$$\alpha_{\rm in}[R] = \alpha_{\rm out}[R]. \quad \rightarrow \quad A = 1 + B.$$

The charge flux must also be continuous: $\rho_f^2 \partial_\xi \alpha_{\rm in} = \rho_i^2 \partial_\xi \alpha_{\rm out}$ at $\xi = R$.
Using

$$\partial_\xi \alpha_{\rm in} = A\omega_i \cos\vartheta, \qquad \partial_\xi \alpha_{\rm out} = \omega_i \left[1 - 3B\right] \cos\vartheta,$$

we get

$$\rho_f^2 A = \rho_i^2 [1 - 3B].$$

Solving together with $A = 1 + B$:

$$B = \frac{\rho_i^2 - \rho_f^2}{\rho_f^2 + 3\rho_i^2}, \qquad A = \frac{4\rho_i^2}{\rho_f^2 + 3\rho_i^2}.$$

Therefore

$$\alpha_{\rm in} = \omega_i \frac{4\rho_i^2}{\rho_f^2 + 3\rho_i^2} \tau, \qquad \alpha_{\rm out} = \omega_i \left[1 + \frac{\rho_i^2 - \rho_f^2}{\rho_f^2 + 3\rho_i^2} \frac{R^4}{\xi^4}\right] \tau.$$

Backup: terminal velocity from a Lorentzian shell

A controlled estimate should be done directly in Lorentzian signature. Assume a late-time self-similar wall,

$$R_M(t) = vt, \qquad \xi = \frac{r}{t}, \qquad \xi_{\rm wall} = v, \qquad \xi_{\rm lc} = 1.$$

Solve the Lorentzian phase equation

$$\partial_t(\epsilon\,\partial_t\alpha) - \frac{1}{r^2}\partial_r\left(r^2\epsilon\,\partial_r\alpha\right) = 0, \qquad \epsilon = \rho^2,$$

with

$$\alpha(t,r) = \Omega t\, f(\xi), \qquad \epsilon(\xi) = \begin{cases} \epsilon_f, & 0 < \xi < v, \\ \epsilon_i, & v < \xi < 1. \end{cases}$$

For constant ϵ,

$$(\xi^2-1)f'' - \frac{2}{\xi}f' = 0, \qquad f(\xi) = c_1 + c_2\left(\xi + \frac{1}{\xi}\right).$$

Regularity and $f(1) = 1$ give

$$f_f(\xi) = f_w, \qquad f_i(\xi) = 1 + C(v)\left(\xi + \frac{1}{\xi} - 2\right).$$

Backup: matching and shell energy

Continuity at the wall gives

$$f_w = 1 + C(v)\left(v + \frac{1}{v} - 2\right).$$

Flux conservation across the moving wall gives

$$\epsilon_f \left[\partial_r \alpha_f + v\partial_t \alpha_f\right] = \epsilon_i \left[\partial_r \alpha_i + v\partial_t \alpha_i\right]_{\xi=v}.$$

Using

$$\partial_t \alpha = \Omega\left(f - \xi f'\right), \qquad \partial_r \alpha = \Omega f',$$

one obtains

$$C(v) = -\frac{\Delta\epsilon\, v}{\Delta\epsilon\, v\left[v + \frac{1}{v} - 2\right] + \epsilon_i \frac{(1-v^2)^2}{v^2}}.$$

The Lorentzian phase-shell energy scales as

$$E_\alpha^M = 4\pi t^3\, \mathcal{E}_\alpha(v),$$

with

$$\mathcal{E}_\alpha(v) = \frac{\Omega^2}{2}\left[I_{\rm in}(v) + \epsilon_i I_{\rm sh}(v)\right].$$

Backup: self-similar terminal velocity

The interior piece is

$$I_{\rm in}(v) = \frac{v^3}{3}\left[\epsilon_f f_w^2 - \epsilon_i\right].$$

The shell piece is

$$I_{\rm sh}(v) = \int_v^1 d\xi\, \xi^2 \left\{\left[f - \xi f'\right]^2 + \left[f'\right]^2 - 1\right\}.$$

For the matched shell solution this integral is analytic:

$$I_{\rm sh}(v) = -\frac{C(v)(v-1)^2}{3v}\left[5C(v)v^2 - 2C(v)v - 3C(v) - 4v^2 - 2v\right].$$

The released vacuum energy is

$$E_{\rm vac} = \frac{4\pi}{3} R_M^3 \Delta V = \frac{4\pi}{3} v^3 t^3 \Delta V.$$

Thus $v_\star$ is fixed by

$$\boxed{\mathcal{E}_\alpha(v_\star) = \frac{\Delta V}{3} v_\star^3}.$$

Equivalently, with

$$E_\alpha^M = 4\pi R_M^3 \gamma\, \alpha_{\rm TW}^{\rm ss}(v), \qquad \rightarrow \qquad \boxed{\gamma_\star\, \alpha_{\rm TW}^{\rm ss}(v_\star) = \frac{\Delta V}{3}}.$$

Backup: Chemical potential for the tunnelling field

For the complex scalar, the **chemical potential is conjugate to the same charge carried by the tunnelling field**. The homogeneous charged state is therefore a rotating configuration,

$$\Phi_i(t) = \frac{\rho_i}{\sqrt{2}} e^{i\omega t}.$$

In a grand-canonical ensemble, an external reservoir fixes

$$\Omega = E - \mu_{\rm ext} Q.$$

For the homogeneous state,

$$\omega_{\rm GC}(\rho_i, \omega; \mu_{\rm ext}) = V(\rho_i^2) + \frac{1}{2}\omega^2 \rho_i^2 - \mu_{\rm ext}\omega\rho_i^2.$$

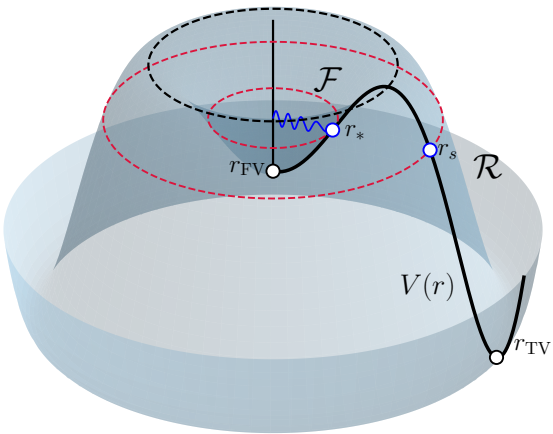
Stationarity with respect to the rotation frequency gives

$$\frac{\partial \omega_{\rm GC}}{\partial \omega} = (\omega - \mu_{\rm ext})\rho_i^2 = 0 \qquad \Longrightarrow \qquad \omega = \mu_{\rm ext}.$$

Hence the **chemical potential fixes the rotation frequency**, while the initial **charge is not independent**

$$Q_i(\mu_{\rm ext}) = V_3\, \mu_{\rm ext}\, \rho_i^2(\mu_{\rm ext}).$$

Mechanical analogue: 2d point particle in a Mexican hat



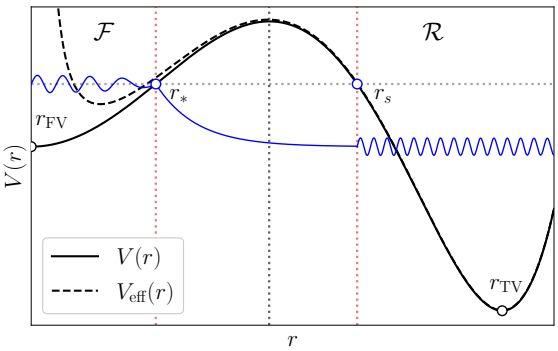


Figure: Illustration of the mechanical model. A point particle moves in a rotationally invariant potential $V(r)$ with a false-vacuum circle at $r = r_{\rm FV}$ and a lower-lying true-vacuum circle at $r = r_{\rm TV}$, with $r_{\rm FV} < r_{\rm TV}$.

See **2604.08660** with T. Steingasser

Picard-Leftschez thimbles 101

Complex saddles and Lefschetz thimbles

Start from an oscillatory path integral and complexify the **integration variables**:

$$I = \int_{\mathcal{C}_{\rm real}} \mathcal{D}X\, e^{F[X]}, \qquad F[X] = \frac{i}{\hbar} S[X].$$

Saddle: a stationary configuration

$$\boxed{\left.\frac{\delta F}{\delta X}\right|_{\bar{X}_\sigma} = 0.}$$

Upward thimble $\mathcal{K}_\sigma$: the dual steepest-ascent cycle.

It determines whether the saddle contributes:

$$\boxed{n_\sigma = \langle \mathcal{C}_{\rm real}, \mathcal{K}_\sigma \rangle.}$$

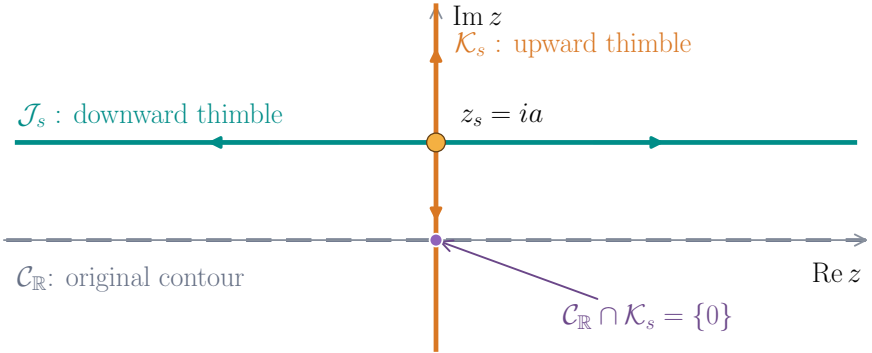
Downward thimble $\mathcal{J}_\sigma$: the steepest-descent integration cycle through $\bar{X}_\sigma$.

$$\boxed{\mathcal{C}_{\rm real} \sim \sum_\sigma n_\sigma \mathcal{J}_\sigma, \qquad I = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathcal{D}X\, e^{F[X]}.}$$

QM: $X = \{x(t)\}$. **QFT:** $X = \{\phi(t, \mathbf{x})\}$.

A Gaussian example: which saddle contributes?

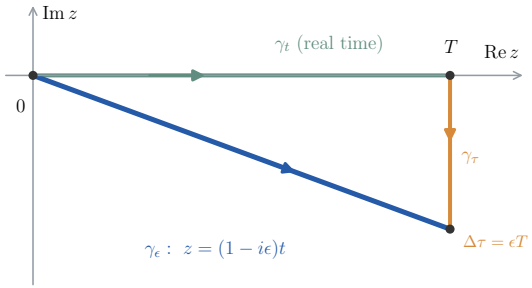
$$I(a) = \int_{-\infty}^{+\infty} dx\, e^{-x^2/2+iax}, \qquad F(z) = -\frac{z^2}{2} + iaz, \qquad F'(z_s) = 0 \Rightarrow z_s = ia.$$



$$\boxed{\mathcal{C}_{\mathbb{R}} \cap \mathcal{K}_s = \{0\} \;\Rightarrow\; n_s = 1, \quad z = ia + u \in \mathcal{J}_s : \quad I(a) = \sqrt{2\pi}\, e^{-a^2/2}.}$$

Steadyons: the time contour is not a thimble

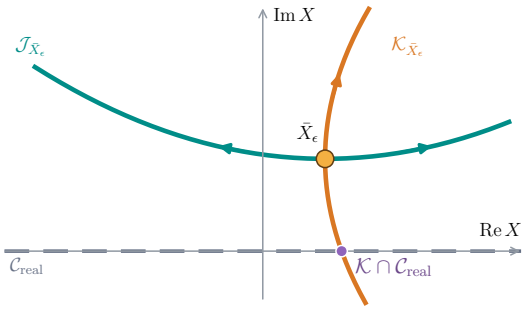
Complex-time plane



$\gamma_\epsilon \subset \mathbb{C}_{\text{time}}$

Tilt the time contour to recover the Euclidean segment.

Complexified path space (schematic)



$\bar{X}_\epsilon \in \mathcal{X}_{\mathbb{C}}, \quad \mathcal{J}_{\bar{X}_\epsilon} \subset \mathcal{X}_{\mathbb{C}}$

Deform the integration cycle to study contributing saddles.

The steadyon is a saddle (an entire complex trajectory), not a point on γ_ϵ.

The illustrated intersection is hypothetical: steadyons do not explicitly compute n_σ.

Numerics

Backup: numerical variables

The independent Euclidean fields are

$$\phi[\tau, r], \qquad \bar{\phi}[\tau, r], \qquad \rho[\tau, r] = \sqrt{2\phi\bar{\phi}}.$$

The homogeneous charged background is

$$\phi_i[\tau] = \varphi_i e^{+\omega\tau}, \qquad \bar{\phi}_i[\tau] = \varphi_i e^{-\omega\tau}, \qquad \varphi_i = \frac{\rho_i}{\sqrt{2}}.$$

To remove the homogeneous exponential, use rotated fields:

$$\phi = e^{+\omega\tau}\phi_{\rm rot}, \qquad \bar{\phi} = e^{-\omega\tau}\bar{\phi}_{\rm rot}.$$

In the rotated frame the homogeneous state is constant:

$$\phi_{{\rm rot},i} = \bar{\phi}_{{\rm rot},i} = \varphi_i.$$

Define regularized unknowns:

$$y = r[\phi_{\rm rot} - \varphi_i], \qquad \bar{y} = r[\bar{\phi}_{\rm rot} - \varphi_i].$$

Equivalently,

$$\phi_{\rm rot} = \frac{y_{\rm tot}}{r}, \qquad \bar{\phi}_{\rm rot} = \frac{\bar{y}_{\rm tot}}{r},$$

with

$$y_{\rm tot} = y + r\varphi_i, \qquad \bar{y}_{\rm tot} = \bar{y} + r\varphi_i, \qquad u = \frac{y_{\rm tot}\bar{y}_{\rm tot}}{r^2}.$$

Backup: residual equations

The factor of r removes the explicit spherical first derivative. The continuum equations solved by the code are

$$0 = \partial_\tau^2 y + \partial_r^2 y + 2\omega\,\partial_\tau y + A[u]\,y_{\rm tot},$$
$$0 = \partial_\tau^2 \bar{y} + \partial_r^2 \bar{y} - 2\omega\,\partial_\tau \bar{y} + A[u]\,\bar{y}_{\rm tot}.$$

Here

$$A[u] = \omega^2 - \frac{1}{\rho[u]}\left.\frac{dU}{d\rho}\right|_{\rho=\rho[u]}, \qquad \rho[u] = \sqrt{2u}.$$

The homogeneous branch satisfies

$$A[\varphi_i^2] = 0.$$

The residuals are discretized with centered finite differences in the bulk.
Important point:

all nonlinear terms and twists act on $y_{\rm tot}, \bar{y}_{\rm tot},$ not only on $y, \bar{y}$.

Otherwise the full fixed-charge closure is not imposed.

Backup: grid and half-domain

The Euclidean solution is solved only on the half-domain

$$r \in [0, L_r], \qquad \tau \in [-\beta/2, 0].$$

The branch at $\tau > 0$ is reconstructed using reflection and exchange:

$$\phi[\tau, r] = \bar{\phi}[-\tau, r], \qquad \bar{\phi}[\tau, r] = \phi[-\tau, r].$$

Use a staggered grid:

$$r_j = [j+1]\Delta r, \qquad \tau_i = -\left[i + \frac{1}{2}\right]\Delta\tau,$$

$$\Delta r = \frac{L_r}{N_r}, \qquad \Delta\tau = \frac{\beta}{2N_\tau}.$$

This avoids placing points directly on

$$\tau = 0, \qquad \tau = -\beta/2,$$

so both time boundaries are implemented by ghost relations.
The residual twist is

$$\eta_0 = \eta_{\mathrm{cl}} - \omega\beta.$$

In the rotated frame:

$$\phi_{\mathrm{rot}}[\tau + \beta, r] = e^{\eta_0}\phi_{\mathrm{rot}}[\tau, r], \qquad \bar{\phi}_{\mathrm{rot}}[\tau + \beta, r] = e^{-\eta_0}\bar{\phi}_{\mathrm{rot}}[\tau, r].$$

Backup: ghost boundary conditions

At the turning slice, reflection and exchange give

$$y_{j,-1} = \bar{y}_{j,0}, \qquad \bar{y}_{j,-1} = y_{j,0}.$$

This implements

$$\phi[0,r] = \bar{\phi}[0,r], \qquad \partial_\tau \phi[0,r] = -\partial_\tau \bar{\phi}[0,r].$$

At the past Euclidean boundary, twist plus reflection gives the total-field ghost rule:

$$y_{\mathrm{tot},j,N_\tau} = e^{-\eta_0}\, \bar{y}_{\mathrm{tot},j,N_\tau-1},$$

$$\bar{y}_{\mathrm{tot},j,N_\tau} = e^{+\eta_0}\, y_{\mathrm{tot},j,N_\tau-1}.$$

In terms of fluctuations:

$$y_{j,N_\tau} = e^{-\eta_0}\,[\bar{y}_{j,N_\tau-1} + r_j\varphi_i] - r_j\varphi_i,$$
$$\bar{y}_{j,N_\tau} = e^{+\eta_0}\,[y_{j,N_\tau-1} + r_j\varphi_i] - r_j\varphi_i.$$

Outer radial boundary:

$$\partial_r \phi_{\mathrm{rot}}[L_r,\tau] = 0, \qquad \partial_r \bar{\phi}_{\mathrm{rot}}[L_r,\tau] = 0.$$

Equivalently,

$$\partial_r y - \frac{y}{L_r} = 0, \qquad \partial_r \bar{y} - \frac{\bar{y}}{L_r} = 0.$$

Backup: Newton solve

All grid values are packed into a real vector

$$\mathbf{x} = [\mathrm{Re}\, y,\ \mathrm{Im}\, y,\ \mathrm{Re}\, \bar{y},\ \mathrm{Im}\, \bar{y}]\,.$$

The nonlinear system is

$$\mathbf{F}[\mathbf{x}; \eta_0] = 0.$$

At fixed η_0, Newton iteration solves

$$\mathbf{J}[\mathbf{x}^{[k]}]\, \delta\mathbf{x}^{[k]} = -\mathbf{F}[\mathbf{x}^{[k]}],$$
$$\mathbf{x}^{[k+1]} = \mathbf{x}^{[k]} + \alpha_k \delta\mathbf{x}^{[k]}.$$

Here

$$\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$$

is an analytic sparse Jacobian, and $\alpha_k \in [0, 1]$ is chosen by backtracking.
Because the residual is local, the Jacobian has a sparse stencil structure. Each grid point carries a local block mixing

$$y, \bar{y}$$

through the nonlinear dependence

$$A[u], \qquad u = \frac{y_{\mathrm{tot}} \bar{y}_{\mathrm{tot}}}{r^2}.$$

Backup: fixed-charge outer loop

The fixed-charge problem is solved by nesting the Newton solve inside a one-dimensional root search for η_0.
For a trial η_0:

$$\text{solve } \mathbf{F}[\mathbf{x}; \eta_0] = 0.$$

Then compute the Euclidean charge:

$$q_E[\tau, r] = \bar{\phi}\, \partial_\tau \phi - \phi\, \partial_\tau \bar{\phi},$$

$$Q[\tau] = 4\pi \int_0^{L_r} dr\, r^2 q_E[\tau, r].$$

The charge is evaluated at the turning slice using the same ghost rules as the residual.
The outer equation is

$$\mathcal{Q}[\eta_0] \equiv Q_{\mathrm{bounce}}[\eta_0] - Q_{\mathrm{target}} = 0.$$

In practice:

$$Q_{\mathrm{target}} \text{ is scanned by continuation,}$$

and each solution is warm-started from the previous one.
Diagnostics:

$$\|\mathbf{F}\| \ll 1, \qquad Q_{\mathrm{bounce}} = Q_{\mathrm{target}},$$

$$\Delta E_{\mathrm{static}}[0] \simeq 0, \qquad S_{Q,\beta} = S_E^b - S_E^i + \eta_0 Q.$$

Backup: seed families

Newton iteration is local: the final branch depends on the seed.
The one-dimensional profiles solve reduced bounce equations and are embedded in the two-dimensional grid.
$O[4]$-**connected seed:**

$$s[r,\tau] = \sqrt{r^2 + \left(\frac{\tau}{\alpha_\tau}\right)^2}.$$

Use the $d = 4$ profile:

$$\phi_{2d}^{[4]}[r,\tau] = \varphi_i + w_\tau[\tau]\left[\varphi_{1d}^{[4]}[s[r,\tau]] - \varphi_i\right].$$

This targets the quantum tunnelling branch connected to Coleman as $Q \to 0$.
$O[3]$-**static seed:**

$$\phi_{2d}^{[3]}[r,\tau] = \varphi_i + A_{\text{seed}}\left[\varphi_{1d}^{[3]}[r] - \varphi_i\right].$$

This probes the static thermal branch.
After building ϕ_{2d}:

$$y = r[\phi_{2d} - \varphi_i], \qquad \bar{y} = y.$$

Then Newton treats y and $\bar{y}$ independently.

Backup: Minkowski continuation

The Euclidean turning slice gives the real-time Cauchy data.
Turning conditions:

$$\phi[0,r] = \bar{\phi}[0,r], \qquad \partial_\tau \phi[0,r] = -\partial_\tau \bar{\phi}[0,r].$$

With the code convention

$$\Phi[t,r] = \phi[it,r], \qquad \Phi^*[t,r] = \bar{\phi}[it,r],$$

the initial data are

$$\Phi[0,r] = \phi[0,r], \qquad \Phi^*[0,r] = \bar{\phi}[0,r],$$
$$\dot{\Phi}[0,r] = i\partial_\tau \phi[0,r], \qquad \dot{\Phi}^*[0,r] = i\partial_\tau \bar{\phi}[0,r].$$

The turning conditions ensure

$$\Phi^*[0,r] = \Phi[0,r]^*, \qquad \dot{\Phi}^*[0,r] = \dot{\Phi}[0,r]^*.$$

In polar variables:

$$\Phi[t,r] = \frac{\rho[t,r]}{\sqrt{2}} e^{i\theta[t,r]}.$$

At $t = 0$:

$$\rho[0,r] = \rho_*[r], \qquad \dot{\rho}[0,r] = 0,$$

but

$$\dot{\theta}[0,r] \neq 0,$$

because the phase carries the fixed charge.

Backup: real-time equations and discretization

The evolution is performed directly in complex fields:

$$\ddot{\phi} = \nabla_r^2 \phi - C[\phi, \bar{\phi}]\phi, \qquad \ddot{\bar{\phi}} = \nabla_r^2 \bar{\phi} - C[\phi, \bar{\phi}]\bar{\phi},$$

with

$$C[\phi, \bar{\phi}] = \frac{1}{\rho}\frac{dU}{d\rho}, \qquad \rho = \sqrt{2\phi\bar{\phi}}.$$

The spherical Laplacian is discretized in finite-volume form:

$$\mathcal{V}_i[\nabla_r^2 f]_i = \mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2},$$

$$\mathcal{F}_{i+1/2} = 4\pi r_{i+1/2}^2 \frac{f_{i+1} - f_i}{r_{i+1} - r_i}.$$

This enforces regularity at the origin through the vanishing area factor.
Time evolution uses velocity-Verlet:

$$\phi^{n+1} = \phi^n + \Delta t\, \dot{\phi}^n + \frac{1}{2}\Delta t^2 a_\phi^n,$$

$$\dot{\phi}^{n+1} = \dot{\phi}^{n+1/2} + \frac{1}{2}\Delta t\, a_\phi^{n+1},$$

and similarly for $\bar{\phi}$.

Backup: real-time diagnostics

Minkowski charge density:

$$q_M = -i\left[\bar{\phi}\,\dot{\phi} - \dot{\bar{\phi}}\,\phi\right].$$

Total charge:

$$Q[t] = 4\pi \int_0^{L_r} dr\, r^2 q_M[t, r].$$

Minkowski energy:

$$E_M = 4\pi \int_0^{L_r} dr\, r^2 \left[\dot{\phi}\,\dot{\bar{\phi}} + \partial_r\phi\,\partial_r\bar{\phi} + U[\rho]\right].$$

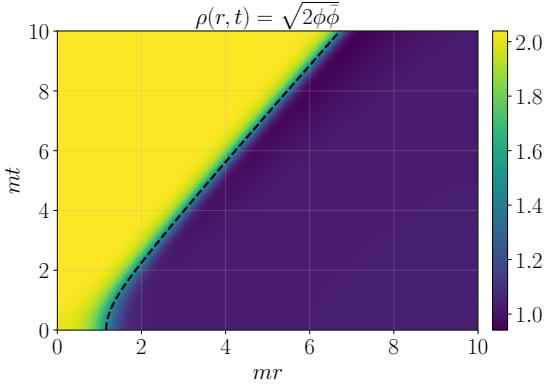
On the physical branch:

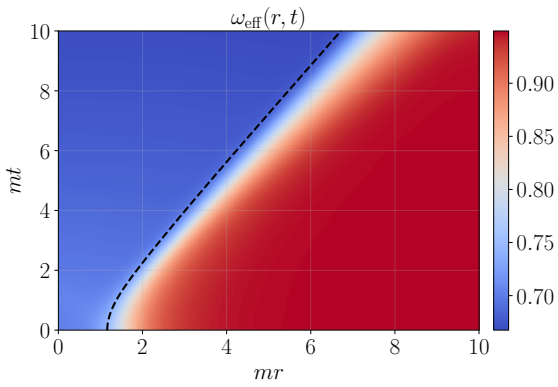
$$E_M = 4\pi \int dr\, r^2 \left[\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\rho'^2 + \frac{1}{2}\rho^2\dot{\theta}^2 + \frac{1}{2}\rho^2\theta'^2 + U[\rho]\right].$$

Bubble observables:

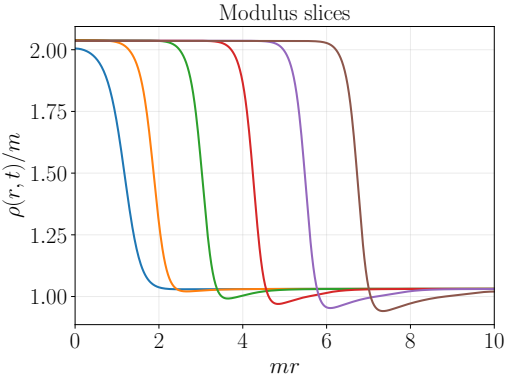
$$R[t] = \text{level set of } \rho[t, r], \qquad v[t] = \dot{R}[t], \qquad \gamma[t] = \frac{1}{\sqrt{1 - v[t]^2}}.$$

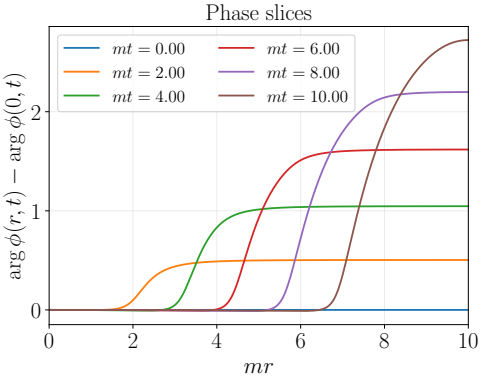
Backup: Real-time observables



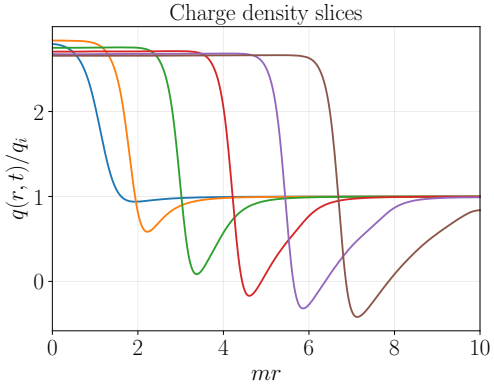


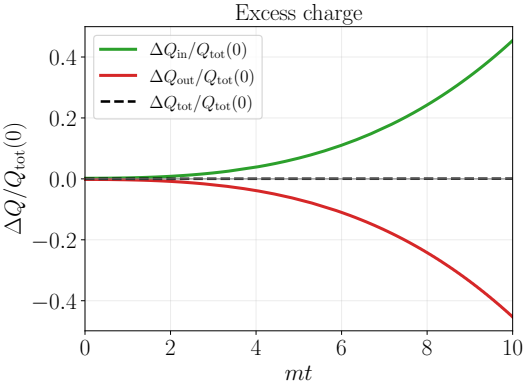
Backup: Real-time observables





Backup: Real-time observables





Backup: wall energy

Decompose the physical energy density:

$$e_{\rm tot} = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\rho'^2 + \frac{1}{2}\rho^2\dot{\theta}^2 + \frac{1}{2}\rho^2\theta'^2 + U[\rho].$$

At each time subtract the bulk plateaux:

$$E_{\rm wall}[t] = 4\pi \int dr\, r^2 \left[e_{\rm tot}[t,r] - e_{\rm bulk}[t,r]\right].$$

Then define

$$\sigma_{\rm lab}[R] = \frac{E_{\rm wall}}{4\pi R^2}, \qquad \sigma_{\rm rest}[R] = \frac{E_{\rm wall}}{4\pi R^2\gamma}.$$

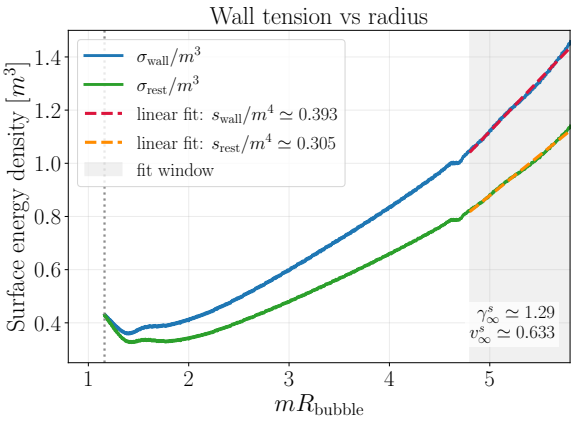
The key numerical observation is

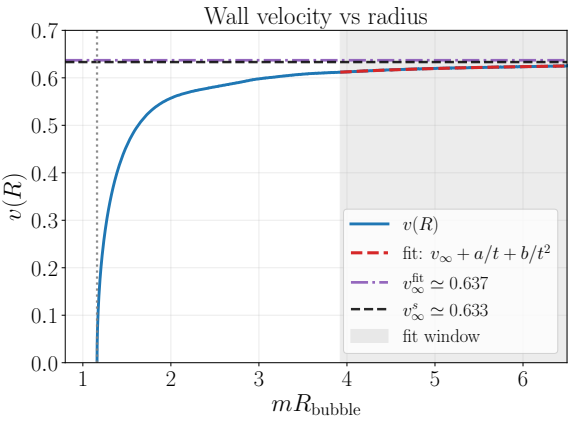
$$\sigma_{\rm rest}[R,Q] \simeq \sigma_0[Q] + \alpha[Q]R.$$

This means the growing energy is not just Lorentz boost:

$\sigma_{\rm rest}$ itself grows with R.

Backup: Surface tension growing





Backup: late-time phase-sector growth

The phase energy splits into

$$e_{\theta,t} = \frac{1}{2}\rho^2\dot{\theta}^2, \qquad e_{\theta,r} = \frac{1}{2}\rho^2\theta'^2.$$

Temporal term:

$$q = \rho^2\omega, \qquad e_{\theta,t} = \frac{1}{2}\rho^2\omega^2 = \frac{q^2}{2\rho^2}.$$

Thus charge is cheaper in the phase with larger ρ:

$$e_{\theta,t}^{[+]} - e_{\theta,t}^{[-]} = \frac{1}{2}q[\omega_+ - \omega_-] = \frac{1}{2}q\Delta\omega.$$

This is a bulk gain:

$$E_{\mathrm{gain},\theta,t} = \frac{4\pi}{3}R^3\left[e_{\theta,t}^{[+]} - e_{\theta,t}^{[-]}\right] = \frac{1}{2}\Delta\omega\, Q_R.$$

Radial term:

$$e_{\theta,r} = \frac{1}{2}\rho^2\theta'^2$$

is localized in a shell near the wall and provides the growing cost.

Backup: Late-time phase sector growth

