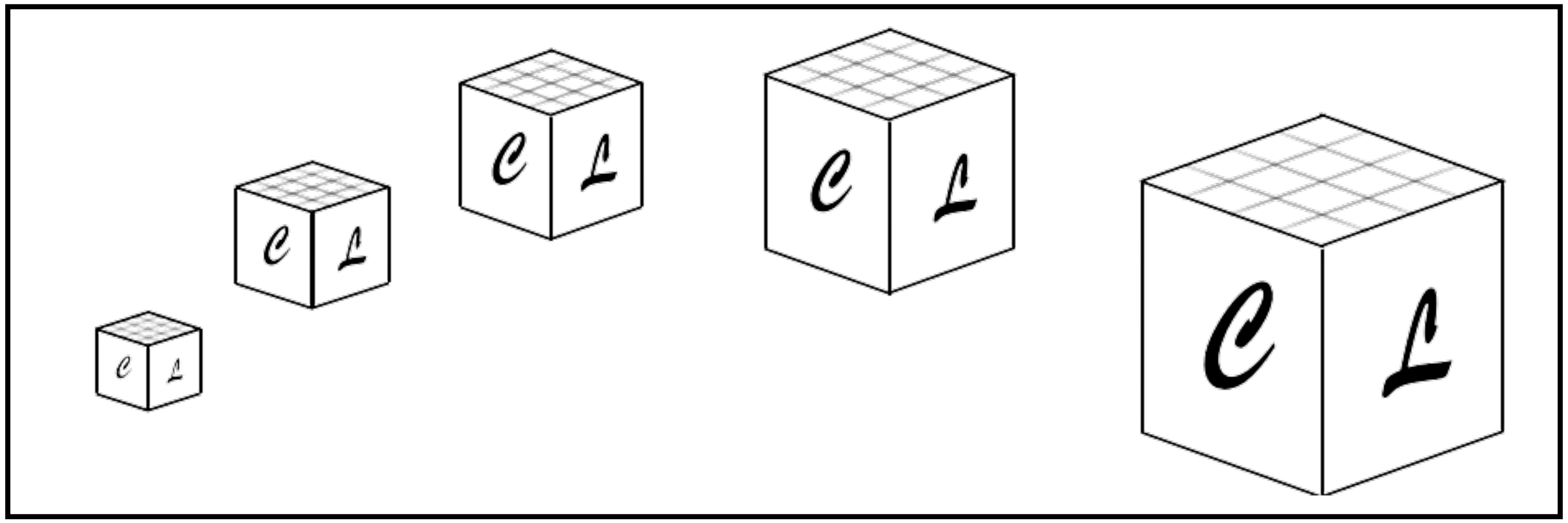

Lecture 3: Lattice formulation of Abelian and non-Abelian gauge theories

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Introduction

WHY DO WE WANT TO SIMULATE GAUGE FIELDS IN THE LATTICE?

➤ **Realistic** physics models must include gauge fields (e.g. the Standard Model).

➤ Gauge field fluctuations can be **significantly amplified** in the early universe (both during and after inflation).

- *Example 1:* Broad parametric resonance of gauge fields coupled to oscillating complex scalars **[TODAY!]**
- *Example 2:* Gauge field production during axion inflation

➤ When $\mathbf{n_k >> 1}$, gauge fields become **classical**, and we can capture their non-linear dynamics with **lattice simulations**.

CAVEAT: Gauge theories must be discretized with **links and plaquettes** in order to preserve **gauge invariance in the lattice** (like in lattice QCD).

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Models with gauge fields

Model 1: Gauge fields coupled to charged scalars (SM-like)

$$S = -\int d^4x\sqrt{-g}\left\{(D_\mu\varphi)^*(D^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(|\varphi|)\right\}$$

$$D_\mu \equiv \partial_\mu - ig_A Q_A A_\mu$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

e.g. Figueroa, Garcia-Bellido & F.T., 1504.04600

THIS LECTURE

Model 2: Gauge fields coupled to axions (during inflation)

$$S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{4f}\phi F_{\mu\nu}\tilde{F}^{\mu\nu} + V(\phi)\right\}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$$

+ self-consistent expansion

Figueroa, Lizarraga, Urio, Urrestilla, 2303.17436

[2512.15627]

[COSMOLATTICE 2.0]

U(1) gauge-invariant action

➤ We are going to learn how to simulate in the lattice the following action:

POTENTIAL

$$\boxed{S = -\int d^4x\sqrt{-g}\left\{(D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\phi,|\varphi|)\right\}}$$

- **Complex scalar:** $\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1)$

  $\varphi_0, \varphi_1 \in \mathscr{R}e$

- **Covariant derivative:** $D_\mu^{\mathrm{A}} \equiv \partial_\mu - ig_A Q_A A_\mu$ ⟶ g_A **Gauge coupling**, Q_A **Abelian charge**

- **Field strength:** $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

  - $E_i \equiv F_{0i}$ **Electric field**
  - $B_i \equiv \frac{1}{2}\epsilon_{ijk}F^{jk}$ **Magnetic field**

➤ **Fields:** $\{\varphi_0, \varphi_1, A_1, A_2, A_3\}$ (we work in the temporal gauge $A_0 = 0$)

Equations of motion (flat spacetime)

$$\boxed{S = -\int d^4x\sqrt{-g}\left\{(D^A_\mu\varphi)^*(D^\mu_A\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(|\varphi|)\right\}}$$

➤ **EOM (flat spacetime)**

$\delta S = 0$

- **Complex scalar:** $D^\mu_A D^A_\mu \varphi = \dfrac{1}{2}\dfrac{\partial V}{\partial|\varphi|}\dfrac{\varphi}{|\varphi|}$
- **U(1) gauge field:** $\partial_\nu F^{\mu\nu} = J^\mu_A; \qquad J^\mu_A \equiv 2g_A Q_A \mathcal{I}m[\varphi^*(D^\mu_A\varphi)]$

  U(1) current

➤ **Action and EOM are invariant under the following gauge transformations:**

$$\varphi(x) \longrightarrow e^{-ig_A Q_A \alpha(x)}\varphi(x)$$

$$A_\mu(x) \longrightarrow A_\mu(x) - \partial_\mu \alpha(x)$$

$$F_{\mu\nu}(x) \longrightarrow F_{\mu\nu}(x)$$

field strength is gauge invariant

Equations of motion (with expansion)

- **FLRW metric:** $ds^2 = -a^{2\alpha}(\eta)d\eta^2 + a^2(\eta)\delta_{ij}dx^i dx^j \qquad d\eta \equiv a^{-\alpha}dt$

- **Dynamical EOM in an expanding background:**

$$\begin{cases} \varphi'' - a^{-2(1-\alpha)}D_A^2\varphi + (3-\alpha)\frac{a'}{a}\varphi' = -\frac{a^{2\alpha}V_{,|\varphi|}}{2}\frac{\varphi}{|\varphi|} \\ \partial_0 F_{0i} - a^{-2(1-\alpha)}\partial_j F_{ji} + (1-\alpha)\frac{a'}{a}F_{0i} = a^{2\alpha}J_i^A \end{cases}$$

U(1) current:

$$J_A^\mu \equiv 2g_A Q_A \mathcal{I}m[\varphi^*(D_A^\mu\varphi)]$$

$$\ddot{A}_i - a^{-2(1-\alpha)}\nabla^2 A_i + \partial_j\partial_i A_j + (1-\alpha)\frac{a'}{a}A_i' = a^{2\alpha}J_i^A$$

- **Gauss's constraint:** **it must be preserved at all times**

$$\partial_i F_{0i} = a^2 J_0^A \longrightarrow \boxed{\nabla\cdot\vec{E} = a^2 J_0^A(\vec{x})} \longrightarrow \int_S \vec{E}\cdot d\vec{S} = a^2 \underbrace{\int_V d^3\vec{x}\, J_0^A(\vec{x})}_{:=\,Q\ (\text{electric charge})} \overset{V\,=\,\text{Lattice}}{=} 0$$

Stress-energy tensor

➤ **Stress-energy tensor:**

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathscr{L})}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathscr{L} - 2\frac{\delta\mathscr{L}}{\delta g^{\mu\nu}} = +\,2\mathrm{Re}\{(D_\mu^A\varphi)^*(D_\nu^A\varphi)\} + g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}$$
$$-g_{\mu\nu}\left(g^{\alpha\beta}\left[(D_\alpha^A\varphi)^*(D_\beta^A\varphi) + \frac{1}{2}(\partial_\alpha\phi)(\partial_\beta\phi)\right] + \frac{1}{4}g^{\alpha\delta}g^{\beta\lambda}F_{\alpha\beta}F_{\delta\lambda} + V\right)$$

$$\bar{\rho} = a^{-2\alpha}\bar{T}_{00} \qquad \bar{p} = \frac{1}{3a^2}\sum_j \bar{T}_{jj}$$

- **Energy density:** $\rho = K_\varphi + G_\varphi + K_{U(1)} + G_{U(1)} + V$
- **Pressure density:** $p = K_\varphi - \frac{1}{3}G_\varphi + \frac{1}{3}(K_{U(1)} + G_{U(1)}) - V$

$$K_\varphi = \frac{1}{a^{2\alpha}}(D_0^A\varphi)^*(D_0^A\varphi)$$
$$G_\varphi = \frac{1}{a^2}\sum_i (D_i^A\varphi)^*(D_i^A\varphi)$$

(Complex scalar)

$$K_{U(1)} = \frac{1}{2a^{2+2\alpha}}\sum_i F_{0i}^2$$
$$G_{U(1)} = \frac{1}{2a^4}\sum_{i,j<i} F_{ij}^2$$

(Electric & Magnetic)

Friedmann equations

- **Friedmann equations:**

$$\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3m_p^2}\langle K_\varphi + G_\varphi + K_{U(1)} + G_{U(1)} + V\rangle$$

$$\frac{a''}{a} = \frac{a^{2\alpha}}{3m_p^2}\langle(\alpha-2)K_\varphi + \alpha G_\varphi + (\alpha+1)V + (\alpha-1)(K_{U(1)} + G_{U(1)})\rangle$$

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Discretization of gauge theories

- For scalar field theories, we just discretize the EOM of the scalar fields by **approximating the derivatives in the continuum** by **finite differences in the discrete**.

Example: $\partial_\mu \varphi(\mathbf{n})$

$$\Delta^+_\mu \varphi \equiv \frac{\varphi_{+\mu} - \varphi}{\delta x^\mu} = \partial_\mu \varphi\,|_{\mathbf{n}+\hat{\mu}(\delta x^\mu/2)} + \mathcal{O}(\delta x^2)$$

$$\Delta^-_\mu \varphi \equiv \frac{\varphi - \varphi_{-\mu}}{\delta x^\mu} = \partial_\mu \varphi\,|_{\mathbf{n}+\hat{\mu}(\delta x^\mu/2)} + \mathcal{O}(\delta x^2)$$

- **CAN WE DO THE SAME FOR GAUGE FIELDS? No.**

- **WHY? This formulation does not preserve gauge invariance in the lattice (and propagates spurious degrees of freedom).**

Gauge invariance in the discrete

$$S = \int d^4x \mathscr{L} \qquad \boxed{-\mathscr{L} = (D_\mu\varphi)^* D^\mu \varphi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\varphi^*\varphi)} \qquad e \equiv g_A Q_A$$

➤ **Gauge transformation in the continuum:** $\begin{cases} \varphi(x) \to e^{-ie\alpha(x)}\varphi(x) \\ A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x) \end{cases}$

$$\boxed{\begin{aligned} (D_\mu\varphi) \equiv \partial_\mu\varphi - ieA_\mu\varphi \quad \longrightarrow \quad & \partial_\mu(e^{-ie\alpha(x)}\varphi(x)) - ie(A_\mu - \partial_\mu\alpha(x))\varphi e^{-ie\alpha(x)} = \\ & = e^{-ie\alpha(x)}\Big(\partial_\mu\varphi(x) - \cancel{ie}(\partial_\mu\alpha)\varphi(x) - ieA_\mu + \cancel{ie}\partial_\mu\alpha(x)\varphi(x)\Big) = e^{-ie\alpha(x)}D_\mu\varphi \end{aligned}}$$

➤ **Gauge transformation in the discrete** **(<u>naive discretization</u>)** $\begin{cases} \varphi(x) \to e^{-ie\alpha(x)}\varphi(x) \\ A_\mu(x) \to A_\mu(x) - \Delta^+_\mu\alpha(x) \end{cases}$

$$\boxed{(D_\mu\varphi) \equiv \Delta^+_\mu\varphi - ieA_\mu\varphi \quad \longrightarrow \quad \Delta^+_\mu(e^{-ie\alpha(x)}\varphi(x)) - ie(A_\mu - \Delta^+_\mu\alpha(x))\varphi e^{-ie\alpha(x)} \neq e^{-ie\alpha(x)}D_\mu\varphi}$$

[The Leibniz rule $(fg)' = fg' + f'g$ does not hold for finite difference operators]

Links and plaquettes

We must discretize the theory with links and plaquettes in order to preserve GAUGE INVARIANCE in the lattice.

- **Parallel transporter:** Connects two points of spacetime $\quad dx^\mu = (d\eta, dx^i)$

$$V(x,y) = \exp\left(-ie\int_x^y dx^\mu A_\mu(x)\right)$$

Gauge transformation:

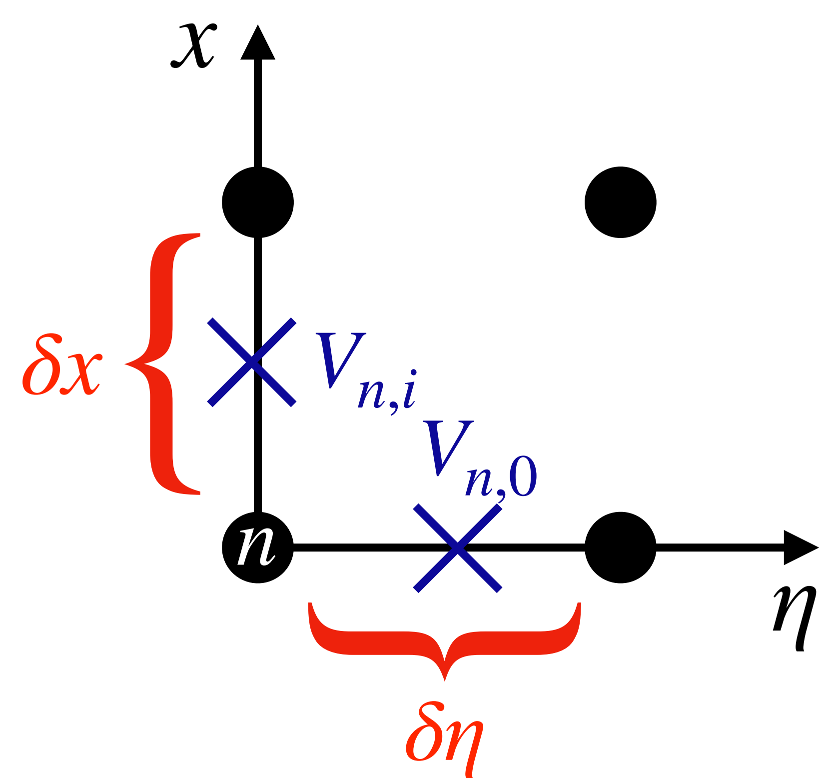
$$V(x,y) \xrightarrow{} V(x,y)e^{-ie(\alpha(x)-\alpha(y))}$$

$[A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha(x)]$

- **Links:** (minimal connectors)

$$V_{n,0} \equiv \exp\left\{-ie\int_{x(n)}^{x(n+\hat{0})} d\eta' A_0\right\} \approx e^{-ie\delta\eta A_0}$$

$$V_{n,i} \equiv \exp\left\{-ie\int_{x(n)}^{x(n+\hat{i})} dx' A_i\right\} \approx e^{-ie\delta x A_i}$$



Links (and gauge fields) live in $n + \hat{\mu}/2$ points

Links and plaquettes

Notation for links
(and gauge field components)

1) We remove the lattice site label ('n')

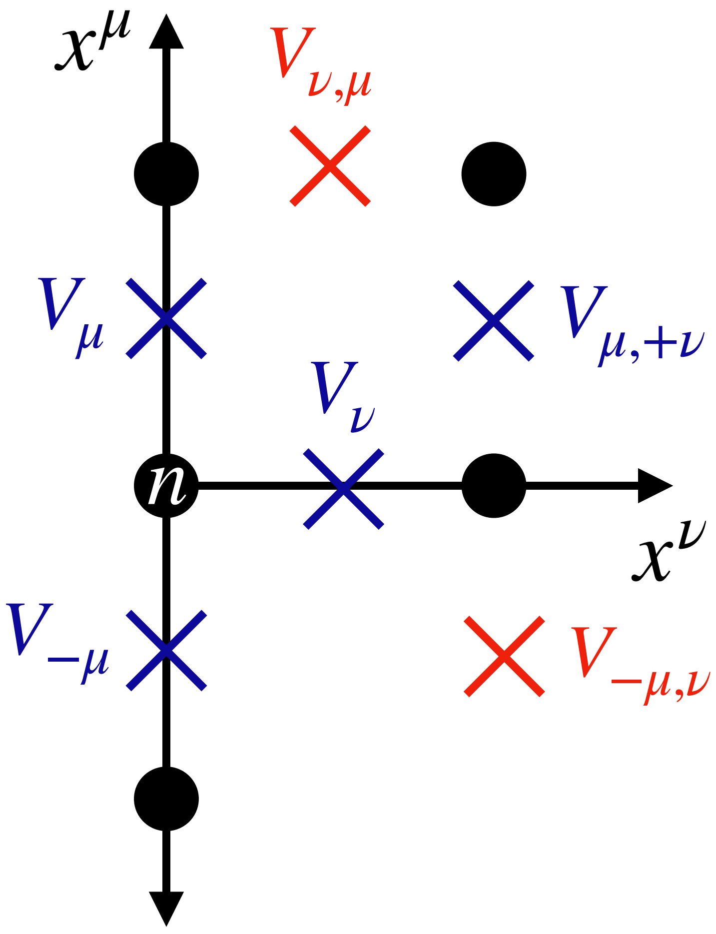
$$V_{n,\mu} \equiv V_{n,\mu}(n + \frac{1}{2}\hat{\mu}) \longrightarrow V_{\mu}$$

2) It follows:

$$V_{\mu,\pm\nu} = V_{\mu,n\pm\nu} = V_{\mu}(n + \frac{1}{2}\hat{\mu} \pm \hat{\nu})$$

3) We also define:

$$V_{-\mu} \equiv V^{\dagger}_{\mu,-\mu} = V^{\dagger}_{\mu}(n - \frac{1}{2}\hat{\mu})$$



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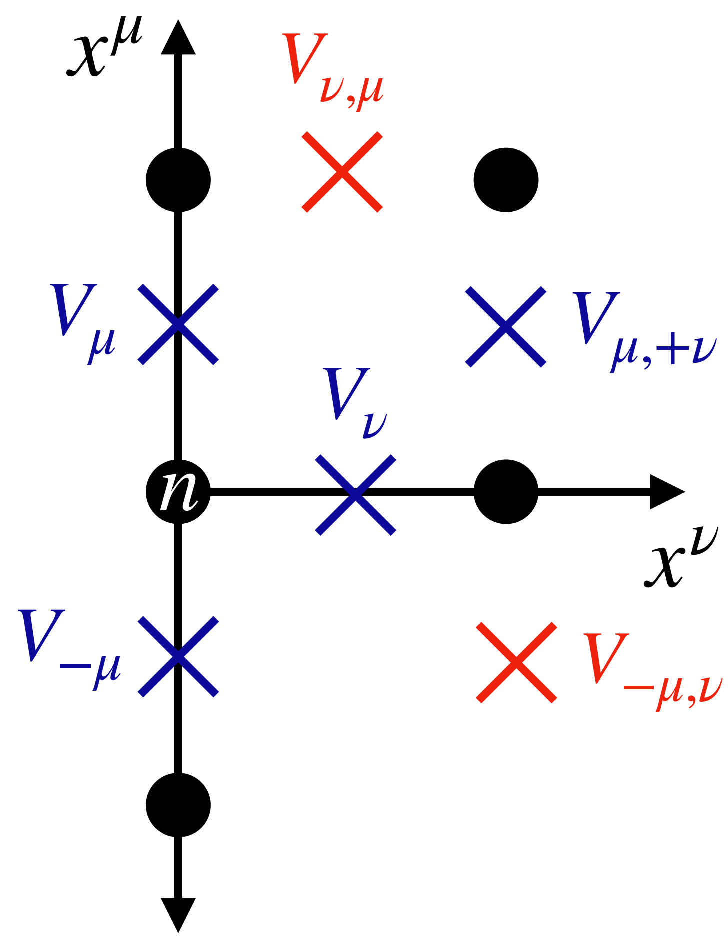

The conjugate operation sets up the "direction" of the link

$$V_\mu \equiv \exp\left\{-ie\int_{x(n)}^{x(n+\hat{\mu})} dx^\mu A_\mu\right\}$$

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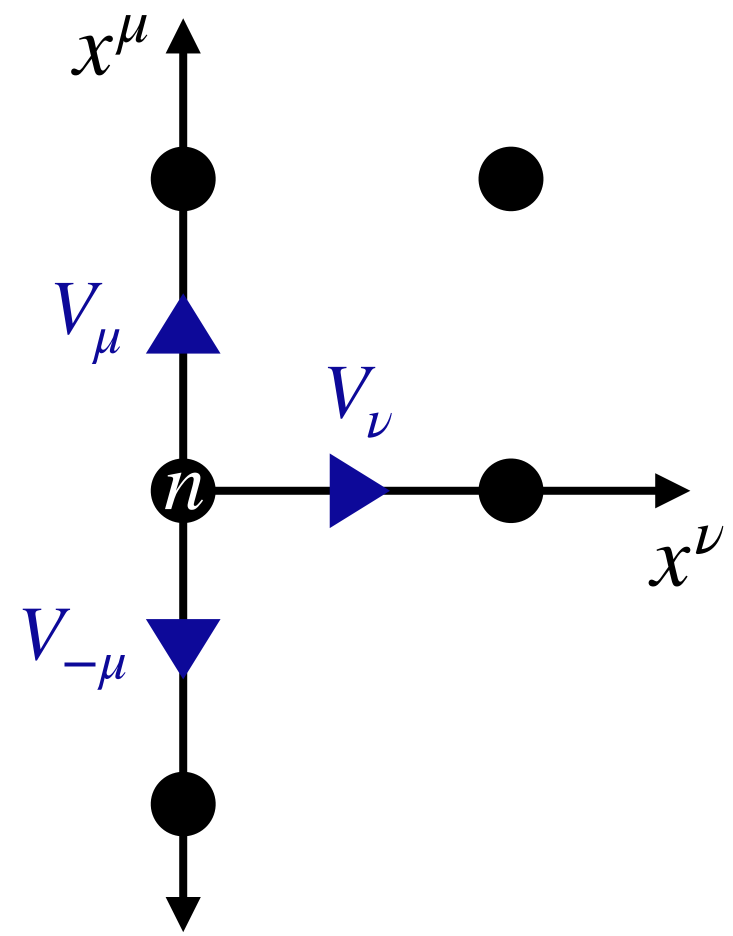

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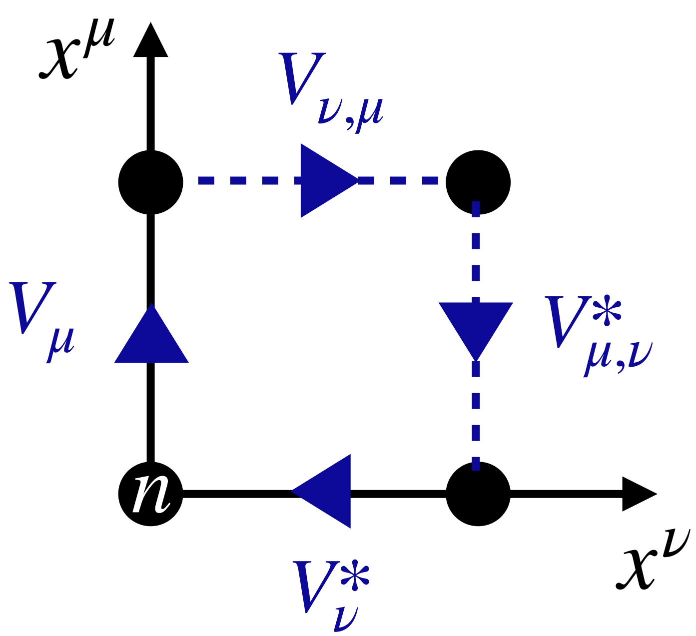
Links and plaquettes

- **Gauge covariant derivative:**

$$\left.\begin{aligned} \varphi &\longrightarrow e^{iea(x)}\varphi \\ A_\mu &\longrightarrow A_\mu - \Delta_\mu^+\alpha \\ V_{\pm\mu} &\longrightarrow V_{\pm\mu}e^{ie(\alpha_{\pm\mu}-\alpha)} \end{aligned}\right\} \Rightarrow \boxed{(D_\mu^\pm\varphi)(\mathbf{l}) = \pm\frac{1}{\delta x^\mu}(V_{\pm\mu}\varphi_{\pm\mu} - \varphi)} \qquad \hat{\mathbf{l}} = \hat{\mathbf{n}} + \frac{1}{2}\hat{\mu}$$

  - ***Expansion:*** $(D_\mu^\pm\varphi)(\mathbf{l}) = (D_\mu\varphi)(\mathbf{l}) + \mathcal{O}(\delta x^2)$
  - ***Gauge transform.:*** $D_\mu^\pm\varphi \rightarrow e^{iea}(D_\mu^\pm\varphi)$

- **Plaquettes:** $\boxed{V_{\mu\nu} \equiv V_\mu V_{\nu,+\mu} V_{\mu,+\nu}^* V_\nu^* \simeq e^{-ie\delta x_\mu \delta x_\nu [F_{\mu\nu} + \mathcal{O}(\mathrm{dx})]}}$



  - ***Expansion:***

$$\mathscr{R}e\{V_{\mu\nu}\} \longrightarrow 1 - \frac{1}{2}\delta x_\mu^2 \delta x_\nu^2 e^2 F_{\mu\nu}{}^2 + \mathcal{O}(\delta x^5)$$

$$\mathscr{I}m\{V_{\mu\nu}\} \longrightarrow -\delta x_\mu \delta x_\nu e F_{\mu\nu} + \mathcal{O}(\delta x^3) \qquad \mathbf{l} = \mathbf{n} + \frac{1}{2}\hat{\mu} + \frac{1}{2}\hat{\nu}$$

  - ***Gauge transform.:*** $V_{\mu\nu} \longrightarrow V_{\mu\nu}$

Compact and non-compact formulations

Two formulations for U(1) gauge fields:

- **Non-compact:** solves the EOM for gauge fields A_μ
- **Compact:** solves the EOM of links V_μ

U(1) toolkit:

$$\text{Links}: V_\mu \equiv e^{-ig_AQ_A\delta x_\mu A_\mu} = \cos(g_AQ_A\delta x_\mu A_\mu) - i\sin(g_AQ_A\delta x_\mu A_\mu); \quad V_{-\mu} \equiv V^*_{\mu,-\mu}; \quad V^*_\mu V_\mu = 1\,;$$

$$\text{Plaquettes}: V_{\mu\nu} \equiv V_\mu V_{\mu,+\mu} V^*_{\mu,+\nu} V^*_\nu \simeq e^{-ig_AQ_A\delta x_\mu\delta x_\nu[F_{\mu\nu}+\mathcal{O}(\delta x)]}; \quad V^*_{\mu\nu} = V_{\nu\mu}\,;$$

$$\text{Covariant Derivs.}: (D^\pm_\mu\varphi)(\mathbf{l}) = \pm\frac{1}{\delta x^\mu}(V_{\pm\mu}\varphi_{\pm\mu} - \varphi)\,, \quad \mathbf{l} = \mathbf{n} \pm \frac{1}{2}\hat{\mu}$$

$$\text{Expansions}: \begin{cases} (D^\pm_\mu\varphi)(\mathbf{l}) \longrightarrow (D_\mu\varphi)(\mathbf{l}) + \mathcal{O}(\delta x^2) & \mathbf{l} = \mathbf{n} \pm \frac{1}{2}\hat{\mu} \\ \mathcal{R}e\{V_{\mu\nu}\} \longrightarrow 1 - \frac{1}{2}\delta x_\mu^2\delta x_\nu^2 g_A^2 Q_A^2 F_{\mu\nu}^2 + \mathcal{O}(\delta x^5)\,, & \mathbf{l} = \mathbf{n} + \frac{1}{2}\hat{\mu} + \frac{1}{2}\hat{\nu} \\ \mathcal{I}m\{V_{\mu\nu}\} \longrightarrow -\delta x_\mu\delta x_\nu g_A Q_A F_{\mu\nu} + \mathcal{O}(\delta x^3)\,, & \mathbf{l} = \mathbf{n} + \frac{1}{2}\hat{\mu} + \frac{1}{2}\hat{\nu} \end{cases}$$

$$\text{Expressions}: \begin{cases} \left.\begin{array}{l} \sum_n \frac{1}{4}F_{\mu\nu}^2 \cong -\frac{1}{2}\sum_n \frac{\mathcal{R}e\{V_{\mu\nu}\}}{\delta x_\mu^2\delta x_\nu^2 g_A^2 Q_A^2} = -\frac{1}{4}\sum_n \frac{(V_{\mu\nu}+V^*_{\mu\nu})}{\delta x_\mu^2\delta x_\nu^2 g_A^2 Q_A^2} + \mathcal{O}(\delta x^2) \\ \sum_n \frac{1}{4}F_{\mu\nu}^2 \simeq \sum_n \frac{1}{4}\frac{\mathcal{I}m^2\{V_{\mu\nu}\}}{\delta x_\mu^2\delta x_\nu^2 g_A^2 Q_A^2} = -\sum_n \frac{1}{4}\frac{(V_{\mu\nu}-V^*_{\mu\nu})^2}{\delta x_\mu^2\delta x_\nu^2 g_A^2 Q_A^2} + \mathcal{O}(\delta x^2) \end{array}\right] & (\texttt{Compact}) \\ \sum_n \frac{1}{4}F_{\mu\nu}^2 \simeq \frac{1}{4}\sum_n(\Delta^+_\mu A_\nu - \Delta^+_\nu A_\mu)^2 + \mathcal{O}(\delta x^2) \;\big] & (\texttt{Non - Compact}) \end{cases}$$

$$\text{Gauge Trans.}: \left.\begin{cases} \phi \longrightarrow e^{+ig_AQ_A\alpha}\phi \\ A_\mu \longrightarrow A_\mu - \Delta^+_\mu\alpha \\ V_{\pm\mu} \longrightarrow V_{\pm\mu}e^{ig_AQ_A(\alpha_{\pm\mu}-\alpha)} \end{cases}\right] \Longrightarrow \begin{cases} D^\pm_\mu\phi \longrightarrow e^{ig_AQ_A\alpha}(D^\pm_\mu\phi) \\ V_{\mu\nu} \longrightarrow V_{\mu\nu}\ (\text{gauge inv.}) \end{cases}$$

NOTE: **SU(2) theories are formulated with a compact formulation**

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Gauge EOM in program variables

- **Program variables:**

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt \qquad d\tilde{x}^i \equiv \omega_* dx^i$$

$$\tilde{\varphi} \equiv \frac{1}{f_*}\varphi \qquad \widetilde{A}_\mu \equiv \frac{1}{\omega_*}A_\mu$$

Link:

$$V_\mu = e^{-i\delta x_\mu A^\mu} = e^{-i\delta\tilde{x}_\mu \tilde{A}^\mu}$$

Program potential:

$$\tilde{V}(|\tilde{\varphi}|) = \frac{1}{f_*^2\omega_*^2}V(f_*|\varphi|)$$

$$\widetilde{F}_{\mu\nu} \equiv F_{\mu\nu}/\omega_*^2$$
$$\widetilde{D}_\mu \equiv D_\mu/\omega_*$$
$$\tilde{J}_A^\mu \equiv 2g_A Q_A \mathscr{I}m[\tilde{\varphi}^*(\tilde{D}^\mu\tilde{\varphi})]$$

- **Field equations (in the continuum):**

$$(a^{3-\alpha}\tilde{\varphi}')' - a^{1+\alpha}\overrightarrow{\widetilde{D}}^2\tilde{\varphi} = -a^{\alpha+3}\widetilde{V}_{,|\tilde{\varphi}|}\frac{\tilde{\varphi}}{2|\tilde{\varphi}|}$$

$$\tilde{\partial}_0(a^{1-\alpha}\widetilde{F}_{0i}) - a^{\alpha-1}\partial_j\widetilde{F}_{ji} = a^{1+\alpha}\tilde{J}_i^A$$

Conjugate momenta:

$$\tilde{\pi}_\varphi \equiv a^{3-\alpha}\tilde{\varphi}'$$

$$(\tilde{\pi}_A)_i \equiv a^{1-\alpha}\widetilde{F}_{0i} = a^{1-\alpha}A_i'$$

$$(\tilde{\pi}_\varphi)' = \mathscr{K}_\varphi[a,\tilde{\varphi},\widetilde{A}_j] \qquad \mathscr{K}_\varphi[a,\tilde{\varphi},\widetilde{A}_j] \equiv -a^{\alpha+3}\widetilde{V}_{,|\tilde{\varphi}|}\frac{\tilde{\varphi}}{2|\tilde{\varphi}|} + a^{1+\alpha}\overrightarrow{\widetilde{D}}^2\tilde{\varphi}$$

$$(\pi_A)_i' = \mathscr{K}_{A_i}[a,\tilde{\varphi},\widetilde{A}_j] \qquad \mathscr{K}_{A_i}[a,\tilde{\varphi},\widetilde{A}_j] \equiv a^{1+\alpha}\tilde{J}_i^A + a^{\alpha-1}\tilde{\partial}_j\widetilde{F}_{ji}$$

$$\tilde{\varphi}' = a^{\alpha-3}\tilde{\pi}_\varphi$$

$$A_i' = a^{\alpha-1}(\tilde{\pi}_A)_i$$

Gauge EOM in program variables

- **Second Friedmann equation (dynamical):** $b \equiv a' = da/d\tilde{\eta}$

$$b' = \mathcal{K}_a\left[a, \widetilde{E}^{\varphi}_K, \widetilde{E}^{\varphi}_G, \widetilde{E}^{\varphi}_V, \widetilde{E}^{A}_K, \widetilde{E}^{A}_G\right]$$

$$\mathcal{K}_a\left[a, \widetilde{E}^{\varphi}_K, \widetilde{E}^{\varphi}_G, \widetilde{E}^{\varphi}_V, \widetilde{E}^{A}_K, \widetilde{E}^{A}_G\right] \equiv \frac{a^{2\alpha+1}}{3}\frac{f_*^2}{m_p^2}\left[(\alpha-2)\widetilde{E}^{\varphi}_K + \alpha\widetilde{E}^{\varphi}_G + (\alpha+1)\widetilde{E}_V + (\alpha-1)(\widetilde{E}^{A}_K + \widetilde{E}^{A}_G)\right]$$

- **First Friedmann equation (constraint):**

$$b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}\left[\widetilde{E}^{\varphi}_K + \widetilde{E}^{\varphi}_G + \widetilde{E}_V + \widetilde{E}^{A}_K + \widetilde{E}^{A}_G\right]$$

with:

$$\widetilde{E}^{\varphi}_K = \frac{1}{a^6}\left\langle \tilde{\pi}^2_{\varphi}\right\rangle \qquad \widetilde{E}^{A}_K = \frac{1}{2a^4}\frac{\omega_*^2}{f_*^2}\sum_{i=1}^{3}\left\langle (\tilde{\pi}_A)^2_i\right\rangle$$

$$\widetilde{E}^{\varphi}_G = \frac{1}{a^2}\langle \sum_i (\widetilde{D}^A_i\varphi)^*(\widetilde{D}^A_i\widetilde{\varphi})\rangle \qquad \widetilde{E}^{A}_G = \frac{1}{2a^4}\frac{\omega_*^2}{f_*^2}\sum_{i,j<i}\langle \widetilde{F}^2_{ij}\rangle \qquad \widetilde{E}_V = \left\langle \widetilde{V}(\tilde{\varphi},\ldots)\right\rangle$$

(Complex scalar: Kinetic and gradient)

(Electric and magnetic)

(Potential)

Discretization of gauge field equations

➤ **We can discretize the field equations by using the U(1) toolkit (non-compact):**

CONTINUOUS	DISCRETE

- ***Kernels:***

$$\mathcal{K}_\varphi[a,\tilde\varphi,\widetilde{A}_j] = -a^{\alpha+3}\widetilde{V}_{,|\tilde\varphi|}\frac{\tilde\varphi}{2|\tilde\varphi|} + a^{1+\alpha}\overrightarrow{\widetilde{D}}^{2}\tilde\varphi \longrightarrow \mathcal{K}_\varphi[a,\tilde\varphi,\widetilde{A}_i] = -a^{\alpha+3}\frac{\widetilde{V}_{,|\tilde\varphi|}}{2}\frac{\tilde\varphi}{|\tilde\varphi|} + a^{1+\alpha}\sum_i \widetilde{D}_i^-\widetilde{D}_i^+\tilde\varphi$$

$$\mathcal{K}_{A_i}[a,\tilde\varphi,\widetilde{A}_j] = 2a^{1+\alpha}g_AQ_A\mathcal{I}m[\tilde\varphi^*(\tilde D_i\tilde\varphi)] + a^{\alpha-1}\tilde\partial_j\widetilde{F}_{ji} \longrightarrow \mathcal{K}_{A_i}[a,\tilde\varphi,\widetilde{A}_j] = a^{1+\alpha}\left(\frac{2g_AQ_A}{\delta\tilde x}\frac{f_*^2}{\omega_*^2}\mathcal{I}m[\tilde\varphi^* e^{-i\delta x\tilde A^i}\tilde\varphi]\right) + a^{\alpha-1}\sum_j\left(\tilde\Delta_j^-\tilde\Delta_j^+\widetilde{A}_i - \tilde\Delta_j^-\tilde\Delta_i^+\widetilde{A}_j\right)$$

- ***Energies:***

$$\widetilde{E}_G^\varphi = \frac{1}{a^2}\langle\sum_i(\widetilde{D}_i^A\varphi)^*(\widetilde{D}_i^A\widetilde{\varphi})\rangle \longrightarrow \widetilde{E}_G^\varphi = \frac{1}{a^2}\sum_i\left\langle(\tilde D_i^+\tilde\varphi)^*(\tilde D_i^+\tilde\varphi)\right\rangle$$

$$\widetilde{E}_G^A = \frac{1}{2a^4}\frac{\omega_*^2}{f_*^2}\sum_{i,j<i}\langle\widetilde{F}_{ij}^2\rangle \longrightarrow \widetilde{E}_G^A = \frac{1}{2a^4}\frac{\omega_*^2}{f_*^2}\sum_{i,j<i}\left\langle(\tilde\Delta_i^+\widetilde{A}_j - \tilde\Delta_j^+\widetilde{A}_i)^2\right\rangle$$

And now we solve them with an appropriate evolution algorithm!

(Leapfrog, velocity-verlet..., see 2006.15122)

Initial conditions for complex scalars

- **Initial condition for complex scalars:**

$$\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1) \quad \longrightarrow \quad \boxed{\begin{aligned} \varphi_n(\mathbf{x}, t_*) &\equiv |\varphi_*| + \delta\varphi_{n_*}(\mathbf{x}) \\ \dot{\varphi}_n(\mathbf{x}, t_*) &\equiv |\dot{\varphi}_*| + \delta\dot{\varphi}_{n*}(\mathbf{x}) \end{aligned}} \qquad n = 0,1$$

Homogeneous mode

Fluctuations

- For **complex scalars** we **try** to set **the same spectrum of initial Gaussian fluctuations** as for scalar singlets (see **Lecture 2**):

$$\boxed{\begin{aligned} \delta\tilde{\varphi}_n(\tilde{\mathbf{n}}) &= \frac{1}{\sqrt{2}}\left(|\delta\tilde{\varphi}_n^{(l)}(\tilde{\mathbf{n}})|\, e^{i\theta_n^{(l)}(\tilde{\mathbf{n}})} + |\delta\tilde{\varphi}_n^{(r)}(\tilde{\mathbf{n}})|\, e^{i\theta_n^{(r)}(\tilde{\mathbf{n}})}\right) \\ \delta\tilde{\varphi}_n{}'(\tilde{\mathbf{n}}) &= \frac{1}{a^{1-\alpha}}\left[\frac{i\tilde{\omega}_{k,\varphi_n}}{\sqrt{2}}\left(|\delta\tilde{\varphi}_n^{(l)}(\tilde{\mathbf{n}})|\, e^{i\theta_n^{(l)}(\tilde{\mathbf{n}})} - |\delta\tilde{\varphi}_n^{(r)}(\tilde{\mathbf{n}})|\, e^{i\theta_n^{(r)}(\tilde{\mathbf{n}})}\right)\right] - \tilde{\mathscr{H}}\,\delta\tilde{\varphi}_n(\tilde{\mathbf{n}}) \end{aligned}}$$

$|\delta\tilde{\phi}^{(l,r)}(\tilde{\mathbf{n}})|$: Rayleigh distribution with expected **(*)**

$\theta^{(l,r)}(\tilde{\mathbf{n}})$: Random phase in range [0,2π]

$$\tilde{\omega}_{k,\varphi_n} \equiv \sqrt{\tilde{k}^2 + a^2(\partial^2\tilde{V}/\partial\tilde{\varphi}_n^2)}$$

$$\tilde{\mathscr{H}} \equiv a^\alpha H/\omega_*$$

(in principle)
8 random functions

Initial conditions for gauge fields

- For **gauge fields** we only set fluctuations to the time-derivative:

$$\boxed{\begin{aligned} A_i(\mathbf{x}, t_*) &\equiv 0 \\ \dot{A}_i(\mathbf{x}, t_*) &\equiv \delta \dot{A}_{i*}(\mathbf{x}) \end{aligned}}$$

→ Initially we have some electric field, but zero magnetic field

- But the fluctuations must preserve the **Gauss constraint**:

$$\partial_i \dot{A}_i(\mathbf{x}) = J_0^A(\mathbf{x}) \xrightarrow{\text{Fourier transform}} -ik^i \dot{A}_i(\mathbf{k}) = J_0^A(\mathbf{k}) \longrightarrow \boxed{\dot{A}_i(\mathbf{k}) = i\frac{k_i}{k^2} J_0^A(\mathbf{k})}$$

$$J_0^A(\mathbf{x}) \equiv 2 g_A Q_A^{(\varphi)} \mathcal{I}m[\varphi^* \varphi']$$

Initial spectrum for electric field

- In the discrete:

$$\sum_i \Delta_i^- \Delta_0^+ A_i(\mathbf{n}) = J_0^A(\mathbf{n}) \longrightarrow \Delta_0^+ A_i(\tilde{\mathbf{n}}) = i \frac{k_{\mathrm{Lat},i}^-}{(k_{\mathrm{Lat},i}^-)^2} J_0^A(\tilde{\mathbf{n}})$$

$$k_{\mathrm{Lat},i}^- = \frac{\sin(2\pi \tilde{n}_i / N)}{\delta x} - i \frac{1 - \cos(2\pi \tilde{n}_i / N)}{\delta x}$$

Initial conditions for gauge fields

- **We want zero electric charge in the lattice, so we require:**

$$Q = J_0^A(\mathbf{k}=\mathbf{0}) \; = \int d^3\mathbf{x} J_0^A(\mathbf{x}) \propto \int d^3\mathbf{k} \mathscr{R}e[\varphi_0^*(\mathbf{k})\varphi_1'(\mathbf{k}) - \varphi_0'(\mathbf{k})\varphi_1^*(\mathbf{k})] \overset{!}{=} 0$$

3 constraints:

$$|\delta\varphi_0^{(l)}(\mathbf{k})| = |\delta\varphi_0^{(r)}(\mathbf{k})|$$

$$|\delta\varphi_1^{(l)}(\mathbf{k})| = |\delta\varphi_1^{(r)}(\mathbf{k})|$$

$$\theta_1^{(r)}(\mathbf{k}) = \theta_0^{(r)}(\mathbf{k}) + \theta_1^{(l)}(\mathbf{k}) - \theta_0^{(l)}(\mathbf{k})$$

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Example: Abelian-Higgs model

$$\boxed{S = -\int d^4x\sqrt{-g}\left\{(D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(|\varphi|)\right\}}$$

$$\boxed{V(|\varphi|) = \lambda|\varphi|^4 = \frac{\lambda}{4}(\varphi_0^2 + \varphi_1^2)^2}$$

$$\ddot{A}_i - a^{-2(1-\alpha)}\nabla^2 A_i + \partial_j\partial_i A_j + (1-\alpha)\frac{a'}{a}A_i' = \overbrace{a^{2\alpha}J_i^A}^{\text{Source term}}$$

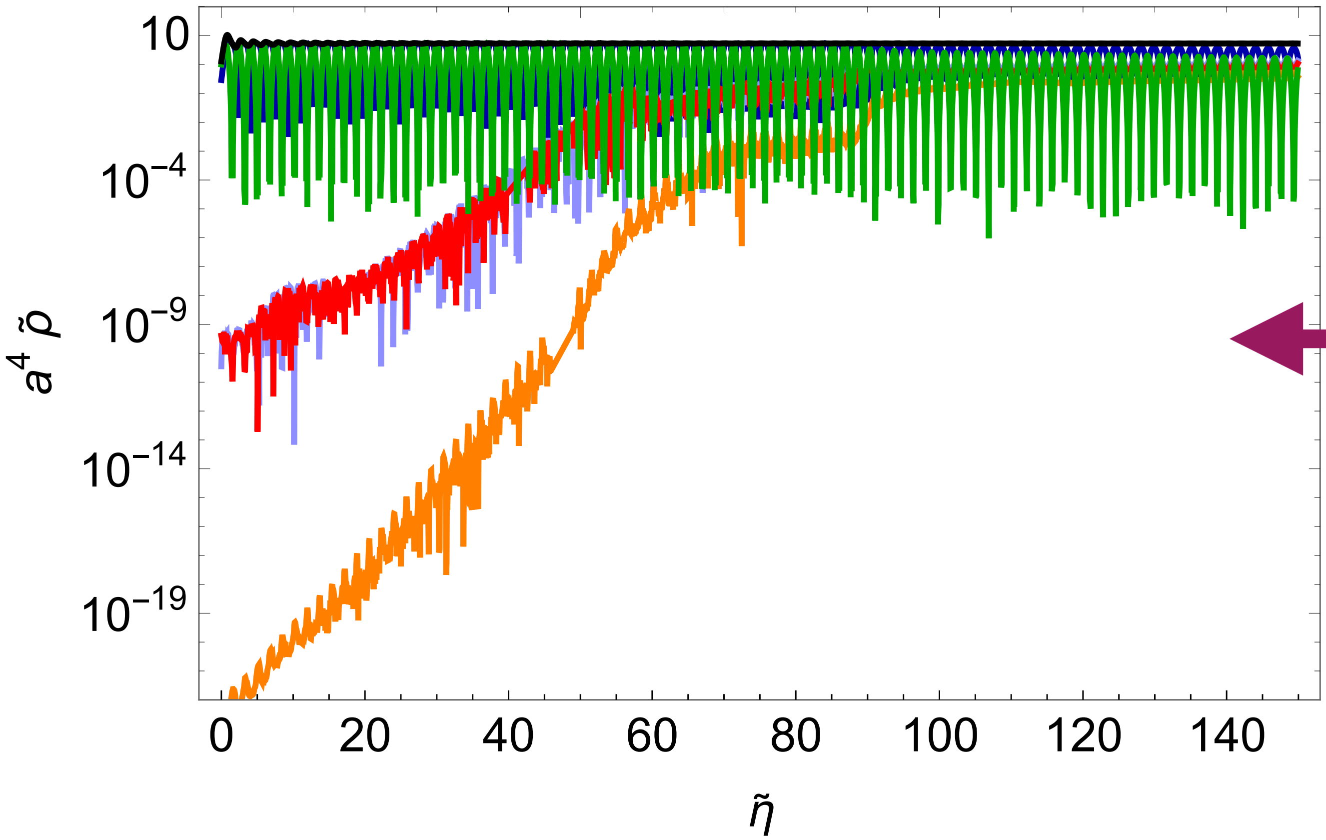
$$J_A^i \equiv 2g_AQ_A\mathscr{I}m[\varphi^*(\partial_i\varphi - ig_AQ_AA_i\varphi)]$$

contains

Similar to the source term of the analogous scalar equation $\longleftarrow$ $g_AQ_A|\varphi|^2A_i$

Gauge fields coupled to charged scalars with a monomial potential **experience parametric resonance**

Example: Abelian-Higgs model



Energy density:

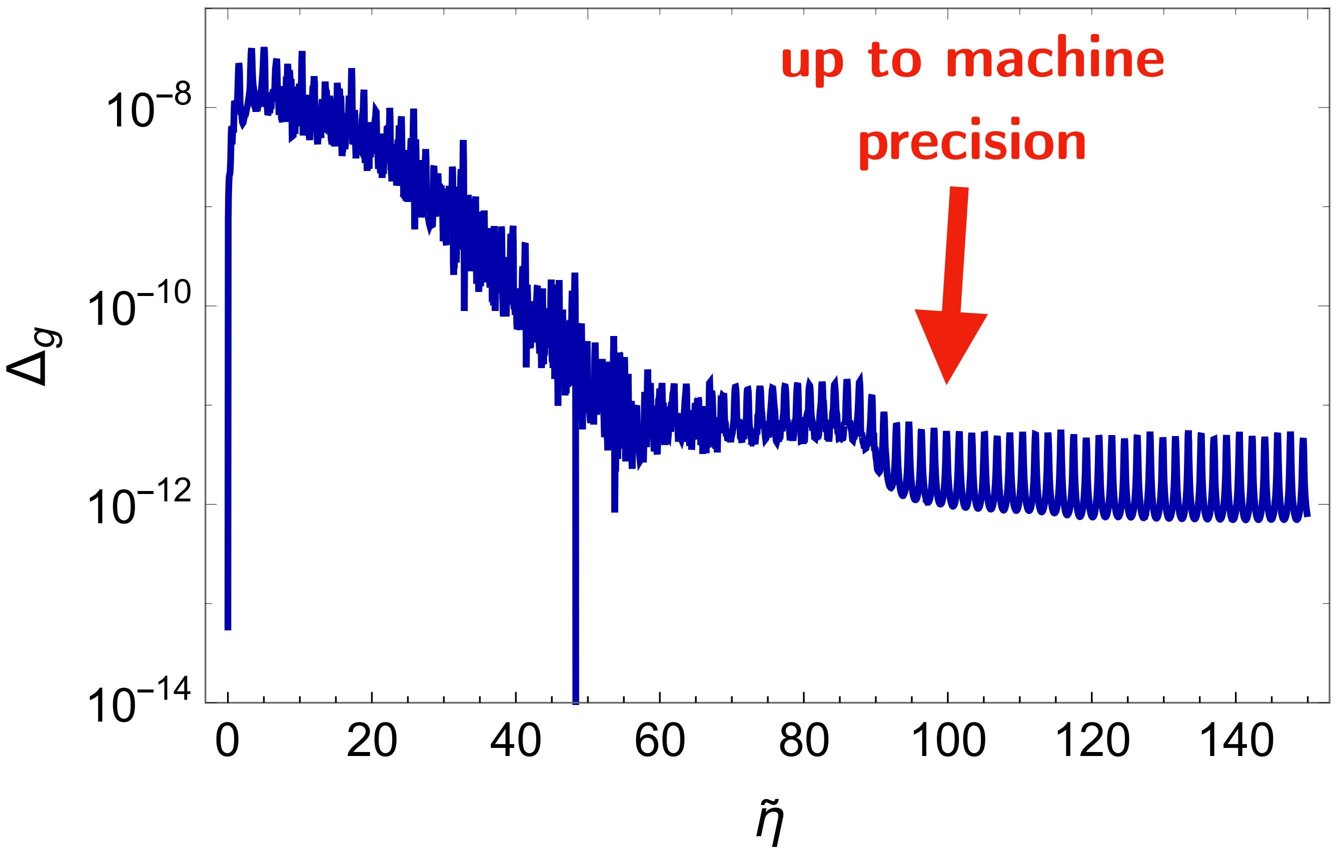
$$\tilde{\rho} = \frac{1}{f_*^2\omega_*^2}\left(\widetilde{E}_K^{\varphi} + \widetilde{E}_G^{\varphi} + \widetilde{E}_K^{A} + \widetilde{E}_G^{A} + \widetilde{E}_V\right)$$

Gauss law conservation:

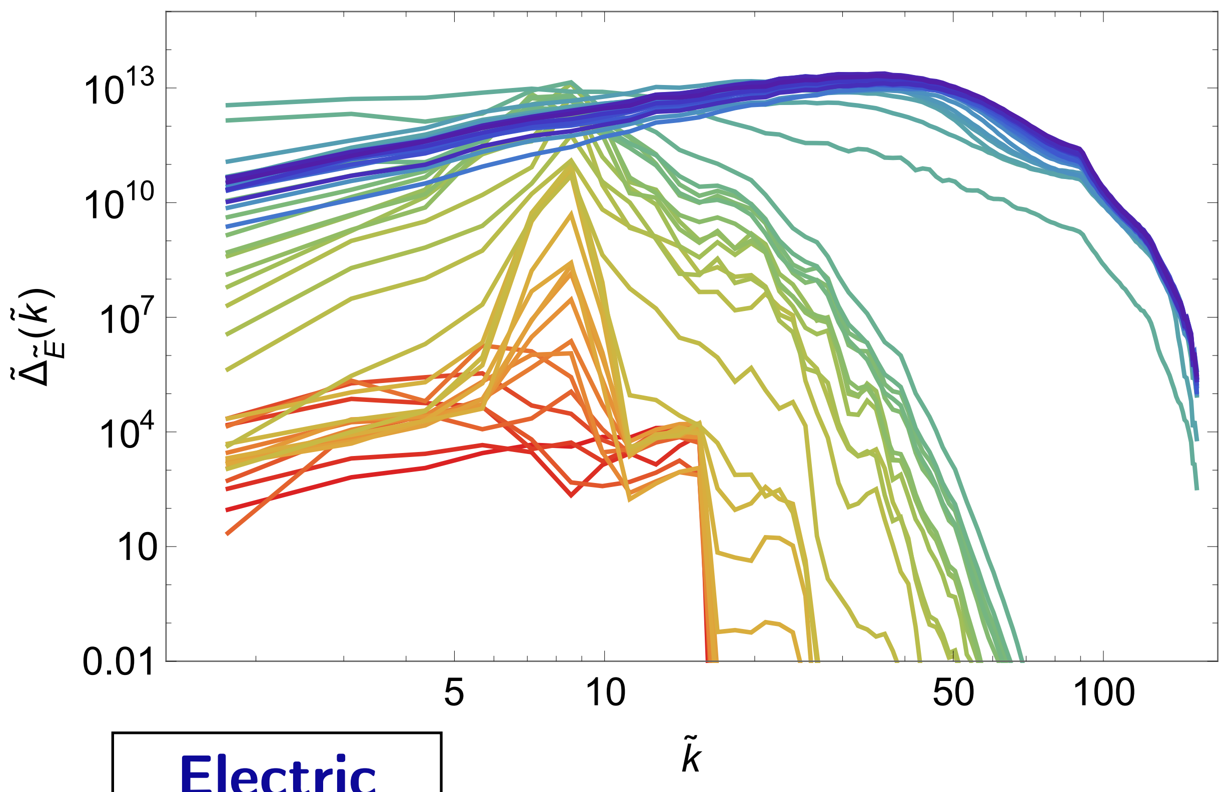
$$\underbrace{\Delta_i^-\Delta_0^+A_i}_{LHS} = \underbrace{a^2J_0^A}_{RHS}$$

$$\Delta_g \equiv \frac{\langle\sqrt{(LHS-RHS)^2}\rangle}{\langle\sqrt{(LHS+RHS)^2}\rangle}$$

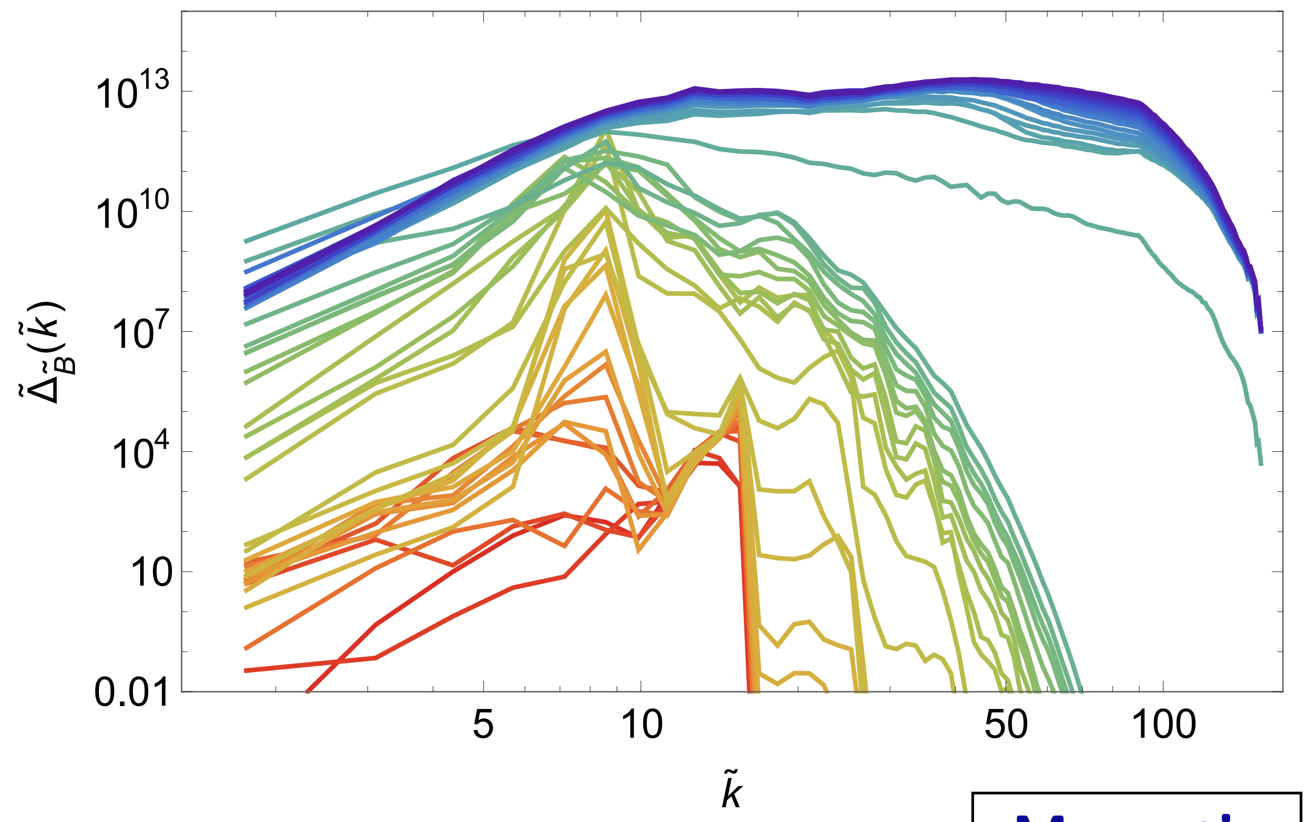
Conserved up to machine precision



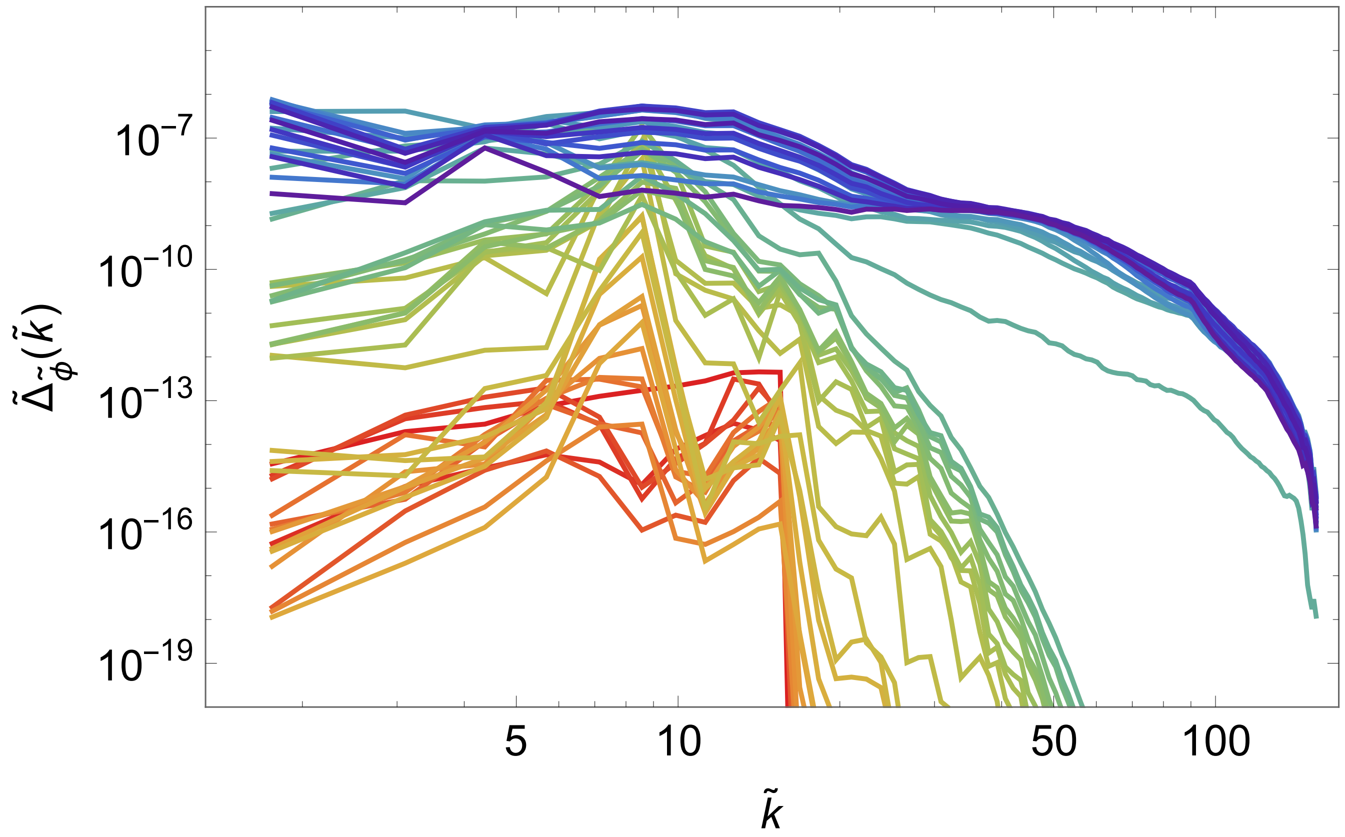
Example: Abelian-Higgs model



Electric spectra



Magnetic spectra



Complex scalar spectra

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SU(2) group and algebra

SU(2) group: $U^\dagger = U^{-1}; \qquad det(U) = 1$

SU(2) algebra: $M = M^\dagger \qquad Tr(M) = 0$

$$M = M^a T^a$$

$$T^a = \frac{\sigma^a}{2} \qquad Tr(T^a T^b) = \frac{1}{2}\delta_{ab};$$

$$U = e^{-i\theta^a T_a}$$

SU(2) gauge-invariant action

$$\boxed{S = -\int d^4x\sqrt{-g}\left[(D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}Tr[G_{\mu\nu}G^{\mu\nu}] + V(|\Phi|)\right]}$$

- **Scalar SU(2) doublet:** $\Phi = \dfrac{1}{\sqrt{2}}\begin{pmatrix}\varphi_0 + i\varphi_1 \\ \varphi_2 + i\varphi_3\end{pmatrix};\quad D_\mu = \partial_\mu 1 - i g_{\mathrm{B}} Q_{\mathrm{B}} B_\mu$

- **SU(2) gauge field:** $B_\mu = B_\mu^a T^a \in \mathfrak{su}(2)$

$$B_\mu = (B_0, B_1, B_2, B_3)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu, B_\nu]$$

- **Electric & magnetic field:** $E_i^{SU(2)} \equiv G_{0i} \qquad B_i^{SU(2)} \equiv \dfrac{1}{2}\varepsilon_{ijk}G^{jk}$

SU(2): gauge invariance

$$\boxed{S = -\int d^4x\sqrt{-g}\left[(D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}Tr[G_{\mu\nu}G^{\mu\nu}] + V(|\Phi|)\right]}$$

Action is invariant under **gauge SU(2) transformations:** $(\Omega(x) \in \mathrm{SU}(2))$

$$\phi(x) \to \Omega(x)\phi(x)$$

$$B_\mu(x) \to \Omega(x)B_\mu(x)\Omega^\dagger(x) - \frac{i}{g_B Q_B}[\partial_\mu\Omega(x)]\Omega^\dagger(x)$$

this implies

$$G_{\mu\nu}(x) \to \Omega(x)G_{\mu\nu}(x)\Omega^\dagger(x)$$

$$D_\mu\Phi(x) \to \Omega(x)D_\mu\Phi(x)$$

As for the U(1) theory, we work in the temporal gauge $(B_0 = 0)$

(4+9=13 degrees of freedom)

SU(2) EOM (in the continuum)

Equations of motion in the continuum are similar to the U(1) case:

$$\Phi'' - a^{-2(1-\alpha)} D_i D_i \Phi + (3-\alpha)\frac{a'}{a}\Phi' = -\frac{a^{2\alpha}}{2}\frac{\partial V}{\partial|\Phi|}\frac{\Phi}{|\Phi|}$$

$$\mathscr{D}_0 G_{0i} - a^{-2(1-\alpha)}\mathscr{D}_j G_{ji} + (1-\alpha)\frac{a'}{a}G_{0i} = a^{2\alpha}J_i$$

Gauss's law: $\mathscr{D}_i G_{0i} = a^2 J_0$

- Some new elements:

$J_i^a = 2g_B Q_B Im[\Phi^\dagger T_a (D_0\Phi)]$ **Current**

$$[\mathscr{D}_\mu G_{\mu\nu}]^a = (\mathscr{D}_\mu)_{ab} G^b_{\mu\nu} \equiv (\delta_{ab}\partial_\mu - f_{abc}B^c_\mu)G_{\mu\nu,b}$$

SU(2) theory discretization

We discretize using **links and plaquettes (compact formulation)**:

Links: $U_\mu(n) = e^{-ie\delta x B_\mu} \in \mathrm{SU}(2)$

Plaquette: $U_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n) \in \mathrm{SU}(2)$

- **Continuum limit:**

$$U_{\mu\nu} \approx 1 - i g_B Q_B \delta x^2 G_{\mu\nu} + \frac{1}{2} g_B^2 Q_B^2 \delta x^4 G_{\mu\nu} G_{\mu\nu} + \ldots \qquad \text{at } n + \frac{\hat{\mu}}{2} + \frac{\hat{\nu}}{2}$$

$$D_\mu \Phi \approx D_\mu^+ \Phi \left(n + \frac{\hat{\mu}}{2}\right) \equiv \frac{1}{\delta x_\mu} \left[U_\mu(n) \Phi(n+\hat{\mu}) - \Phi(n) \right]$$

$$\mathscr{D}_i G_{ij} \approx \frac{1}{\delta x} \left[U_{ij}(n) - U_i^\dagger(n-\hat{\imath}) U_{ij}(n-\hat{\imath}) U_i(n-\hat{\imath}) \right]$$

SU(2) toolkit and full lattice treatment in 2006.15122

Thank you!

Investigación confinanciada por la Comunidad de Madrid en el marco de la convocatoria correspondiente al año 2025 de las ayudas a la atracción de talento investigador “César Nombela” (**2025-T1/COM-36104**).

Backup slides

Example: Abelian-Higgs model

- As an example, we are going to simulate an **Abelian gauge model**:

$$\boxed{S = -\int d^4x\sqrt{-g}\left\{(D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(|\varphi|)\right\}}$$

+ self-consistent expansion

$$\boxed{V(|\varphi|) = \lambda|\varphi|^4 = \frac{\lambda}{4}(\varphi_0^2 + \varphi_1^2)^2}$$

$$\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1)$$

- We chose **program variables** analogously to scalar model:

$$f_* = |\varphi_*|$$
$$\omega_* = \sqrt{\lambda}|\varphi_*|$$
$$\alpha = 1$$

PROGRAM POTENTIAL:

$$\tilde{\varphi} \equiv \varphi/f_*$$

$$\boxed{\tilde{V}(|\tilde{\varphi}|) \equiv \frac{1}{f_*^2\omega_*^2}V(f_*|\tilde{\varphi}|) = |\tilde{\varphi}|^4}$$

- **Initial conditions:** We distribute the complex scalar amplitude equally between its components:

$$\varphi_0 = \varphi_1 = \phi_*/\sqrt{2} \qquad \dot{\varphi}_0 = \dot{\varphi}_1 = \dot{\phi}_*/\sqrt{2}$$

amplitude for the end of inflation in λφ⁴ potential

Evolution algorithms for gauge theories

(Non-compact) Staggered leapfrog algorithm

- ***Initial conditions:***

$$\left\{a, \tilde{\varphi}, \widetilde{A}_i\right\} \text{ at } \tilde{\eta}_0, \quad \left\{b_{-1/2}, (\tilde{\pi}_\varphi)_{-1/2}, (\tilde{\pi}_A)_{i,-1/2}\right\} \text{ at } \tilde{\eta}_0 - \frac{\delta\tilde{\eta}}{2}.$$

- ***Evolution:***

$$\begin{aligned}
(\tilde{\pi}_\varphi)_{+1/2} &= (\tilde{\pi}_\varphi)_{-1/2} + \delta\tilde{\eta}\mathcal{K}_\varphi[a, \tilde{\varphi}, \widetilde{A}_i]\,, \\
(\tilde{\pi}_A)_{i,+1/2} &= (\tilde{\pi}_A)_{i,-1/2} + \delta\tilde{\eta}\mathcal{K}_{A_i}[a, \tilde{\varphi}, \widetilde{A}_j]\,, \\
b_{+1/2} &= b_{-1/2} + \delta\tilde{\eta}\mathcal{K}_a\left[a, \overline{\widetilde{E}_K^\varphi}, \widetilde{E}_G^\varphi, \widetilde{E}_V^\varphi, \overline{\widetilde{E}_K^A}, \widetilde{E}_G^A\right]\,, \\
a_{+0} &= a + \delta\tilde{\eta}b_{+1/2}\,, \\
a_{+1/2} &= (a_{+0} + a)/2\,, \\
\tilde{\varphi}_{+0} &= \tilde{\varphi} + \delta\tilde{\eta}a_{+1/2}^{-(3-\alpha)}\,(\tilde{\pi}_\varphi)_{+1/2}\,, \\
\widetilde{A}_{i,+0} &= \widetilde{A}_i + \delta\tilde{\eta}a_{+1/2}^{-(1-\alpha)}\,(\tilde{\pi}_A)_{i,+1/2}\,,
\end{aligned}$$

- ***Hubble constraint:***

$$b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}\left[\overline{\widetilde{E}_K^\varphi} + \widetilde{E}_G^\varphi + \widetilde{E}_V + \overline{\widetilde{E}_K^A} + \widetilde{E}_G^A\right]$$

Evolution algorithms for gauge theories

(Non-compact) Velocity-Verlet algorithm

➤ *Initial conditions:* $\left\{a, b, \tilde{\varphi}, \tilde{\pi}_\varphi, \widetilde{A}_i, (\tilde{\pi}_A)_i\right\}$ at η_0.

➤ *Evolution:*

$$
\begin{aligned}
(\tilde{\pi}_\varphi)_{+1/2} &= \tilde{\pi}_\varphi + \frac{\delta\tilde{\eta}}{2}\mathcal{K}_\varphi[a, \tilde{\varphi}, \widetilde{A}_j]\,, \\
(\tilde{\pi}_A)_{i,+1/2} &= (\tilde{\pi}_A)_i + \frac{\delta\tilde{\eta}}{2}\mathcal{K}_{A_i}[a, \tilde{\varphi}, \widetilde{A}_j]\,, \\
b_{+1/2} &= b + \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a\left[a, \widetilde{E}_K^\varphi, \widetilde{E}_G^\varphi, \widetilde{E}_V^\varphi, \widetilde{E}_K^A, \widetilde{E}_G^A\right], \\
a_{+0} &= a + \delta\tilde{\eta} b_{+1/2}\,, \\
a_{+1/2} &= \frac{a_{+0} + a}{2}\,, \\
\tilde{\varphi}_{+0} &= \tilde{\varphi} + \delta\tilde{\eta}\frac{(\tilde{\pi}_\varphi)_{+1/2}}{a_{+1/2}^{3-\alpha}}\,, \\
\widetilde{A}_{i,+0} &= \widetilde{A}_i + \delta\tilde{\eta}\frac{(\tilde{\pi}_A)_{+1/2}}{a_{+1/2}^{1-\alpha}}\,, \\
(\tilde{\pi}_\varphi)_{+0} &= (\tilde{\pi}_\varphi)_{+1/2} + \frac{\delta\tilde{\eta}}{2}\mathcal{K}_\varphi[a_{+0}, \tilde{\varphi}_{+0}, \widetilde{A}_{j,+0}]\,, \\
(\tilde{\pi}_A)_{i,+0} &= (\tilde{\pi}_A)_{i,+1/2} + \frac{\delta\tilde{\eta}}{2}\mathcal{K}_{A_i}[a_{+0}, \tilde{\varphi}_{+0}, \widetilde{A}_{j,+0}]\,, \\
b_{+0} &= b_{+1/2} + \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a\left[a_{+0}, \widetilde{E}_{K,+0}^\varphi, \widetilde{E}_{G,+0}^\varphi, \widetilde{E}_{V,+0}^\varphi, \widetilde{E}_{K,+0}^A, \widetilde{E}_{G,+0}^A\right],
\end{aligned}
$$

Generalization to VV4 - VV10 implemented in CosmoLattice (and explained in the "Art")

➤ *Hubble constraint:* $b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}\left[\widetilde{E}_K^\varphi + \widetilde{E}_G^\varphi + \widetilde{E}_V + \widetilde{E}_K^A + \widetilde{E}_G^A\right]$

Example: Abelian-Higgs model

This model is implemented in the file `lphi4U1.h` of CosmoLattice.

- **In the model file (`lphi4U1.h`):**

```
struct ModelPars : public TempLat::DefaultModelPars{
    static constexpr size_t NScalars = 0;
    static constexpr size_t NCScalars = 1;
    static constexpr size_t NU1Flds = 1;
    static constexpr size_t NSU2Doublet = 0;
    static constexpr size_t NSU2Flds = 0;
    static constexpr size_t NPotTerms = 1;

    // Coupling managers:  they deal with the different couplings between
    // the gauge fields and complex scalars/SU2 doublets
    //  --> If a type of interaction is not present, comment the corresponding line
    typedef TempLat::CouplingsManager<NCScalars, NU1Flds,true> CsU1Couplings;
  // typedef TempLat::CouplingsManager<NSU2Doublet, NU1Flds, true> SU2DoubletU1Couplings;
  // typedef TempLat::CouplingsManager<NSU2Doublet, NSU2Flds,true> SU2DoubletSU2Couplings;
};
```

Number of fields for each species

Coupling managers (sets which complex scalars are coupled to the U(1) gauge sector)

Example: Abelian-Higgs model

- **In the parameter file** (lphi4U1.in):

```
#IC
kCutOff = 6
initial_amplitudes = 0
initial_momenta = 0
cmplx_field_initial_norm = 5.25151e18      # homogeneous amplitudes in GeV
cmplx_momentum_initial_norm = -4.45258e30 # homogeneous amplitudes in GeV2

#Gauge couplings and effectiveCharges
CSU1Charges = 1
gU1s = 1.11037e-05

#Model Parameters
lambda = 9e-14
```

$\rightarrow |\varphi_*|, |\dot{\varphi}_*|$

$\rightarrow Q_A, g_A$

$\rightarrow \lambda$

- **In the model file** (lphi4U1.h):

```
// COMPLEX SCALAR NORM: initial homogeneous amplitude and derivative
double normCmplx0 = parser.get<double>("cmplx_field_initial_norm");
double normPiCmplx0 = parser.get<double>("cmplx_momentum_initial_norm");

// We distribute the norm equally between the two components using the "Complexify" function
fldCS0(0_c) = Complexify(normCmplx0 / sqrt(2.0), normCmplx0 / sqrt(2.0));
piCS0(0_c)  = Complexify(normPiCmplx0 / sqrt(2.0), normPiCmplx0 / sqrt(2.0));
```

Reads initial homogeneous modes from input file

Sets the initial values for complex scalars: $\varphi = \left(\frac{|\varphi_*|}{\sqrt{2}}, \frac{|\varphi_*|}{\sqrt{2}}\right)$ $\dot{\varphi} = \left(\frac{|\dot{\varphi}_*|}{\sqrt{2}}, \frac{|\dot{\varphi}_*|}{\sqrt{2}}\right)$

Example: Abelian-Higgs model

- In the model file (lphi4U1.h):

```
alpha = 1;
fStar = normCmplx0;
omegaStar = sqrt(lambda) * normCmplx0;
```

$$f_* = |\varphi_*|, \quad \omega_* = \sqrt{\lambda}\,|\varphi_*|, \quad \alpha = 1$$

```
auto potentialTerms(Tag<0>) // Term 0: Qua
{
    return  pow<4>(norm(fldCS(0_c)));
}
```

$$\tilde{V}(|\tilde{\varphi}|) = |\tilde{\varphi}|^4$$

```
auto potDerivNormCS(Tag<0>) // Derivative
{
    return 4 * pow<3>(norm(fldCS(0_c)));
}
```

$$\tilde{V}_{,|\tilde{\varphi}|}(|\tilde{\varphi}|) = 4|\tilde{\varphi}|^3$$

```
auto potDeriv2NormCS(Tag<0>) // 2nd derivati
{
    return  12 * pow<2>(norm(fldCS(0_c)));
}
```

$$\tilde{V}_{,|\tilde{\varphi}||\tilde{\varphi}|}(|\tilde{\varphi}|) = 12|\tilde{\varphi}|^2$$