Simulating the Early Universe: CosmoLattice Workshop

Lecture 4

Gravitational Waves emission

From scalars to gauge fields

Nicolas Loayza Romero (CEICO-FZU, Czech Academy of Science)

Slides have been created in collaboration with Jorge Baeza-Ballesteros

Lecture is based on:

The art of simulating the early Universe Part II. Non-canonical cases & Gravitational Waves

Section 8

https://arxiv.org/pdf/2512.15627

Part I: Theoretical basis of Gravitational Waves

Gravitational Wave equation of motion

FLRW background + tensor perturbations

$$ds^2 = -dt^2 + a(t)^2(\delta_{ij} + h_{ij})dx_i dx_j \qquad h_{ii} = \partial_i h_{ij} = 0$$

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}$$

Gravitational Wave equation of motion

FLRW background + tensor perturbations

$$ds^2 = -dt^2 + a(t)^2(\delta_{ij} + h_{ij})dx_i dx_j \qquad h_{ii} = \partial_i h_{ij} = 0$$

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}$$

Latin indices $\{i, j, ..\}$ spatial components

Summation over repeated indices

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}$$

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}_{\text{TT part of Anisotropic stress tensor}}$$

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}$$

TT part of Anisotropic stress tensor

$$\partial_i \Pi_{ij}^{TT} = \Pi_{ii}^{TT} = 0 \;\; \text{hold} \;\; \forall \mathbf{x}, \forall \mathbf{t}$$

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}$$

TT part of Anisotropic stress tensor

$$\partial_i \Pi_{ij}^{TT} = \Pi_{ii}^{TT} = 0 \text{ hold } \forall \mathbf{x}, \forall \mathbf{t}$$

To obtain TT part of a tensor in real space is a **non-local operation!**

TT projection in Fourier Space

$$f\left(\mathbf{x},t\right)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\;e^{i\mathbf{x}\cdot\mathbf{k}}\;\tilde{f}\left(\mathbf{k},t\right)$$

Continuum Fourier Transform

TT projection in Fourier Space

$$f(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\; e^{i\mathbf{x}\cdot\mathbf{k}}\; \tilde{f}(\mathbf{k},t)$$

Continuum Fourier Transform

$$\Pi^{TT}_{ij}(\mathbf{k}) = \Lambda_{ijkl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k},t)$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\,,\, \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

TT projection in Fourier Space

$$f(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\; e^{i\mathbf{x}\cdot\mathbf{k}}\; \tilde{f}(\mathbf{k}, t)$$

Continuum Fourier Transform

$$\Pi^{TT}_{ij}(\mathbf{k}) = \Lambda_{ij,kl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k}, t)$$

check $\Pi^{TT}_{ij}(\mathbf{k})$ satisfies $k_i \Pi^{TT}_{ij} = \Pi^{TT}_{ii} = 0$

$$\Lambda_{ij,lm}(\mathbf{k}) \equiv P_{il}(\mathbf{k})P_{jm}(\mathbf{k}) - \frac{1}{2}P_{ij}(\mathbf{k})P_{lm}(\mathbf{k}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \mathbf{k}_i\mathbf{k}_j\,,\ \mathbf{k}_i = \mathbf{k}_i/k$$

TT projection in Fourier Space

$$f(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \; e^{i\mathbf{x}\cdot\mathbf{k}} \; \tilde{f}(\mathbf{k}, t)$$

Continuum Fourier Transform

$$\Pi_{ij}^{TT}(\mathbf{k}) = \Lambda_{ijkl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k}, t)$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j \,, \; \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

$$\boxed{\Pi_{ij}^{TT}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, e^{i\mathbf{x}\cdot\mathbf{k}} \Lambda_{ijkl}(\hat{\mathbf{k}}) \int d^3\mathbf{x} \, e^{-i\mathbf{k}\cdot\mathbf{x}} \, \Pi_{kl}(\mathbf{x}, t)}$$

TT projection in Fourier Space

$$f(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\; e^{i\mathbf{x}\cdot\mathbf{k}}\; \tilde{f}(\mathbf{k},t)$$

Continuum Fourier Transform

$$\Pi^{TT}_{ij}(\mathbf{k}) = \Lambda_{ijkl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k},t)$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\,,\ \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

Problem!

$$\boxed{\Pi^{TT}_{ij}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{x}\cdot\mathbf{k}}\Lambda_{ijkl}(\hat{\mathbf{k}}) \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,\Pi_{kl}(\mathbf{x},t)}$$

Anisotropic stress tensor

$$\Pi_{\mu\nu} \equiv T_{\mu\nu} - (T_{\mu\nu})_{perfect\,fluid}$$

Spatial Component $\quad \Pi_{ij} \equiv T_{ij} - p g_{ij}$

p = homogeneous background pressure

$$g_{ij} = a^2(t)(\delta_{ij} + h_{ij})$$

Anisotropic stress tensor

$$\Pi_{\mu\nu} \equiv T_{\mu\nu} - (T_{\mu\nu})_{perfect\,fluid}$$

Spatial Component $\quad \Pi_{ij} \equiv T_{ij} - p g_{ij}$

p = homogeneous background pressure

$$g_{ij} = a^2(t)(\delta_{ij} + h_{ij})$$

Source of GWs equation

$$\Pi_{\mu\nu} \equiv T_{\mu\nu} - (T_{\mu\nu})_{perfect\,fluid}$$ Spatial Component $$\Pi_{ij} \equiv T_{ij} - p g_{ij}$$

Let us consider most general theory with Scalars, Charged Scalars and U(1) gauge sector

$$\boxed{\mathcal{S} = -\int dx^4 \sqrt{-g}\mathscr{L} = -\int dx^4 \sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu\varphi)^*(D^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\phi, |\varphi|)\right\}}$$

$$\phi \in \mathbb{R}\,, \quad \varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \in \mathbb{C}\,, \quad \text{with} \quad \phi_1, \phi_2 \in \mathbb{R}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad \text{and} \quad D_\mu\varphi = \partial_\mu\varphi - i g_A Q_A A_\mu\varphi$$

Source of GWs equation

$$\boxed{\mathcal{S} = -\int dx^4 \sqrt{-g}\mathcal{L} = -\int dx^4 \sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu\varphi)^*(D^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\phi, |\varphi|)\right\}}$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L} - 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}$$

$$= -g_{\mu\nu}\left(g^{\alpha\beta}\left[(D_\alpha\varphi_b)^*(D_\beta\varphi_b) + \frac{1}{2}(\partial_\alpha\phi_a)(\partial_\beta\phi_a)\right] + \frac{1}{4}g^{\alpha\delta}g^{\beta\lambda}(F_{\alpha\beta}F_{\delta\lambda}) + V(\phi_a, \varphi_b)\right)$$

$$+(\partial_\mu\phi_a)(\partial_\nu\phi_a) + [2\mathrm{Re}\{(D_\mu\varphi_b)^*(D_\nu\varphi_b)\} + g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}]$$

Source of GWs equation

$$\boxed{\mathcal{S} = -\int dx^4 \sqrt{-g}\,\mathscr{L} = -\int dx^4 \sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu\varphi)^*(D^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\phi, |\varphi|)\right\}}$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L})}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathscr{L} - 2\frac{\delta\mathscr{L}}{\delta g^{\mu\nu}}$$

$$= -g_{\mu\nu}\left(g^{\alpha\beta}\left[(D_\alpha\varphi_b)^*(D_\beta\varphi_b) + \frac{1}{2}(\partial_\alpha\phi_a)(\partial_\beta\phi_a)\right] + \frac{1}{4}g^{\alpha\delta}g^{\beta\lambda}(F_{\alpha\beta}F_{\delta\lambda}) + V(\phi_a, \varphi_b)\right)$$

$$+(\partial_\mu\phi_a)(\partial_\nu\phi_a) + [2\mathrm{Re}\{(D_\mu\varphi_b)^*(D_\nu\varphi_b)\} + g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}]$$

$$\boxed{\Pi_{ij}^{\mathrm{eff}} = \partial_i\phi_a\,\partial_j\phi_a + 2\mathrm{Re}\{(D_i\varphi_b)^*\,(D_j\varphi_b)\} - \left(E_iE_j + a^{-2}B_iB_j\right)}$$

Source of GWs equation

$$\boxed{\mathcal{S} = -\int dx^4 \sqrt{-g}\mathcal{L} = -\int dx^4 \sqrt{-g}\left\{\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + (D_\mu \varphi)^*(D^\mu \varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\phi, |\varphi|)\right\}}$$

$$\boxed{\Pi_{ij}^{\text{eff}} = \partial_i \phi_a \, \partial_j \phi_a + 2\text{Re}\{(D_i \varphi_b)^* \, (D_j \varphi_b)\} - \left(E_i E_j + a^{-2} B_i B_j\right)}$$

given

$$E_i \equiv F_{0i} \,, \quad \text{and} \quad B_i \equiv \frac{1}{2}\epsilon_{ijk}F^{jk}$$

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\rangle_V$$

$\langle \ldots \rangle_V$ Volume average to encompass all relevant wavelengths

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x}, t)\dot{h}_{ij}(\mathbf{x}, t)\rangle_V$$

$$\approx \frac{1}{32\pi G V}\int_V \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{h}_{ij}(\mathbf{k}, t)\dot{h}^*_{ij}(\mathbf{k}, t)$$

Fourier transform of $h_{ij}(\mathbf{x}, t)$,
Approximation valid for
$kV^{1/3} \gg 1$

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\rangle_V$$

$$\approx \frac{1}{32\pi G V}\int_V \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)$$

$$\rho_{\rm GW}(t) = \int \frac{d\rho_{\rm GW}(k,t)}{d\log k}\frac{dk}{k}$$

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\rangle_V$$

$$\approx \frac{1}{32\pi G V}\int_V \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)$$

Energy density per logarithmic interval

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^3}{(4\pi)^3 G V}\int \frac{d\Omega_k}{4\pi}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)$$

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x}, t)\dot{h}^*_{ij}(\mathbf{x}, t)\rangle$$

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}^*_{ij}(\mathbf{x},t)\rangle$$

For stochastic sources use ensemble average $\langle \ldots \rangle$

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}^*_{ij}(\mathbf{x},t)\rangle$$

$$\langle \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k}',t)\rangle = (2\pi)^3 P_{\dot{h}}(k,t)\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

$$\rho_{\rm GW}(t) \equiv \frac{1}{(4\pi)^3 G}\int \frac{dk}{k} k^3 P_{\dot{h}}(k,t)$$

$P_{\dot{h}}$ power spectrum of $\dot{h}$

Energy density of Gravitational Wave Background (GWB)

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}^*_{ij}(\mathbf{x},t)\rangle$$

$$\langle \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k}',t)\rangle = (2\pi)^3 P_{\dot{h}}(k,t)\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

$$\rho_{\rm GW}(t) \equiv \frac{1}{(4\pi)^3 G}\int \frac{dk}{k} k^3 P_{\dot{h}}(k,t)$$

Energy density per logarithmic interval

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^3}{(4\pi)^3 G} P_{\dot{h}}(k,t)$$

Energy density of Gravitational Wave Background (GWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\mathrm{GW}}}{d\log k} = \frac{k^3}{(4\pi)^3 GV} \int \frac{d\Omega_k}{4\pi} \dot{h}_{ij}(\mathbf{k}, t) \dot{h}^*_{ij}(\mathbf{k}, t)$$

Critical energy density

$$\rho_{\mathrm{c}} \equiv 3H^2/8\pi G$$

GW energy density per critical energy density

$$\Omega_{\mathrm{GW}} = \frac{1}{\rho_{\mathrm{c}}} \frac{d\rho_{\mathrm{GW}}}{d\log k}$$

Gravitational Waves equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}_{\text{Transverse-Traceless Tensor}}$$

$\partial_i \Pi_{ij}^{TT}$ **Problem!** $\mathbf{x}, \forall \mathbf{t}$

To obtain TT part of a tensor in real space is a **non-local operation!**

$$\boxed{\Pi_{ij}^{TT}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{x}\cdot\mathbf{k}}\Lambda_{ijkl}(\hat{\mathbf{k}}) \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \Pi_{kl}(\mathbf{x},t)}$$

Evolution of GW modes

non-local operations are computationally expensive!

Every Time-step! ➡

$$\boxed{\Pi_{ij}^{TT}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{x}\cdot\mathbf{k}} \Lambda_{ijkl}(\hat{\mathbf{k}}) \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \Pi_{kl}(\mathbf{x},t)}$$

Evolution of GWs modes

non-local operations are computationally expensive!

Solution: we define a set of unphysical tensor modes u's

1) Evolve equation of motion of u's

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}$$

Garcia-Bellido, Figueroa & Sastre '07

Evolution of GWs modes

non-local operations are computationally expensive!

Solution: we define a set of unphysical tensor modes u's

1) Evolve equation of motion of u's

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}$$

2) When needed, (compute power spectrum energy density) we apply transformation

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

Garcia-Bellido, Figueroa & Sastre '07

Why it works?

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}$$

Linear in Fourier Space
=> commute

Why it works?

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

$$\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt} + \frac{k^2}{a^2}\right)h_{ij}(k,t) = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}(k,t)$$

Why it works?

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

$$\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt} + \frac{k^2}{a^2}\right)\Lambda_{ijkl}u_{kl}\,(k,t) = \frac{2}{m_p^2a^2}\Lambda_{ijkl}\Pi_{kl}(k,t)$$

Why it works?

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}$$

Every Time-step!

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

Few Times!

What are we giving up?

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}^{TT}$$

2 polarizations or 2 *dofs*

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}^{\mathrm{eff}}$$

6 *dofs*

Less complexity but
more memory needed!

Can we reduce it further?

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}^{\text{eff}}$$

6 *dofs*

$$u_{33} = -(u_{11} + u_{22})$$

5 *dofs*

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\left(\Pi_{ij}^{\text{eff}} - \frac{1}{3}\delta_{ij}\Pi_{kk}^{\text{eff}}\right)$$

Energy density of Gravitational Wave Background (GWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\mathrm{GW}}}{d\log k} = \frac{k^3}{(4\pi)^3 GV}\int \frac{d\Omega_k}{4\pi}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)$$

Energy density of Gravitational Wave Background (GWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\mathrm{GW}}}{d\log k} = \frac{k^3}{(4\pi)^3 GV}\int \frac{d\Omega_k}{4\pi} \boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)}$$

$$\begin{aligned}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) &= (\Lambda_{ijkl}(\mathbf{k})u_{kl}(\mathbf{k},t))(\Lambda_{ijmn}(\mathbf{k})u^*_{mn}(\mathbf{k},t)) \\ &= \Lambda_{klmn}(\mathbf{k})u_{kl}(\mathbf{k},t)u^*_{mn}(\mathbf{k},t)\end{aligned}$$

Energy density of Gravitational Wave Background (GWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^3}{(4\pi)^3 GV}\int \frac{d\Omega_k}{4\pi} \boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)}$$

$$\begin{aligned}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) &= (\Lambda_{ijkl}(\mathbf{k})u_{kl}(\mathbf{k},t))(\Lambda_{ijmn}(\mathbf{k})u^*_{mn}(\mathbf{k},t)) \\ &= \Lambda_{klmn}(\mathbf{k})u_{kl}(\mathbf{k},t)u^*_{mn}(\mathbf{k},t)\end{aligned}$$

$$\boxed{\Lambda_{klmn}(\mathbf{k}) = \Lambda_{klpq}(\mathbf{k})\Lambda_{pqmn}(\mathbf{k})}$$

Energy density of Gravitational Wave Background (GWB)

$$\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) = \mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) - \frac{1}{2}\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}})\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*)$$

$$(\dot{\mathtt{u}})_{ij} = \dot{u}_{ij} \qquad (\mathtt{P})_{ij} = P_{ij}$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\,,\ \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

Energy density of Gravitational Wave Background (GWB)

$$\boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) = \mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) - \frac{1}{2}\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}})\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*)}$$

$$v_{ij} \equiv P_{ik}\dot{u}_{kj} \qquad \tilde{v}_{ij} \equiv P_{ik}\dot{u}^*_{kj}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) = v_{11}\tilde{v}_{11} + v_{22}\tilde{v}_{22} + v_{33}\tilde{v}_{33} + v_{12}\tilde{v}_{21} + v_{21}\tilde{v}_{12} + v_{13}\tilde{v}_{31} + v_{31}\tilde{v}_{13} + v_{23}\tilde{v}_{32} + v_{32}\tilde{v}_{23}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}) = v_{11} + v_{22} + v_{33}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*) = \tilde{v}_{11} + \tilde{v}_{22} + \tilde{v}_{33}$$

Energy density of Gravitational Wave Background (GWB)

$$\boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) = \mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) - \frac{1}{2}\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}})\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*)}$$

$$v_{ij} \equiv P_{ik}\dot{u}_{kj} \qquad \tilde{v}_{ij} \equiv P_{ik}\dot{u}^*_{kj}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) = v_{11}\tilde{v}_{11} + v_{22}\tilde{v}_{22} + v_{33}\tilde{v}_{33} + v_{12}\tilde{v}_{21} + v_{21}\tilde{v}_{12} + v_{13}\tilde{v}_{31} + v_{31}\tilde{v}_{13} + v_{23}\tilde{v}_{32} + v_{32}\tilde{v}_{23}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}) = v_{11} + v_{22} + v_{33}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*) = \tilde{v}_{11} + \tilde{v}_{22} + \tilde{v}_{33}$$

Part II: GWs in the lattice

Recap of lattice discretization (I)

We work in a real periodic lattice with spacing δx

$$\boldsymbol{n} = (n_1, n_2, n_3), \qquad n_i = 0, 1, ..., N-1$$

Recap of lattice discretization (I)

We work in a real periodic lattice with spacing δx

$$\boldsymbol{n} = (n_1, n_2, n_3), \qquad n_i = 0, 1, ..., N-1$$

Fourier modes also live in a periodic lattice with spacing $k_{\rm IR} = 2\pi/L$

$$\tilde{\boldsymbol{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3), \qquad \tilde{n}_i = -\frac{N}{2} + 1, ..., -1, 0, 1, ..., \frac{N}{2}$$

Recap of lattice discretization (I)

We work in a real periodic lattice with spacing δx

$$\boldsymbol{n} = (n_1, n_2, n_3), \qquad n_i = 0, 1, ..., N-1$$

Fourier modes also live in a periodic lattice with spacing $k_{\rm IR} = 2\pi/L$

$$\tilde{\boldsymbol{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3), \qquad \tilde{n}_i = -\frac{N}{2}+1, ..., -1, 0, 1, ..., \frac{N}{2}$$

Related by discrete Fourier transform

$$f(\boldsymbol{n}) = \frac{1}{N^3}\sum_{\tilde{\boldsymbol{n}}} \mathrm{e}^{\frac{2\pi i}{N}\tilde{\boldsymbol{n}}\boldsymbol{n}} f(\tilde{\boldsymbol{n}}) \qquad f(\tilde{\boldsymbol{n}}) = \sum_{\boldsymbol{n}} \mathrm{e}^{-\frac{2\pi i}{N}\tilde{\boldsymbol{n}}\boldsymbol{n}} f(\boldsymbol{n})$$

Recap of lattice discretization (II)

Recap of lattice discretization (II)

We define a lattice momentum

$$[\nabla_i f](\tilde{\boldsymbol{n}}) = -i\boldsymbol{k}_{\mathrm{L}}(\tilde{\boldsymbol{n}})f(\tilde{\boldsymbol{n}})$$

Recap of lattice discretization (II)

We define a lattice momentum

$$[\nabla_i f](\tilde{\boldsymbol{n}}) = -i \boldsymbol{k}_{\mathrm{L}}(\tilde{\boldsymbol{n}}) f(\tilde{\boldsymbol{n}})$$

Different derivative discretizations

$$[\nabla_i^0 f](\boldsymbol{n}) = \frac{f(\boldsymbol{n}+\hat{i}) - f(\boldsymbol{n}-\hat{i})}{2\delta x} \quad \longrightarrow \quad k_{\mathrm{L},i}^0 = \frac{\sin(2\pi \tilde{n}_i/N)}{\delta x}$$

$$[\nabla_i^\pm f](\boldsymbol{n}) = \frac{\pm f(\boldsymbol{n}\pm\hat{i}) \mp f(\boldsymbol{n})}{\delta x} \quad \longrightarrow \quad k_{\mathrm{L},i}^\pm = 2 e^{\mp i\pi\tilde{n}_i/N} \frac{\sin(\pi \tilde{n}_i/N)}{\delta x}$$

GWs in the lattice

GWs in the lattice

We work with five **unphysical** real degrees of freedom

$$\ddot{u}_{ij}(\boldsymbol{n}) + 3H\dot{u}_{ij}(\boldsymbol{n}) - \frac{\nabla_{\rm L}^2}{a^2}u_{ij}(\boldsymbol{n}) = \frac{2}{m_{\rm Pl}^2 a^2}\left(\Pi_{ij}(\boldsymbol{n}) - \frac{1}{3}\delta_{ij}\Pi_{kk}(\boldsymbol{n})\right)$$

GWs in the lattice

We work with five **unphysical** real degrees of freedom

$$\ddot{u}_{ij}(\boldsymbol{n}) + 3H\dot{u}_{ij}(\boldsymbol{n}) - \frac{\nabla^2_{\rm L}}{a^2}u_{ij}(\boldsymbol{n}) = \frac{2}{m_{\rm Pl}^2 a^2}\left(\Pi_{ij}(\boldsymbol{n}) - \frac{1}{3}\delta_{ij}\Pi_{kk}(\boldsymbol{n})\right)$$

Physical fields projected **only** when measuring

$$h_{ij}(\tilde{\boldsymbol{n}}, t) = \Lambda^{\rm L}_{ij,kl}(\tilde{\boldsymbol{n}})u_{kl}(\tilde{\boldsymbol{n}}, t)$$

GWs in the lattice

We work with five **unphysical** real degrees of freedom

$$\ddot{u}_{ij}(\boldsymbol{n}) + 3H\dot{u}_{ij}(\boldsymbol{n}) - \frac{\nabla_{\mathrm{L}}^2}{a^2}u_{ij}(\boldsymbol{n}) = \frac{2}{m_{\mathrm{Pl}}^2 a^2}\left(\Pi_{ij}(\boldsymbol{n}) - \frac{1}{3}\delta_{ij}\Pi_{kk}(\boldsymbol{n})\right)$$

Physical fields projected **only** when measuring

$$h_{ij}(\tilde{\boldsymbol{n}}, t) = \Lambda^{\mathrm{L}}_{ij,kl}(\tilde{\boldsymbol{n}})u_{kl}(\tilde{\boldsymbol{n}}, t)$$

Need to choose a particular lattice derivative

$$\Lambda^{\mathrm{L}}_{ij,kl} = P_{il}P_{jm} - \frac{1}{2}P_{ij}P_{lm} \qquad P_{ij} = \delta_{ij} - \frac{k_{\mathrm{L},i}k_{\mathrm{L},j}}{k_{\mathrm{L}}^2}$$

Ensures $h_{ii} = 0$ and $\nabla_{\mathrm{L},i}h_{ij} = 0$ [Figueroa, García-Bellido, Ranjantie, (2011), 1110.0337]

Properties of the TT projector

For the **forward/backward** derivatives,

$$\sum_i k_{\mathrm{L},i} P_{ij} = 0 \qquad \sum_i k^*_{\mathrm{L},i} P_{ij} \neq 0$$

$$\sum_j k_{\mathrm{L},j} P_{ij} \neq 0 \qquad \sum_j k^*_{\mathrm{L},j} P_{ij} = 0$$

$$P^*_{ij} = P_{ji} \qquad P_{ij}(-\tilde{\boldsymbol{n}}) = P_{ij}(\tilde{\boldsymbol{n}})$$

$$P_{ij} P_{jk} = P_{ik} \qquad P_{ij} P_{ki} \neq P_{ik}$$

[Figueroa, García-Bellido, Ranjantie, (2011), 1110.0337]

Properties of the TT projector

For the **forward/backward** derivatives,

$$\sum_i k_{\mathrm{L},i} P_{ij} = 0 \qquad \sum_i k^*_{\mathrm{L},i} P_{ij} \neq 0$$

$$\sum_j k_{\mathrm{L},j} P_{ij} \neq 0 \qquad \sum_j k^*_{\mathrm{L},j} P_{ij} = 0$$

$$\boxed{P^*_{ij} = P_{ji}} \qquad P_{ij}(-\tilde{\boldsymbol{n}}) = P_{ij}(\tilde{\boldsymbol{n}})$$

$$P_{ij}P_{jk} = P_{ik} \qquad P_{ij}P_{ki} \neq P_{ik}$$

[Figueroa, García-Bellido, Ranjantie, (2011), 1110.0337]

Properties of the TT projector

For the **forward/backward** derivatives,

$$\sum_i k_{\mathrm{L},i} P_{ij} = 0 \qquad \sum_i k^*_{\mathrm{L},i} P_{ij} \neq 0$$

$$\sum_j k_{\mathrm{L},j} P_{ij} \neq 0 \qquad \sum_j k^*_{\mathrm{L},j} P_{ij} = 0$$

$$\boxed{P^*_{ij} = P_{ji}} \qquad P_{ij}(-\tilde{\boldsymbol{n}}) = P_{ij}(\tilde{\boldsymbol{n}})$$

$$\boxed{P_{ij} P_{jk} = P_{ik}} \qquad P_{ij} P_{ki} \neq P_{ik}$$

[Figueroa, García-Bellido, Ranjantie, (2011), 1110.0337]

Properties of the TT projector

For the **forward/backward** derivatives,

$$\sum_i k_{\mathrm{L},i} P_{ij} = 0 \qquad \sum_i k^*_{\mathrm{L},i} P_{ij} \neq 0$$

$$\sum_j k_{\mathrm{L},j} P_{ij} \neq 0 \qquad \sum_j k^*_{\mathrm{L},j} P_{ij} = 0$$

$$\boxed{P^*_{ij} = P_{ji}} \qquad P_{ij}(-\tilde{\boldsymbol{n}}) = P_{ij}(\tilde{\boldsymbol{n}})$$

$$\boxed{P_{ij} P_{jk} = P_{ik}} \qquad P_{ij} P_{ki} \neq P_{ik}$$

Similar set of properties for real projector

[Figueroa, García-Bellido, Ranjantie, (2011), 1110.0337]

Properties of the TT projector

For the **forward/backward** derivatives,

$$\sum_i k_{\mathrm{L},i} P_{ij} = 0 \qquad \sum_i k^*_{\mathrm{L},i} P_{ij} \neq 0$$

$$\sum_j k_{\mathrm{L},j} P_{ij} \neq 0 \qquad \sum_j k^*_{\mathrm{L},j} P_{ij} = 0$$

$$\boxed{P^*_{ij} = P_{ji}} \qquad P_{ij}(-\tilde{\boldsymbol{n}}) = P_{ij}(\tilde{\boldsymbol{n}})$$

$$\boxed{P_{ij} P_{jk} = P_{ik}} \qquad P_{ij} P_{ki} \neq P_{ik}$$

Similar set of properties for real projector

[Figueroa, García-Bellido, Ranjantie, (2011), 1110.0337]

GW energy density power spectrum

GW energy density power spectrum

We define the GW energy density as a volume average

$$\rho_{\rm GW}(t) \quad = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V$$

GW energy density power spectrum

We define the GW energy density as a volume average

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t)$$

GW energy density power spectrum

We define the GW energy density as a volume average

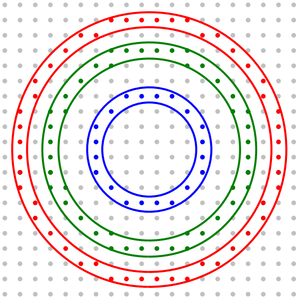
$$\begin{aligned}\rho_{\rm GW}(t) &= \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t)\\ &= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\end{aligned}$$

Fourier transform
Parseval's theorem

GW energy density power spectrum

We define the GW energy density as a volume average

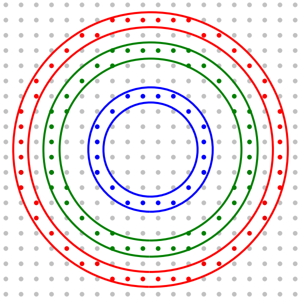
$$
\begin{aligned}
\rho_{\rm GW}(t) \quad &= \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{n}}\sum_{R(\tilde{n})} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)
\end{aligned}
$$



GW energy density power spectrum

We define the GW energy density as a volume average

$$\begin{aligned}\rho_{\rm GW}(t) &= \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t)\\ &= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\\ &= \frac{1}{32\pi G N^6}\sum_{\tilde{n}}\sum_{R(\tilde{n})} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\end{aligned}$$

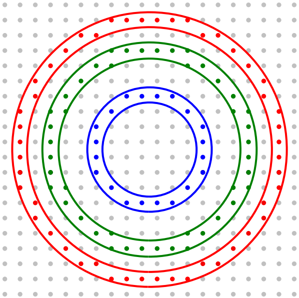


Multiplicity of shells
$\approx 4\pi\tilde{n}^2$

GW energy density power spectrum

We define the GW energy density as a volume average

$$\begin{aligned}
\rho_{\mathrm{GW}}(t) &= \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{n}} 4\pi\tilde{n}^2\langle \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde{n})}
\end{aligned}$$



Multiplicity of shells
$\approx 4\pi\tilde{n}^2$

GW energy density power spectrum

We define the GW energy density as a volume average

$$
\begin{aligned}
\rho_{\rm GW}(t) &= \frac{1}{32\pi G}\langle \dot h_{ij}\dot h_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot h_{ij}(\boldsymbol{n},t)\dot h_{ij}(\boldsymbol{n},t)\\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot h_{ij}(\tilde{\boldsymbol{n}},t)\dot h^*_{ij}(\tilde{\boldsymbol{n}},t)\\
&= \frac{1}{32\pi G N^6}\sum_{\tilde n} 4\pi \tilde n^2 \langle \dot h_{ij}(\tilde{\boldsymbol{n}},t)\dot h^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde n)}\\
&= \sum_{\tilde n}\left[\frac{\delta x^6}{(4\pi)^3 G L^3}k^3(\tilde n)\langle \dot h_{ij}(\tilde{\boldsymbol{n}},t)\dot h^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde n)}\right]\underset{\displaystyle k_{\rm IR}/k(\tilde n)}{\underset{\big\downarrow}{\Delta\log k}}
\end{aligned}
$$

GW energy density power spectrum

We define the GW energy density as a volume average

$$\begin{aligned}
\rho_{\mathrm{GW}}(t) &= \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{n}} 4\pi\tilde{n}^2\langle \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde{n})} \\
&= \sum_{\tilde{n}} \underbrace{\left[\frac{\delta x^6}{(4\pi)^3 G L^3} k^3(\tilde{n})\langle \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde{n})}\right]}_{\frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}\log k}} \underset{k_{\mathrm{IR}}/k(\tilde{n})}{\underset{\downarrow}{\Delta \log k}}
\end{aligned}$$

GW energy density power spectrum

We define the GW energy density as a volume average

$$
\begin{aligned}
\rho_{\mathrm{GW}}(t) &= \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t) \\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{n}} 4\pi\tilde{n}^2\langle \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde{n})} \\
&= \sum_{\tilde{n}} \underbrace{\left[\frac{\delta x^6}{(4\pi)^3 G L^3} k^3(\tilde{n})\langle \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde{n})}\right]}_{\frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}\log k}} \underset{k_{\mathrm{IR}}/k(\tilde{n})}{\underset{\downarrow}{\Delta \log k}}
\end{aligned}
$$

Using the TT projector

$$
\dot{h}_{ij}\dot{h}^*_{ij} = \Lambda_{ij,kl}\dot{u}_{kl}\Lambda^*_{ij,mp}\dot{u}^*_{mp} = \mathrm{Tr}[P\dot{u}P\dot{u}^*] - \frac{1}{2}\mathrm{Tr}[P\dot{u}]\mathrm{Tr}[P\dot{u}^*]
$$

We **only** project when we want to measure

GW energy density power spectrum

We define the GW energy density as a volume average

$$\begin{aligned}
\rho_{\rm GW}(t) &= \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle_V = \frac{1}{32\pi G N^3}\sum_{\boldsymbol{n}} \dot{h}_{ij}(\boldsymbol{n},t)\dot{h}_{ij}(\boldsymbol{n},t)\\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{\boldsymbol{n}}} \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\\
&= \frac{1}{32\pi G N^6}\sum_{\tilde{n}} 4\pi\tilde{n}^2\langle \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde{n})}\\
&= \sum_{\tilde{n}} \underbrace{\left[\frac{\delta x^6}{(4\pi)^3 G L^3} k^3(\tilde{n})\langle \dot{h}_{ij}(\tilde{\boldsymbol{n}},t)\dot{h}^*_{ij}(\tilde{\boldsymbol{n}},t)\rangle_{R(\tilde{n})}\right]}_{\frac{d\rho_{\rm GW}}{d\log k}} \underset{k_{\rm IR}/k(\tilde{n})}{\underset{\downarrow}{\Delta \log k}}
\end{aligned}$$

In general we compute the **normalized** energy density power spectrum

$$\Omega_{\rm GW} = \frac{1}{\rho_c}\frac{d\rho_{\rm GW}}{d\log k}$$

Thanks!