The Fate of Ultra-Collinear Modes in On-Shell Massive Sudakov Form Factors

Jakob Schoenleber (TUM)

Talk at QCD Evolution 2026

[Schnubel, JS, Szafron, 2026] (arXiv: 2604.02859)

Massification: Obtain massive result from the dim. reg. result.

DIMENSIONAL REGULARIZATION OF INFRARED DIVERGENCES

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An analysis of the application of dimensional regularization to infrared divergences in lowest order radiative corrections is presented. The main emphasis of the paper is to show explicitly how dimensional regularization can lead in some cases of considerable interest to very simple and elegant evaluations of infrared divergent contributions and their associated finite parts, and to pinpoint the mathematical reason for the equivalence with the traditional method of regularization.

$$\ln \lambda_{\min} \leftrightarrow -\left(\frac{1}{n-4} + c\right)$$

Massification = Mass factorization

$$\underbrace{1+\frac{\alpha}{\pi}\ln\frac{m^2}{Q^2}}_{\text{massive amplitude}} = \underbrace{\left\{1+\frac{\alpha}{\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{Q^2}\right)\right\}}_{\text{massless amplitude}}\underbrace{\left\{1-\frac{\alpha}{\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{m^2}\right)\right\}}_{\text{massification ``}Z\text{-factor''}}+\mathcal{O}(\alpha^2).$$

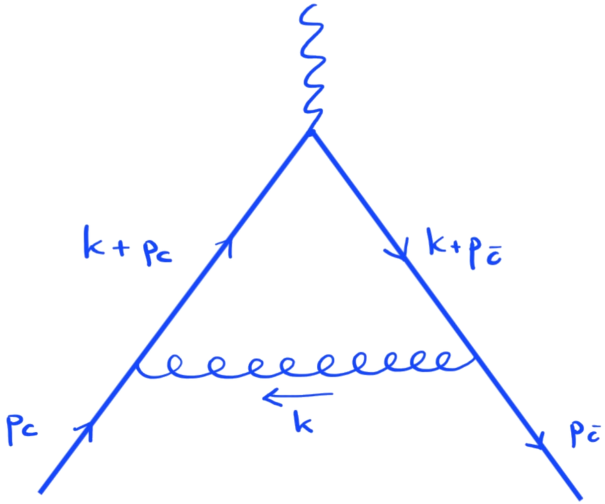
- Z factors are universal $\Rightarrow$ Can restore massive amplitude without explicit calculation (within a power accuracy)
- Factorization enables resummation of (potentially large) $\alpha^i \ln^j \frac{m^2}{Q^2}$

- Expand amplitude in small parameter $\lambda = m/Q$
- Light cone basis $v^\mu = n_+ \cdot v \, \frac{n_-^\mu}{2} + n_- \cdot v \, \frac{n_+^\mu}{2} + v_\perp^\mu$
- Momentum scaling/modes $p = (n_+ p, p_\perp, n_- p)$

$p_h \sim (1,1,1)Q$	hard
$p_c \sim (1,\lambda,\lambda^2)Q$	collinear
$p_{\bar{c}} \sim (\lambda^2,\lambda,1)Q$	anti-collinear
$p_s \sim (\lambda,\lambda,\lambda)Q$	soft

Expand amplitude in small parameter $\lambda = \frac{m}{Q}$ **Method of regions**

$$\int d^dk\, f(k) = \int d^dk\, f(k)|_{k\sim p_h} + \int d^dk\, f(k)|_{k\sim p_c} + \int d^dk\, f(k)|_{k\sim p_{\bar{c}}} + \int d^dk\, f(k)|_{k\sim p_s} + \mathcal{O}(\lambda^2)$$



$$f(k) = \frac{1}{k^2[(k+p_c)^2 - m^2][(k+p_{\bar{c}})^2 - m^2]}$$

$$f(k)|_{k\sim p_h} = \frac{1}{k^2(k + n_+\cdot p_c\, n_-/2)^2(k + n_-\cdot p_{\bar{c}}\, n_+/2)^2}$$

$$f(k)|_{k\sim p_c} = \frac{1}{k^2[(k+p_c)^2 - m^2](n_+\cdot k\, n_-\cdot p_{\bar{c}})}$$

$$f(k)|_{k\sim p_{\bar{c}}} = \frac{1}{k^2(n_-\cdot k\, n_+\cdot p_c)[(k + p_{\bar{c}}^2) - m^2]}$$

$$f(k)|_{k\sim p_s} = \frac{1}{k^2(n_+\cdot p_c\, n_-\cdot k)(n_-\cdot p_{\bar{c}}\, n_+\cdot k)}$$

- SCET performs the λ expansion at the Lagrangian level

$$\psi = \xi_c + \xi_{\bar{c}} + \psi_s, \qquad A = A_c + A_{\bar{c}} + A_s$$

$$\mathcal{L}(\psi, A) = \mathcal{L}_c(\xi_c, A_c) + \mathcal{L}_{\bar{c}}(\xi_{\bar{c}}, A_{\bar{c}}) + \mathcal{L}_s(\psi_s, A_s) + \mathcal{O}(\lambda^2)$$

- Factorization

$$\langle p_{\bar{c}}|\bar{\psi}\gamma^\mu\psi|p_c\rangle = C \times \underbrace{\langle p_{\bar{c}}|\bar{\xi}_{\bar{c}}W_{\bar{c}}|0\rangle}_{=\sqrt{Z}} \times \underbrace{\langle 0|S_+^\dagger S_-|0\rangle}_{=S} \times \underbrace{\langle 0|W_c^\dagger\xi_c|p_c\rangle}_{=\sqrt{Z}} + \mathcal{O}(\lambda^2)$$

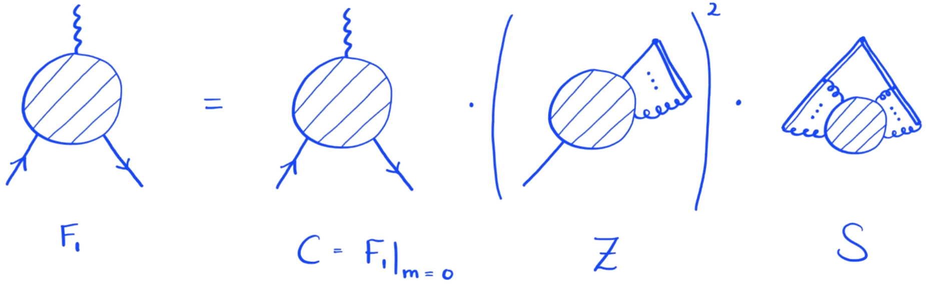
- C is a Wilson coefficient. Coincides with massless Sudakov form factor and depends on the hard current. Z and S contain mass dependence and are universal (i.e. do not depend on the hard local current but only the external state)

- Standard example: On-shell Sudakov form factor $Q^2 = -(p_c - p_{\bar{c}})^2, \quad m^2 = p_c^2 = p_{\bar{c}}^2$

$$\langle p_{\bar{c}} | \bar{\psi} \gamma^\mu \psi | p_c \rangle = \bar{u}(p_{\bar{c}}) \gamma^\mu u(p_c) F_1(Q^2, m^2) + \ldots$$

- Factorization theorem (SCETII)

$$F_1(m/Q) = C(\mu/Q)\, Z(\mu/m, \nu/Q)\, S(\mu/m, \nu/m) + \mathcal{O}(\lambda^2)\,,$$



μ = RGE scale

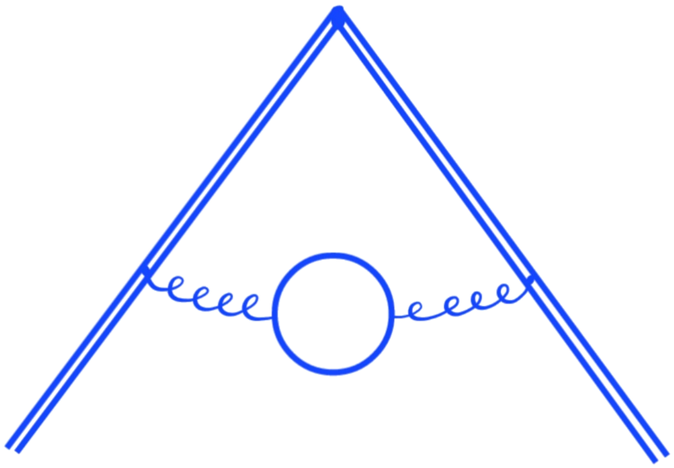
ν = rapidity RGE scale

We use η regulator
[Chiu, Jain, Neill, Rothstein, 2012]

- Since the massive (3 loops) and massless (4 loops) result for the quark SFF is known, we just have to calculate S in order to determine $Z = F_1/(CS)$

- S is zero at scaleless at one-loop. Non-zero contributions are due to closed massive fermion loops

$$S = 1 + \left(\frac{\alpha_0}{4\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}}{m^2}\right)^{2\epsilon} w^2(\nu) \left(\frac{\nu^2}{m^2}\right)^{\eta/2} C_F T_F \Bigg\{ -\frac{8}{3\eta}(\Gamma(\epsilon)e^{\epsilon\gamma_E})^2 \frac{1+\epsilon}{1+\frac{8}{3}\epsilon+\frac{4}{3}\epsilon^2} + \frac{2}{\epsilon^3} - \frac{10}{9\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{56}{27}+\frac{\pi^2}{3}\right) + \frac{328}{27} - \frac{5\pi^2}{27} + 4\zeta_3 + O(\eta,\epsilon)\Bigg\} + O(\alpha_0^3).$$



- Obtained from the massive 2-loop result [Bernreuther et al., 2006]
- This is **universal.** For each external quark leg multiply by $\sqrt{Z}$ to convert the massless result to a massive result*
- *Still need to multiply by the soft function however. This is also universal but depends on the number of legs
- See also [Becher, Melnikov, 2007] [Hoang, Pathak, Pietrulewicz, Stewart, 2015]

$$
\begin{aligned}
Z = 1 &+ \frac{\alpha_0}{4\pi}\left(\frac{4\pi e^{-\gamma_E}}{m^2}\right)^{\epsilon} C_F Z^{(1)} \\
&+ \left(\frac{\alpha_0}{4\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}}{m^2}\right)^{2\epsilon} w^2(\nu)\left(\frac{\nu^2}{Q^2}\right)^{\eta/2} C_F \left\{ C_F Z_F^{(2)} + C_A Z_A^{(2)} + T_F Z_T^{(2)} + O(\eta;\epsilon)\right\} \\
&+ O(\alpha_0^3),
\end{aligned}
$$

$$
Z^{(1)} = \frac{2}{\epsilon^2} + \frac{1}{\epsilon} + 4 + \frac{\pi^2}{6} + \epsilon\left(8 + \frac{\pi^2}{12} - \frac{2}{3}\zeta_3\right) + \epsilon^2\left(16 + \frac{\pi^2}{3} - \frac{1}{3}\zeta_3 + \frac{\pi^4}{80}\right) + O(\epsilon^3),
$$

$$
\begin{aligned}
Z_F^{(2)} &= \frac{2}{\epsilon^4} + \frac{2}{\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{17}{2} + \frac{\pi^2}{3}\right) + \frac{1}{\epsilon}\left(\frac{83}{4} - \frac{2\pi^2}{3} + \frac{32}{3}\zeta_3\right) \\
&\quad + \frac{561}{8} + \frac{61\pi^2}{12} - \frac{22}{3}\zeta_3 - \frac{77\pi^4}{180} - 8\pi^2 \ln 2,
\end{aligned}
$$

$$
\begin{aligned}
Z_A^{(2)} &= \frac{11}{6\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{50}{9} - \frac{\pi^2}{6}\right) + \frac{1}{\epsilon}\left(\frac{1957}{108} + \frac{67\pi^2}{36} - 15\zeta_3\right) \\
&\quad + \frac{31885}{648} + \frac{89\pi^2}{27} + \frac{67}{9}\zeta_3 - \frac{47\pi^4}{180} + 4\pi^2 \ln 2,
\end{aligned}
$$

$$
\begin{aligned}
Z_T^{(2)} &= \frac{8}{3\eta}(\Gamma(\epsilon)e^{\epsilon\gamma_E})^2 \frac{1+\epsilon}{1+\frac{8}{3}\epsilon + \frac{4}{3}\epsilon^2} - \frac{8}{3\epsilon^3} + \frac{2}{3\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{17}{3} - \frac{8\pi^2}{9}\right) \\
&\quad + \frac{1411}{162} - \frac{11\pi^2}{9} - \frac{32}{9}\zeta_3.
\end{aligned}
$$

- Z, S contain $1/\epsilon_{UV},\ 1/\epsilon_{IR}$ and $1/\eta$ divergences $\rightarrow$ Not possible to separate, but can subtract everything together using MSbar

- C has the usual $1/\epsilon_{IR}$ poles

- F_1 is UV finite, but still contains IR divergences due massless gluon/photon

$$\mathrm{F}_1 = F_1^{\mathrm{IR}} F_1^{\mathrm{fin}}, \qquad Z = \mathrm{Z}_Z Z_R, \qquad S = \mathrm{Z}_S S_R, \qquad C = \mathrm{Z}_C C_R$$

- $F_1^{IR} = Z_C Z_S Z_Z$ are the “true” IR divergences (for massive fermions). All $1/\epsilon_{UV},\ 1/\eta$ poles must cancel in the product $Z_C Z_S Z_Z$.

- We know how to resum logs in F_1^{fin}, but how can we resum logs in F_1^{IR} ? (This will be answered later)

- How do we know that these modes (collinear, anti-collinear and soft) describe the low energy ($\ll Q$) physics completely?

We do not!

Problems of the Strategy of Regions

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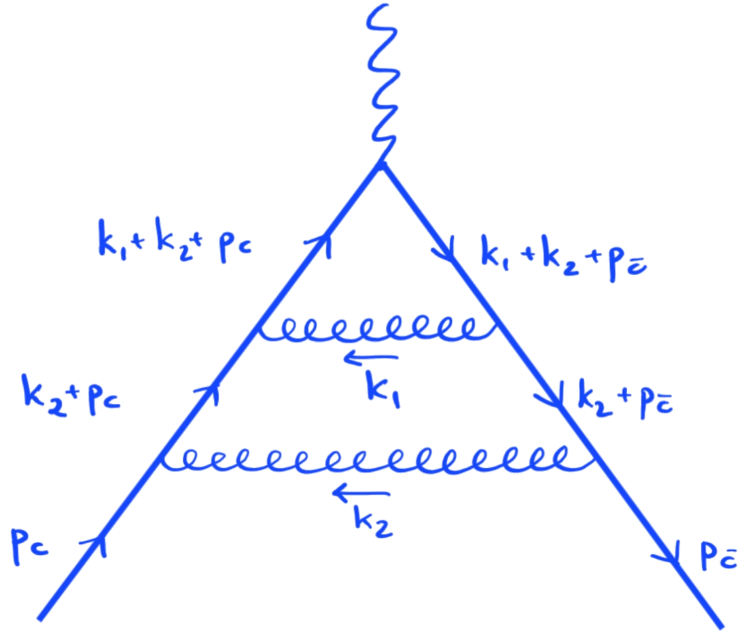
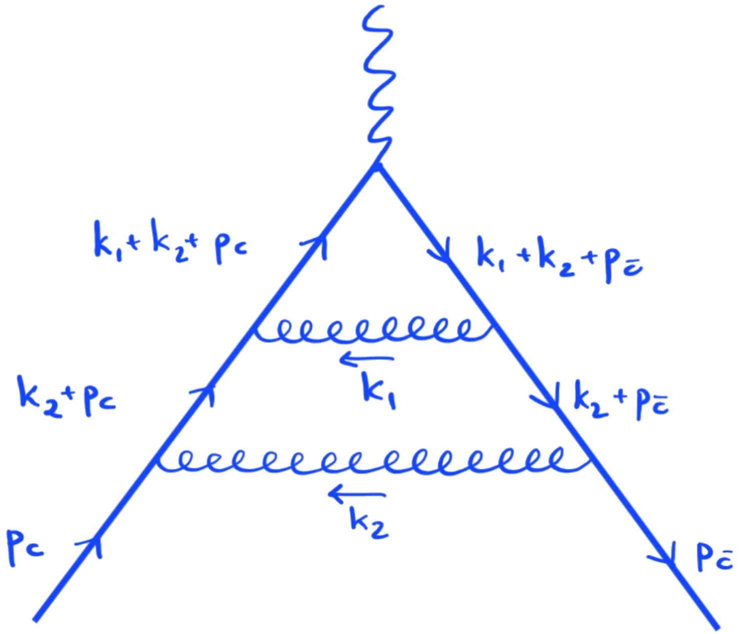
Abstract

Problems that arise in the application of general prescriptions of the so-called strategy of regions for asymptotic expansions of Feynman integrals in various limits of momenta and masses are discussed with the help of characteristic examples of two-loop diagrams. The strategy is also reformulated in the language of alpha parameters.





In addition to collinear, anti-collinear and soft, this diagram has the following regions

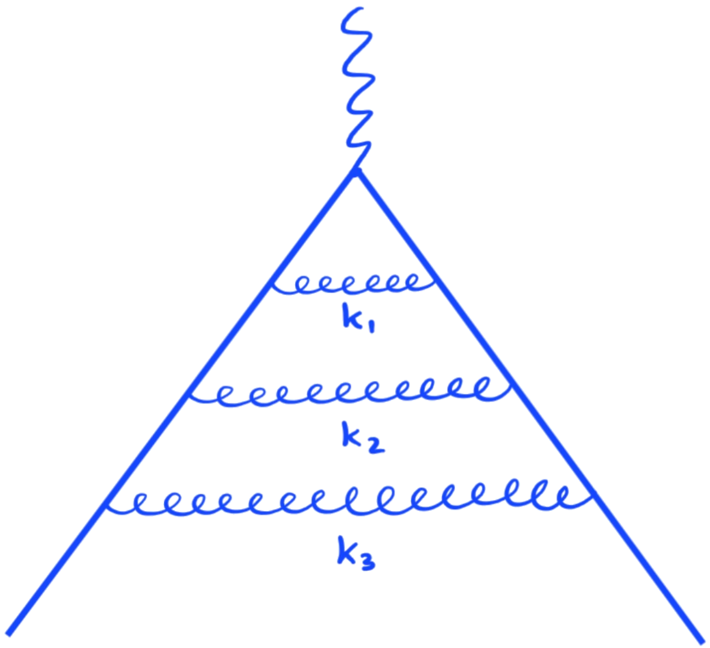
1. $k_1 \sim p_{\bar{c}}, \qquad k_2 \sim p_{uc}$
2. $k_1 \sim p_c, \qquad k_2 \sim p_{\overline{uc}}$

where

$$p_{uc} \sim Q(\lambda^2, \lambda^3, \lambda^4) \sim \lambda^2 p_c$$

$$p_{\overline{uc}} \sim Q(\lambda^4, \lambda^3, \lambda^2) \sim \lambda^2 p_{\bar{c}}$$

It gets worse…



$$1.\ k_1 \sim p_c, \quad k_2 \sim p_{\bar{c}_1}, \quad k_3 \sim p_{c_2}$$

$$2.\ \ldots$$

where

$$p_{c_j} \sim Q(1, \lambda, \lambda^2)\lambda^{2j}, \qquad p_{\bar{c}_j} \sim Q(\lambda^2, \lambda, 1)\lambda^{2j}$$

$$(uc \equiv c_1,\ \overline{uc} \equiv \bar{c}_1)$$

- In fact there is an infinite **cascade of regions** that contribute at increasing orders in the coupling [Ma, 2026] [Jaskiewicz, Jones, Szafron, Ulrich, 2025]

- The good news: The contribution from these regions seems to cancel in the sum over graphs, which was recently (2024) verified at two loops in QED:

Region analysis of QED massive fermion form factor

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ABSTRACT: We perform an analysis of the one- and two-loop massive quark form factor in QED in a region expansion, up to next-to-leading power in the quark mass. This yields an extensive set of regional integrals, categorized into three topologies, against which factorization theorems at next-to-leading power could be tested. Our analysis reveals a number of subtle aspects involving rapidity regulators, as well as additional regions that manifest themselves only beyond one loop, at the level of single diagrams, but which cancel in the form factor.

This work: Shows this to **all orders** using EFT techniques (in QED and QCD)

- c_j modes sit at the scale $\lambda^{1+2j} Q$ where $j \in \mathbb{N}$
- Construct an infinite tower of EFTs that describe physics at those scales
- What is the EFT for the ultra-collinear modes?

$$(p_c + p_{uc})^2 - m^2 \sim 2\, p_c \cdot p_{uc}, \quad (p_{\bar{c}} + p_{uc})^2 - m^2 \sim n_+ \cdot p_{uc}\, n_- \cdot p_{\bar{c}}$$

- ultra-collinear interacts with collinear similar to heavy quark expansion
  $\rightarrow$ **boosted heavy quark effective theory (bHQET)**
- ultra-collinear interacts with anti-collinear like an ultra-soft mode
  $\rightarrow$ can be factorized using decoupling transformation

- bHQET effective fields: $h_{uc}(x) = \sqrt{\frac{2}{n_+ v_-}}\, e^{imv_-\cdot x}\,\xi_c(x), \qquad v_\pm^\mu = \frac{Q}{m}\frac{n_\pm^\mu}{2} + \frac{m}{Q}\frac{n_\mp^\mu}{2}$

- Effective Lagrangian: $\mathcal{L}_{uc} = \bar{h}_{uc} i v_- \cdot D_{uc} \frac{n_+}{2} h_{uc}$

- Decoupling transformation

$$\xi_{\bar{c}}(x) = W_{uc}(x_+)\,\xi_{\bar{c}}^{(0)}(x), \qquad iD_{\bar{c}}^\mu(x) = W_{uc}(x_+)\,iD_{\bar{c}}^{(0)\mu}(x)\,W_{uc}^\dagger(x_+)$$

$$W_{uc}(x) = \mathcal{P}\exp\left(ig\int_{-\infty}^0 ds\, n_+ \cdot A_{uc}(x + sn_+)\right)$$

- Factorization:

$$\langle p_{\bar{c}}|\bar{\psi}\gamma^\mu\psi|p_c\rangle = C\langle p_{\bar{c}}|\bar{\xi}_{\bar{c}}^{(0)}W_{\bar{c}}W_{uc}^\dagger S_-^\dagger\gamma_\perp^\mu S_+ W_{uc}W_c^\dagger\xi_c^{(0)}|p_c\rangle + \mathcal{O}(\lambda)$$
$$= C\,Z\,S\;\langle p_{\bar{c}}|\bar{h}_{uc}W_{uc}|0\rangle\gamma_\perp^\mu\langle 0|W_{uc}^\dagger h_{uc}|p_c\rangle + \mathcal{O}(\lambda)$$

- From this viewpoint Z and S are also Wilson coefficients (similar to C). The arise in the matching of SCET onto bHQET.

The punchline: All loop corrections to the ultra-collinear matrix elements vanish (they are scaleless)

$$\langle 0|W_{uc}^\dagger h_{uc}|p_c\rangle = u(p_c)$$

- This can be straightforwardly generalized to the infinite cascade of modes

Theory	QCD	SCET_{II}	bHQET_1	...	bHQET_j	...
Scale	Q^2	m^2	m^4/Q^2	...	m^{2j+2}/Q^{2j}	...
Fields	ψ, A	$\xi_c, A_c, \xi_{\bar{c}}, A_{\bar{c}}, q_s, A_s$	$h_{c_1}, A_{c_1}, h_{\bar{c}_1}, A_{\bar{c}_1}$	...	$h_{c_j}, A_{c_j}, h_{\bar{c}_j}, A_{\bar{c}_j}$	...

- Conclusion: The original factorization theorem holds (or at least, the cascade of regions is absent)

$$F_1(Q^2, m^2) = C(Q^2)\, Z(m^2)\, S(m^2) \prod_{j=1}^{\infty} \underbrace{\mathfrak{Z}_j}_{=1} + \mathcal{O}(\lambda^2)\,.$$

- In some sense the ultra-collinear modes are the modes describing the IR divergences F_1^{IR} of QED/QCD when all fermions are massive

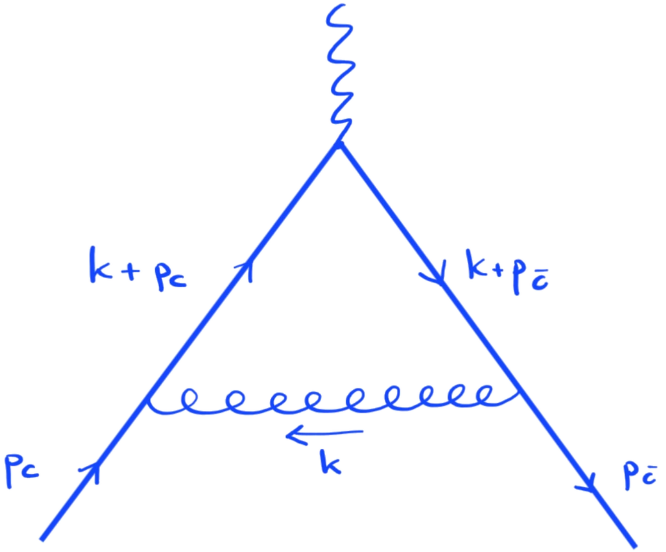
- The full IR structure is only revealed by introducing an IR cutoff, or equivalently, a momentum scale for the ultra-collinear modes

- This IR scale may be observable, e.g. an energy cutoff on radiation

- For now, focus on a particular (unphysical) implementation (most convenient for calculation) which is a gauge boson mass $m_g \ll m$. The gauge boson propagator Feynman rule gets modified as

$$\frac{-ig^{\mu\nu}}{k^2} \rightarrow \frac{-ig^{\mu\nu}}{k^2 - m_g^2}$$

- Then we find, already for the one-loop vertex graph, the following regions (in addition to the usual hard, (anti-)collinear and soft)



$$k \sim p_{uc} \sim \left(\frac{m_g Q}{m}, m_g, \frac{m m_g}{Q}\right) \sim \frac{m_g}{m} p_c$$

$$k \sim p_{\overline{uc}} \sim \left(\frac{m m_g}{Q}, m_g, \frac{m_g Q}{m}\right) \sim \frac{m_g}{m} p_{\bar{c}}$$

$$k \sim p_{us} \sim (m_g, m_g, m_g) \sim \frac{m_g}{m} p_s$$

- Factorization is achieved by the sequence of EFTs

$$\mathrm{QCD}(Q^2) \to \mathrm{SCET_{II}}(m^2) \to \mathrm{bHQET}(m_g^2)$$

$$F_1(m^2, m_g^2, Q^2) = C(Q^2)\, S(m^2)\, Z(m^2)\, \mathfrak{S}(m_g^2)\, \mathfrak{Z}(m_g^2) + \mathcal{O}\left(\frac{m^2}{Q^2}, \frac{m_g^2}{m^2}\right)$$

- New functions are:

$$\sqrt{\mathfrak{Z}}(m_g^2) = \langle 0 | W_{uc}^\dagger h_{uc} | p_c \rangle, \qquad \mathfrak{S}(m_g^2) = \frac{1}{N_c} \mathrm{tr} \langle 0 | \mathfrak{S}_-^\dagger(0) \mathfrak{S}_+(0) | 0 \rangle$$

- Let us consider now QED. $m_g = m_\gamma$ is the photon mass regulator. It has been known since ancient times that the IR divergences exponentiate (YFS exponentiation)

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The Infrared Divergence Phenomena and High-Energy Processes*

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$$M = \exp(\alpha B) \sum_{n=0}^{\infty} m_n$$

- Diagrammatic proof is rather complicated. Can we see this in an easier way using EFTs? **Yes!**

- The bHQET field h_{uc} can be decoupled from the photon field A_{uc}

$$h_{uc}(x) = Y_{v_-}(x)\, h_{uc}^{(0)}(x)$$

$$Y_{v_-}(x) = \exp\left(i\tilde{e}_0 \int_{-\infty}^{0} ds\, v_- A_{uc}(x + s v_-) \right)$$

$$\sqrt{\mathfrak{Z}}(m_g^2) = \langle 0 | W_{uc}^\dagger Y_{v_-} | 0 \rangle$$

- $h_{uc}^{(0)}$ is a sterile (non-interacting) field and the photon is not self-interacting $\rightarrow$ low energy EFT $bHQET(m_\gamma^2)$ is a free field theory!

- This implies that $\mathfrak{Z}$ and $\mathfrak{S}$ are *exactly* the exponentials of their one-loop value

$$\mathfrak{Z} = \exp\left\{\frac{\alpha(\mu)}{2\pi}\left[w^2(\nu)\frac{2}{\eta}e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\frac{\mu^2}{m_\gamma^2}\right)^\epsilon - \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\frac{m^2\nu^2}{Q^2m_\gamma^2} - \ln\frac{\mu^2}{m_\gamma^2} + 1\right) - \frac{1}{2}\ln^2\frac{\mu^2}{m_\gamma^2} + \ln\frac{\mu^2}{m_\gamma^2} + \ln\frac{\mu^2}{m_\gamma^2}\ln\frac{m^2\nu^2}{Q^2m_\gamma^2} + \frac{\pi^2}{12} + \mathcal{O}(\eta,\epsilon)\right]\right\}.$$

$$\mathfrak{S} = \exp\left\{\frac{\alpha(\mu)}{2\pi}\left[-w^2(\nu)\frac{2}{\eta}\left(\frac{\mu^2}{m_\gamma^2}\right)^\epsilon\Gamma(\epsilon)e^{\epsilon\gamma_E} + \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\frac{\mu^2}{m_\gamma^2} - \ln\frac{\nu^2}{m_\gamma^2}\right) + \frac{1}{2}\ln^2\frac{\mu^2}{m_\gamma^2} - \ln\frac{\mu^2}{m_\gamma^2}\ln\frac{\nu^2}{m_\gamma^2} - \frac{\pi^2}{12} + \mathcal{O}(\eta,\epsilon)\right]\right\}.$$

- YFS exponentiation of IR divergences in QED is an immediate corollary of the EFT approach

$$F_1^{\rm IR} = \mathfrak{Z}^R\left(m_\gamma^2, \mu, \frac{m\nu}{Qm_\gamma}\right)\mathfrak{S}^R\left(m_\gamma^2, \mu, \frac{\nu}{m_\gamma}\right) = \exp\left\{-\frac{\alpha(\mu)}{2\pi}\ln\frac{m_\gamma^2}{\mu^2}\left(1 + \ln\frac{m^2}{Q^2}\right)\right\}$$

- This also implies the exponentiation of the dim. reg. result

$$F_1^{{\rm IR}, m_\gamma=0} = \mathrm{Z}_C\mathrm{Z}_S\mathrm{Z}_Z = \frac{1}{\mathrm{Z}_{\mathfrak{S}}\mathrm{Z}_{\mathfrak{Z}}} = \exp\left\{\frac{\alpha(\mu)}{2\pi}\frac{1}{\epsilon}\left(\ln\frac{m^2}{Q^2} + 1\right)\right\}.$$

- In QCD, the exponentiation fails (already at 2 loops), since gluons are self-interacting
- Factorization only allows us to resum

$$F_1^{\mathrm{IR,LL}} = \left(\frac{\alpha(\mu)}{\alpha(m_g)}\right)^{-\frac{8\pi C_F}{\beta_0^2}\left(\frac{1}{\alpha(m)} - \frac{1}{\alpha(Q)}\right) - \frac{2C_F}{\beta_0}}$$

$$\simeq \exp\left\{-\frac{\alpha(\mu)C_F}{2\pi}\ln\frac{m_g^2}{\mu^2}\left(1 + \ln\frac{m^2}{Q^2}\right)\right\}$$

Summary of results

- **We have shown that an infinite tower of modes that give non-zero regions to individual Feynman graphs, drop out at the amplitude level**

- We have calculated new results relevant for massification using the η regulator.

1) Soft and collinear function for multiple hierarchical masses

2) Soft and collinear massification Z factors for external gluons

- The results enable a fully consistent way of resumming logarihms $ln\, m^2/Q^2$ at the amplitude level using the regular and rapidity RGE

- Using a gauge boson mass regulator we have rederived the exponentiation of IR divergences in QED using the EFT formalism

Outlook

- Next-to-leading power in m^2/Q^2 expansion (cancellation of ultra-collinear happens and YFS exponentiation holds to all powers in m^2/Q^2)

- Mass factorization for N-jet operator (rapidity regulator?)

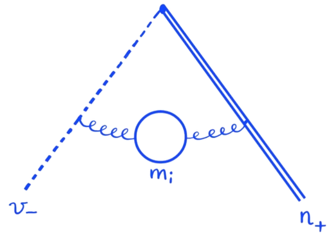
Back-up

- Generalization to hierarchy of masses $Q \gg M_1 \gg \cdots \gg M_{nh} \gg m \gg m_1 \gg \cdots \gg m_{n\ell} \gg 0$

$$F_1(Q^2, \{M_i\}, m, \{m_i\}) = C^{(1+n_h+n_\ell)}(Q^2) \left[\prod_{j=1}^{n_h} S^{(1+n_h+n_\ell-j)}(M_j^2)\, \widetilde{Z}^{(1+n_h+n_\ell-j)}(M_j^2) \right]$$
$$\times\, S^{(n_\ell)}(m^2)\, Z^{(n_\ell)}(m^2) \left[\prod_{i=1}^{n_\ell} S^{(n_\ell-i)}(m_i^2)\, \widetilde{\mathfrak{Z}}^{(n_\ell-i)}(m_i^2) \right]$$

- Ultra-collinear modes contribute, since massive light quark masses m_i provide a scale!

$$p_{uc,i} \sim \left(\frac{m_i Q}{m},\, m_i,\, \frac{m_i m}{Q} \right), \qquad \widetilde{\mathfrak{Z}}(m_i^2) = \langle 0 | W_{uc,i} h_{uc,i} | p_c \rangle$$



Gluon massification Z factor with the η regulator, extracted from [Wang, Xia, Yang, Ye, 2024]

$$
\begin{aligned}
Z_g(m^2) =& 1 + \frac{\alpha_0}{4\pi}\left(\frac{4\pi e^{-\gamma_E}}{m^2}\right)^{\epsilon} Z_g^{(1)} + \left(\frac{\alpha_0}{4\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}}{m^2}\right)^{2\epsilon} w^2(\nu)\left(\frac{\nu^2}{Q^2}\right)^{\frac{\eta}{2}} Z_g^{(2)} + \mathcal{O}(\alpha^3), \\
Z_g^{(1)} =& T_F\left(-\frac{4}{3}\right)\Gamma(\epsilon)e^{\epsilon\gamma_E}, \\
Z_g^{(2)} =& T_F C_A \frac{8}{3\eta}(\Gamma(\epsilon)e^{\epsilon\gamma_E})^2 \frac{1+\epsilon}{1+\frac{8}{3}\epsilon+\frac{4}{3}\epsilon^2} + T_F C_A\left(-\frac{8}{3\epsilon}+\frac{10}{9}-8\zeta_3\right) \\
& + T_F C_F\left(-\frac{2}{\epsilon}-15\right) + T_F^2\left(\frac{16}{9\epsilon^2}+\frac{8\pi^2}{27}\right) + O(\eta,\epsilon).
\end{aligned}
$$