All-orders Evolution of Parton Distributions

J. Rodríguez-Quintero

in collaboration with ***C. D. Roberts, Zhen-Ni Xu, Z-Q Yao ...***

QCD Evolution 2026

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Introduction: QCD running coupling

The field-theory lagrangians invariant under U(1) (**QED**) and SU(3) gauge transformations (**QCD**)

$$\mathcal{L}_{QCD} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \partial_\mu \bar{c}_a \partial^\mu c_a + g\, f_{abc} A_\mu^a (\partial^\mu \bar{c}_b) c_c + \bar{\psi}\,(i\gamma_\mu D_\mu - m)\,\psi$$

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Superfluous gauge degrees of freedom can be suppressed by constraining the lagrangian, I.e. by introducing a **gauge-fixing** term. This implies a new vertex in **QCD**, owing to its **non-abelian** character, while no ghost couples to photon field in **QED**.

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The quantization of the theory plagues the theory with divergencies that should be subsequently absorbed when relating the lagrangian and physical quantities (**renormalization**)

$$\begin{aligned} A_R &= Z_A^{-1/2} A_0 \\ \psi_R &= Z_\psi^{-1/2} \psi_0 \\ \Gamma_R &= Z_\Gamma \Gamma_0 \longleftarrow (\times 1) \\ e_R &= Z_e^{-1} e_0 \end{aligned}$$

$$\begin{aligned} (\times 4) \longrightarrow \quad A_R &= Z_A^{-1/2} A_0 \\ \psi_R &= Z_\psi^{-1/2} \psi_0 \\ \Gamma_R &= Z_\Gamma \Gamma_0 \\ c_R &= Z_c^{-1/2} c_0 \\ g_R &= Z_g^{-1} g_0 \end{aligned}$$

$$Z_e = Z_e(Z_\Gamma, Z_\psi, Z_A)$$

$$Z_e = \frac{Z_\Gamma}{Z_\psi^1 Z_A^{1/2}}$$

$$Z_g = Z_g(Z_\Gamma, Z_\psi, Z_A, Z_c)$$

$$Z_g = \frac{Z_{\Gamma_{3g}}}{Z_A^{3/2}} = \frac{Z_{\Gamma_{4g}}^{1/2}}{Z_A} = \frac{Z_{\Gamma_{gcc}}}{Z_c Z_A^{1/2}} = \frac{Z_{\Gamma_{qgg}}}{Z_\psi Z_A^{1/2}}$$

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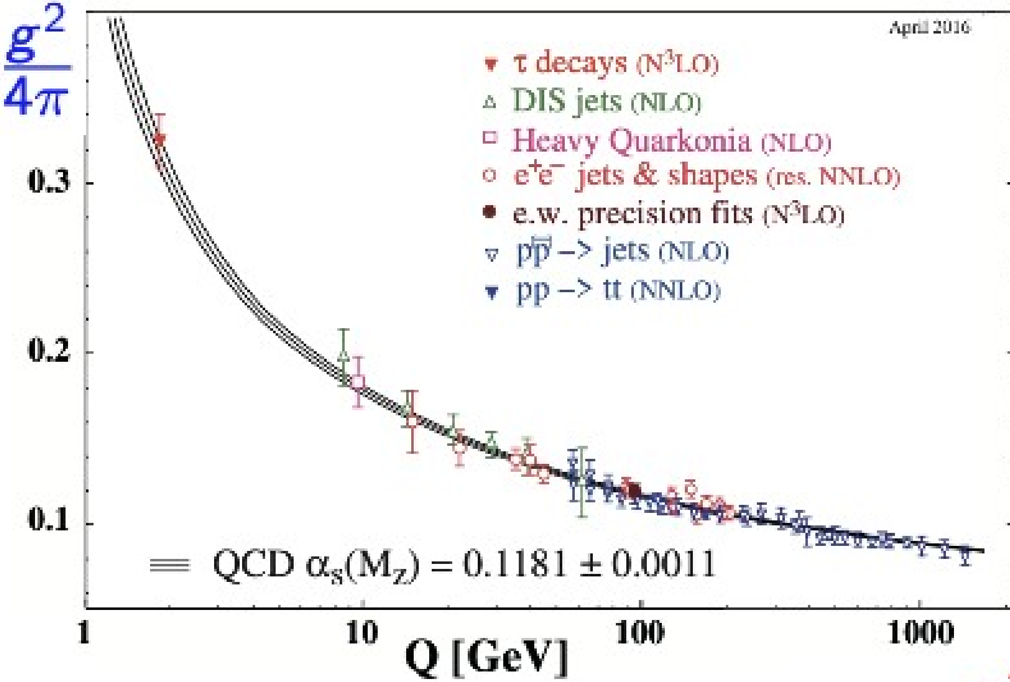
Ward replaced by **Slavnov-Taylor** identities in **QCD**, preventing from simple, direct connections of vertices and propagators, thus requiring the choice of a vertex (and its kinematics) to fully define a **renormalized running coupling**.

Introduction: QCD running coupling

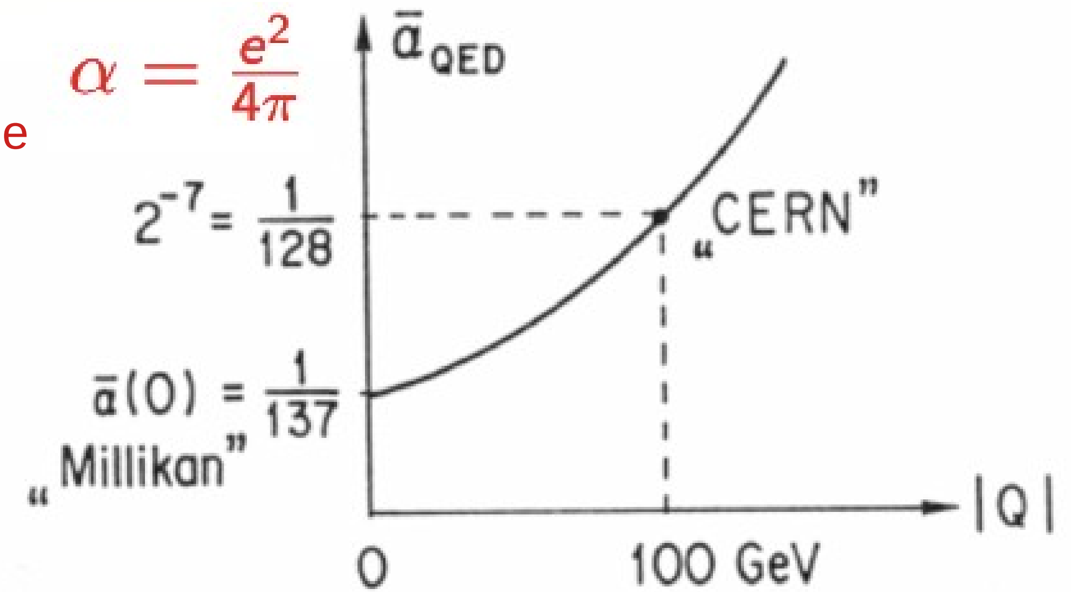
$$\alpha_s = \frac{g^2}{4\pi}$$



In QCD, the charge diminishes with the probing energy (anti-screening), due to its **non-abelian** character.
The theory is asymptotically free.

While, in QED, an abelian theory, the charge grows with the probing energy (screening).

$$\alpha = \frac{e^2}{4\pi}$$



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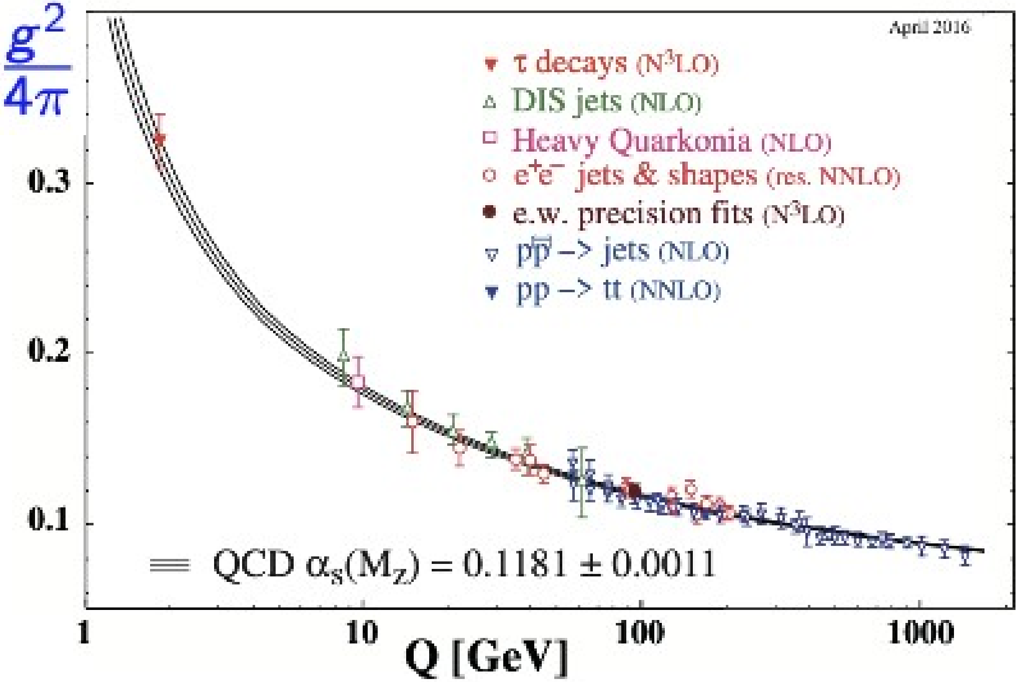

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$$\frac{d}{dt}\alpha(t) = -\frac{\alpha^2(t)}{4\pi}\beta_0 + \ldots$$

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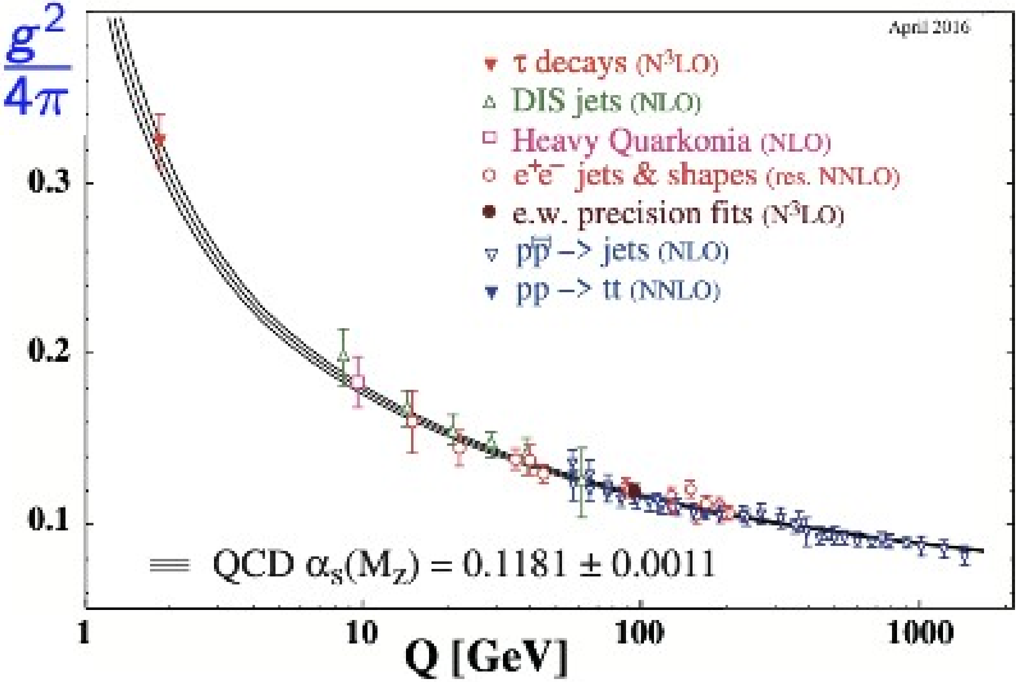

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$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \dots = \frac{4\pi c}{\beta_0}$$

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \dots\right)$$

$$\alpha(t) = \frac{4\pi}{\beta_0 t} + \dots$$

$$t = \ln\left(\frac{\zeta^2}{\Lambda^2}\right)$$

QCD fundamental scale

The perturbative running of couplings in different schemes is driven by their expansion one in terms of each other, but the first coefficient of the expansion defines by itself how their QCD fundamental scales relate.

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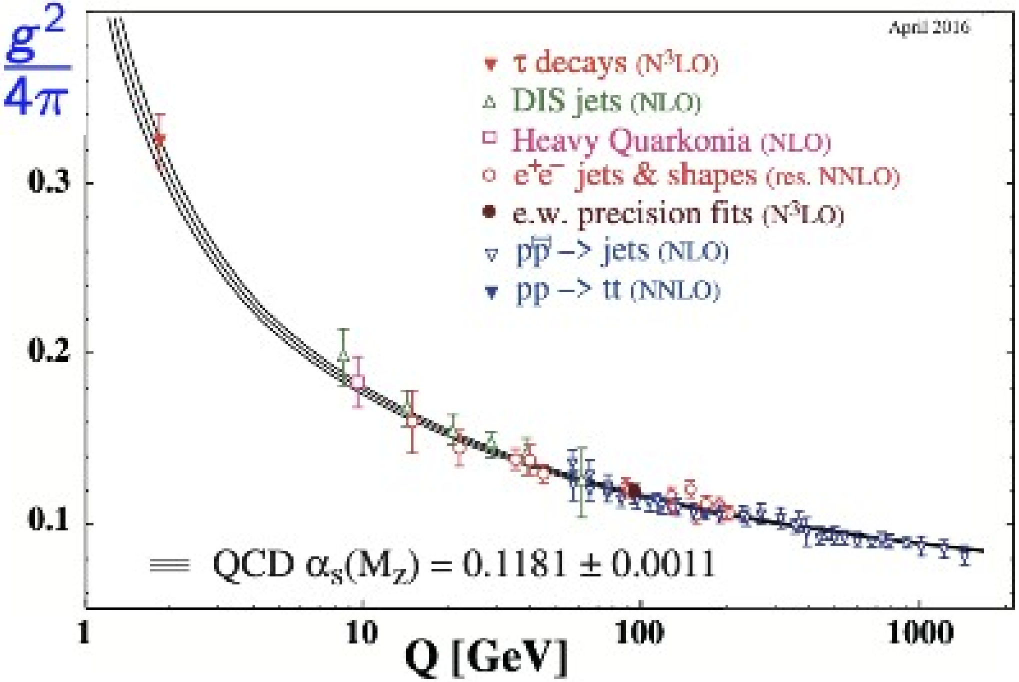

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The perturbative running of couplings in different schemes is driven by their expansion one in terms of each other, but the first coefficient of the expansion defines by itself how their QCD fundamental scales relate. A new different coupling can be perturbatively defined by imposing an exact expansion up to this coefficient.

Effective charges: generalities

A perturbative scheme does not eliminate the **Landau pole**, and cannot hence cure the *fake infrared problem,* but **Grunberg's** notion of **effective charge** extended to the non-perturbative domain does.

G. Grunberg, Phys.Lett.B95, 70 (1980); Phys.Rev.D29, 2315 (1984)

Let's illustrate it with the one based on the **Bjorken sum rule**

J. D. Bjorken, Phys.Rev.D1, 1376(1970)

$$\Gamma_1^{p-n}(Q^2) := \int_0^1 dx\,\left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] \overset{\Lambda_s^2/Q^2\simeq 0}{=} \frac{g_A}{6}$$

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$$\Gamma_1^{\mathrm{p-n}}(Q^2) = \frac{g_{\mathrm{A}}}{6}\Bigg[1 - \frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi} - 3.58\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^2 - 20.21\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^3 - 175.7\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^4 + \sim -893.38\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^5 + \mathcal{O}\left(\left(\alpha_{\mathrm{s}}^{\mathrm{pQCD}}\right)^6\right)\Bigg] + \sum_{n>1}\frac{\mu_{2n}(Q^2)}{Q^{2n-2}}.$$

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$$\Gamma_1^{p-n}(Q^2) =: \frac{g_A}{6}\left(1 - \frac{\alpha_{g_1}(Q^2)}{\pi}\right)$$

$\alpha_{g_1}(Q^2)$ is an effective charge that naturally extends to cover the non-perturbative domain, incorporating both short-distance and long-distance effects, and can be interpreted as a running coupling under a non-perturbative renormalization prescription.

A. Deur et al., Prog.Part.Nucl.Phys. 134(2024)

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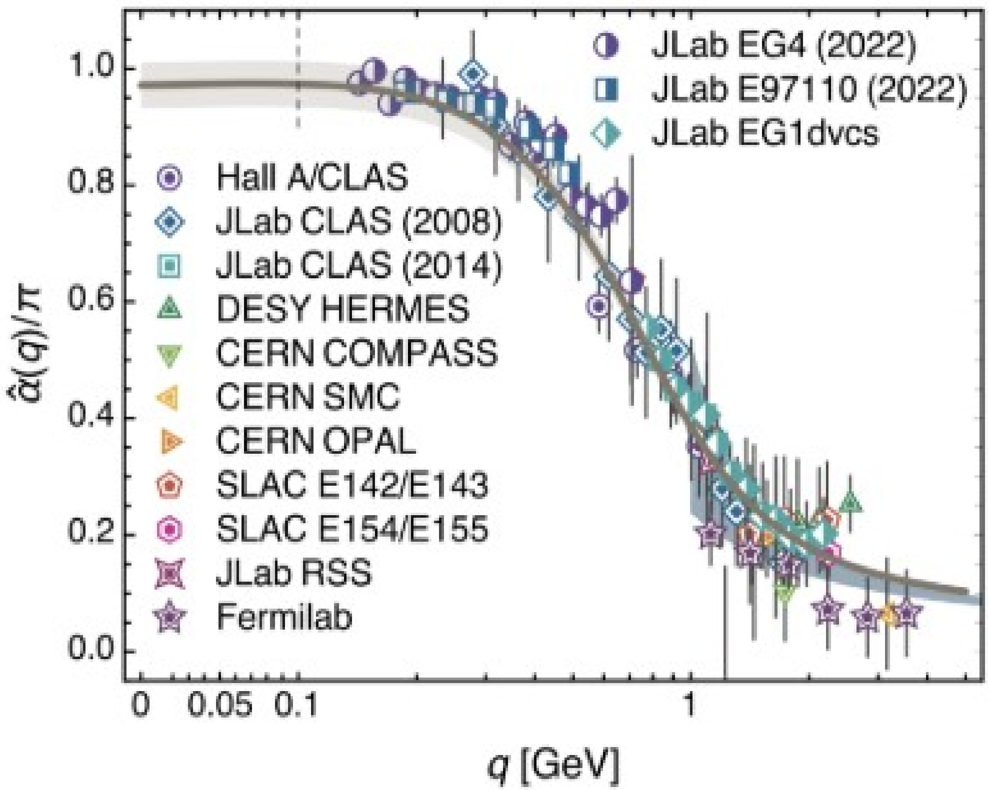

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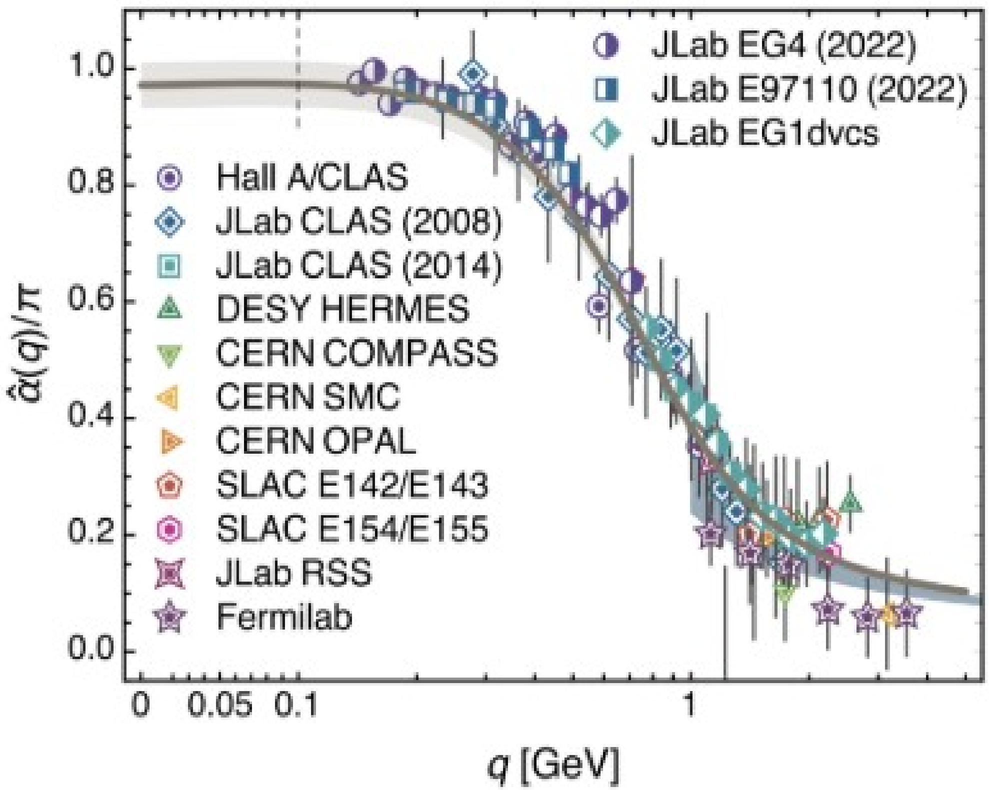

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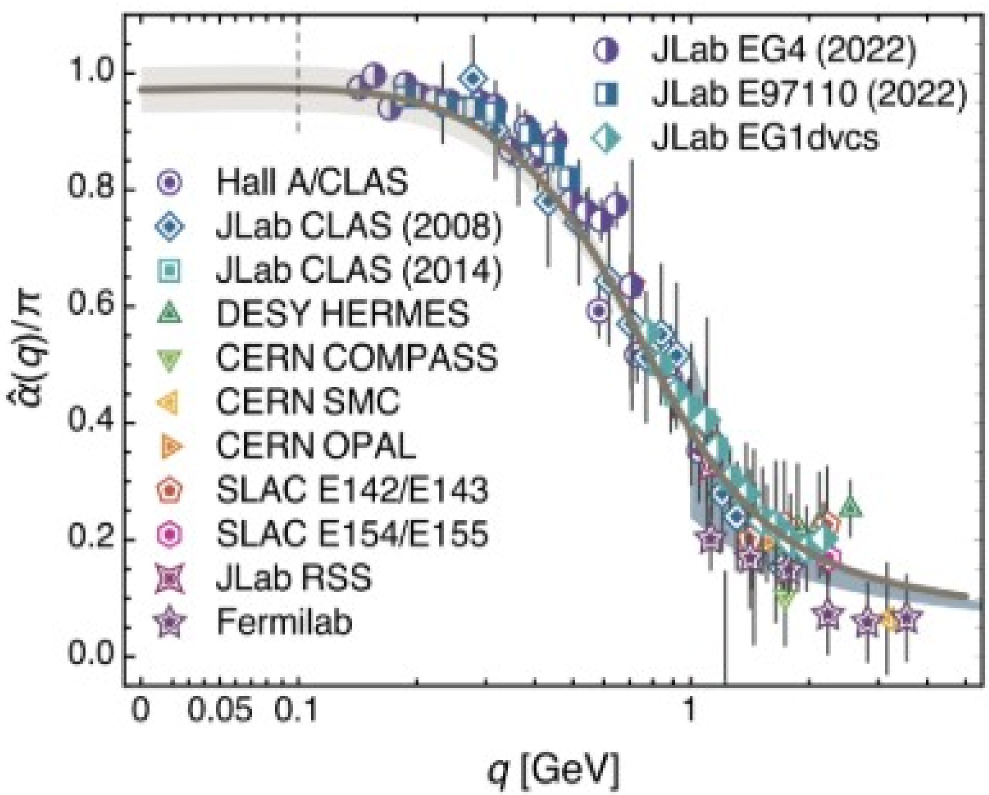
It connects to other infrared non-perturbative calculations of the strong running coupling, as the one derived in the context of the **HLFQCD**.



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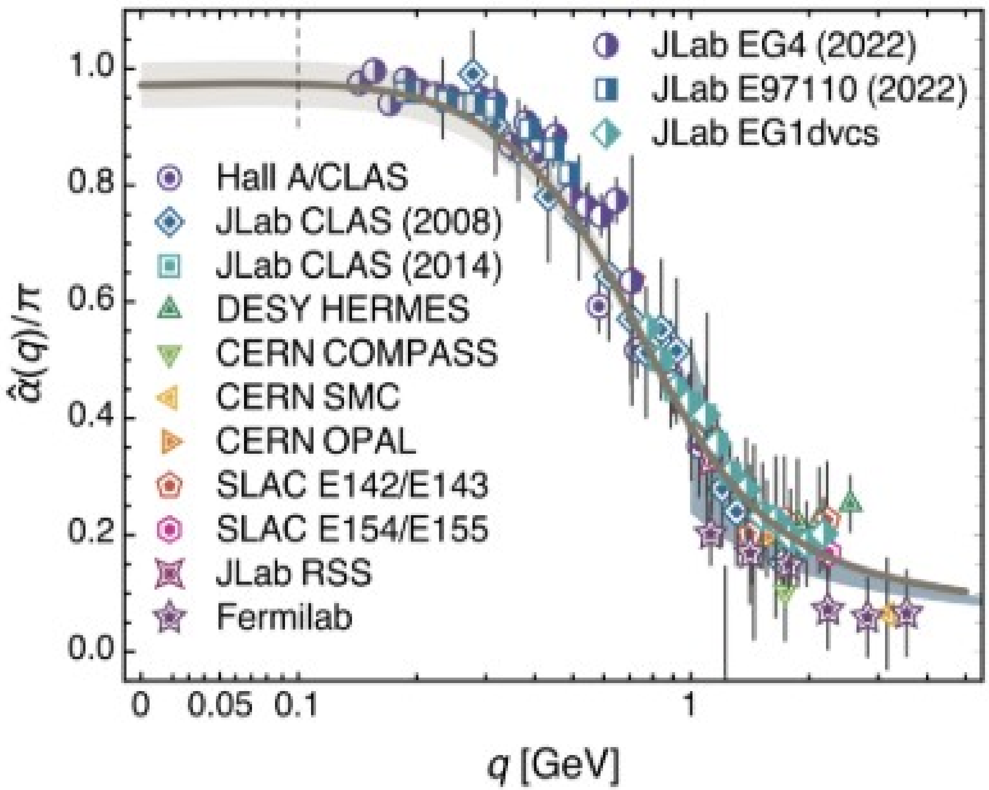

A **running interaction**, defined for the gap equation **kernel** within the ("abelianized") **PT-BFM** scheme for DSEs, results from the (massive) gluon vacuum polarization;

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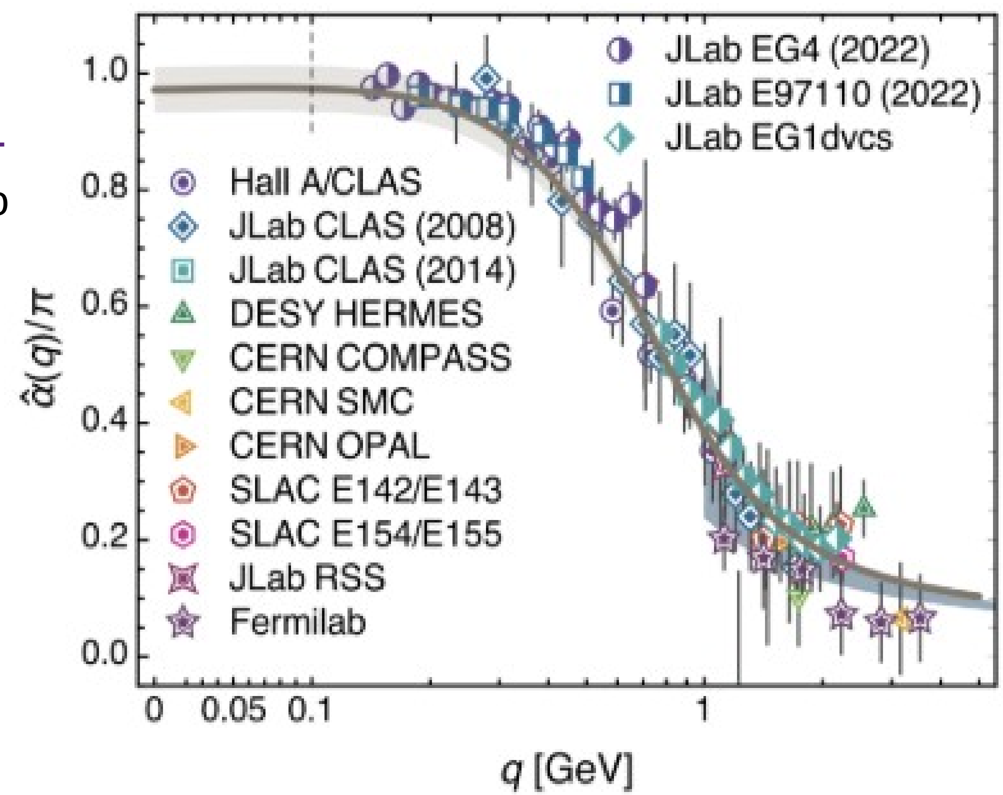

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$$\hat{d}(k^2) = \hat{\alpha}(k^2)\mathcal{D}(k^2)$$

A **running interaction**, defined for the gap equation **kernel** within the ("abelianized") **PT-BFM** scheme for DSEs, results from the (massive) gluon vacuum polarization; expressed in terms of **gluon** and **ghost** 2-point Schwinger functions. The charge is obtained by factorizing a RGI function, derived from the gluon propagator, which expresses its massive nature in the IR and is proportional to k^2 in the UV.

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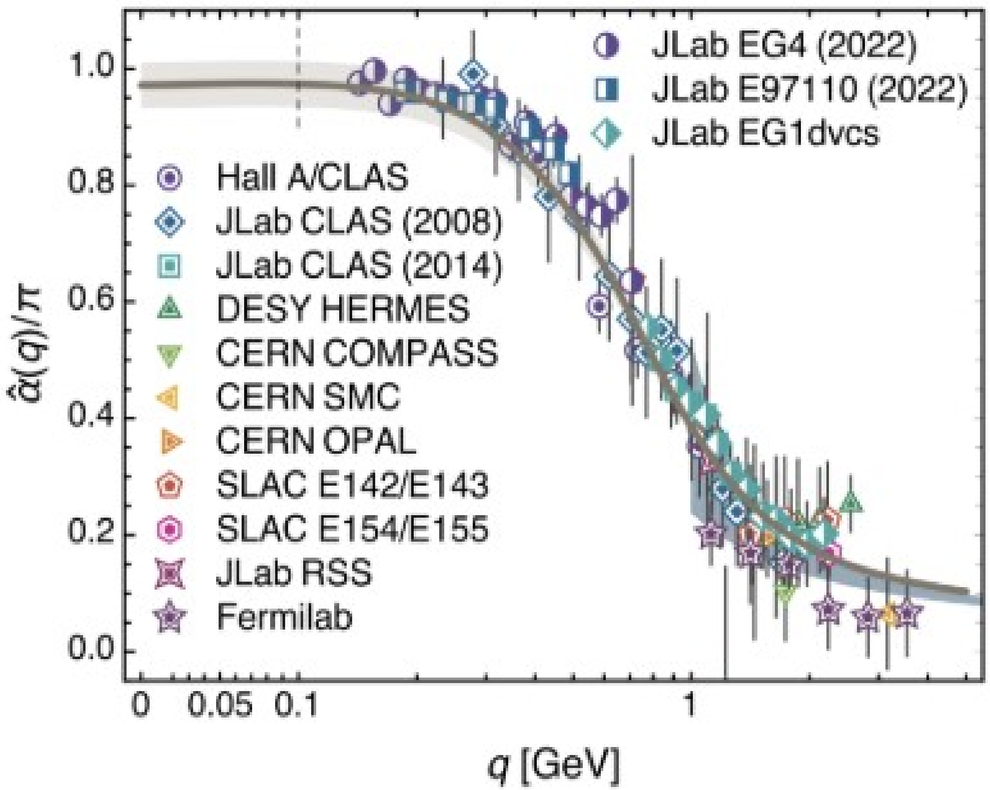

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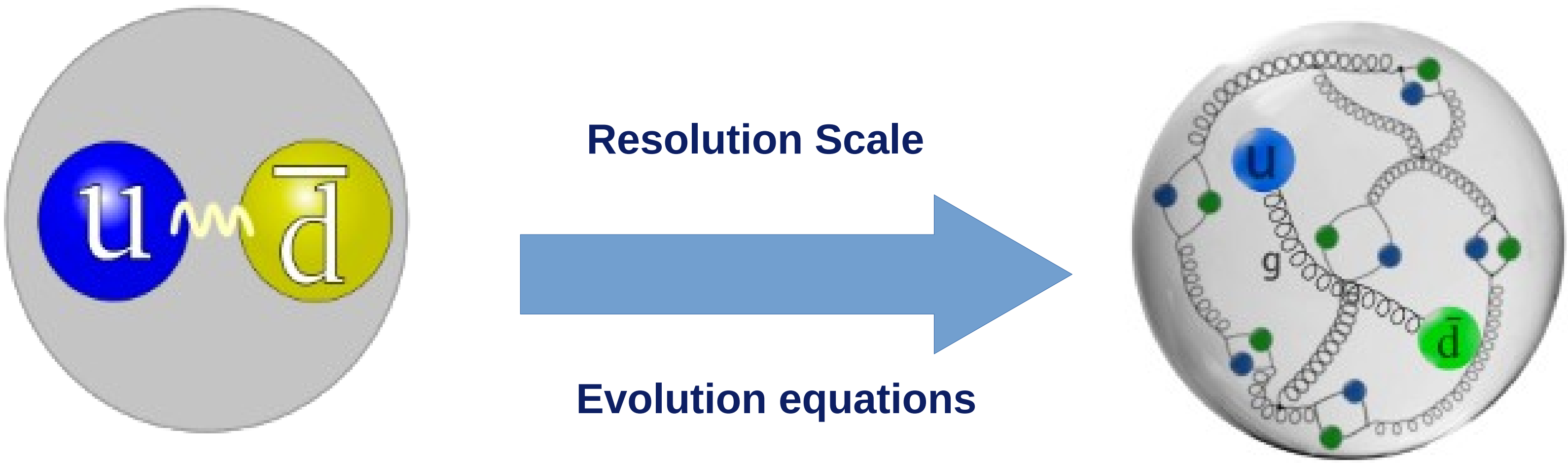
Parton distributions: energy scales

ζ_H $\qquad\qquad\qquad\qquad$ $\zeta > \zeta_H$



- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here (e.g. large-x behavior of the PDF)

$$u^{\pi}(x;\zeta) \overset{x\simeq 1}{\approx} (1-x)^{\beta = 2+\gamma(\zeta)}$$

- **Unveiling of glue and sea d.o.f.**
  - **Experimental** data is given **here**.
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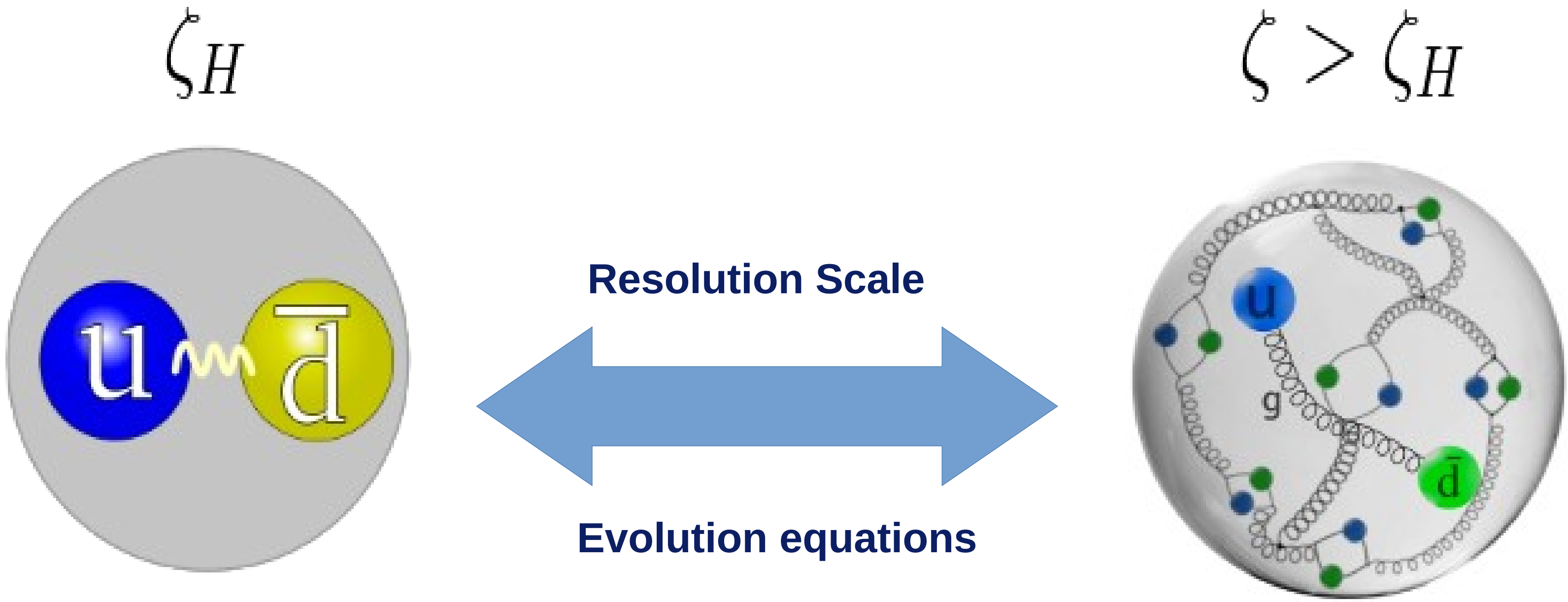

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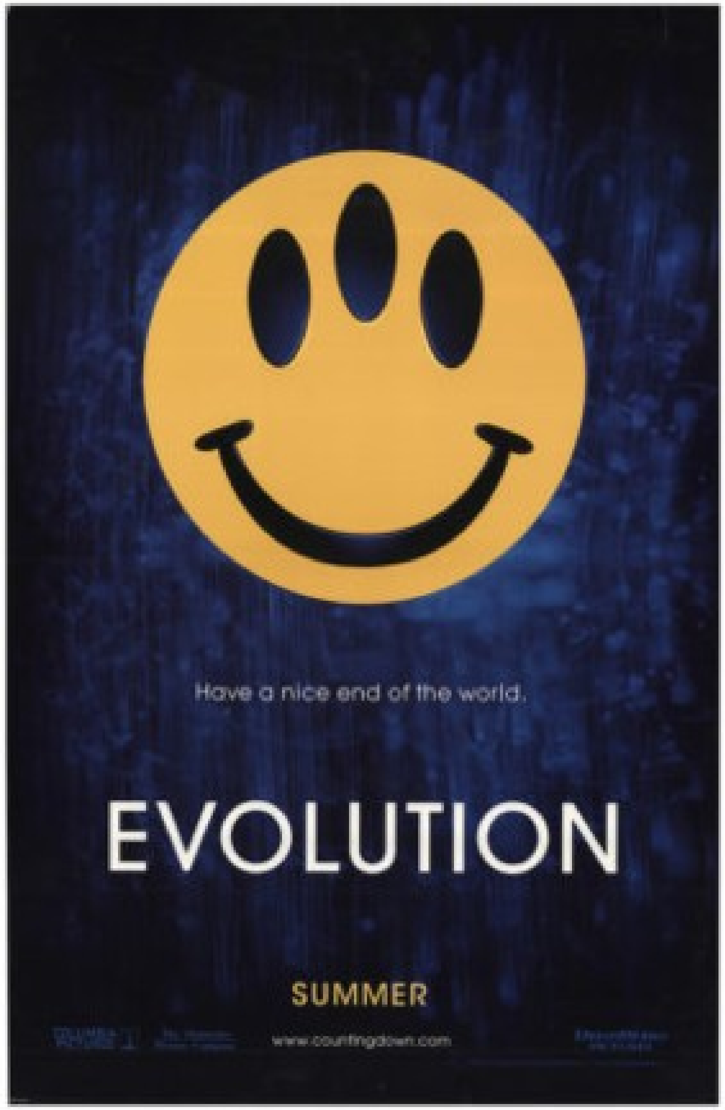
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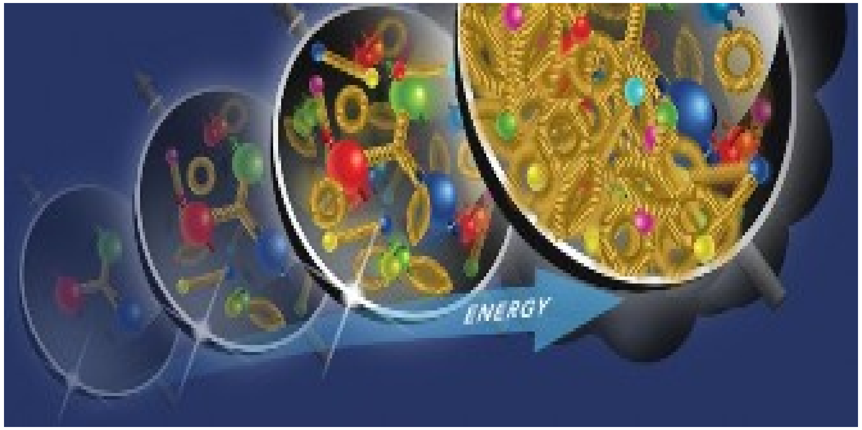
DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)
K. Raya et al. Chin.Phys.C46 013105(2022)

$$\left\{\zeta^2\frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x)\;-\;\frac{\alpha(\zeta^2)}{4\pi}\int_x^1\frac{dy}{y}\begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\rm NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta)\end{pmatrix}\;=\;0$$

DGLAP leading-order evolution equations



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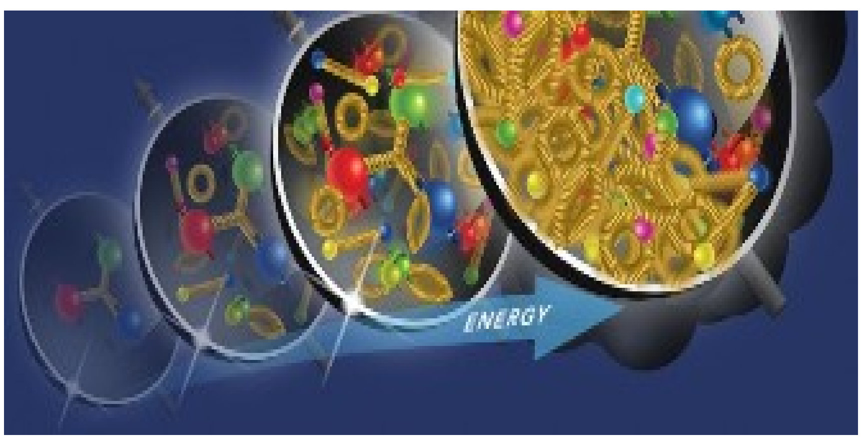
Assumption: define an **effective** charge such that

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)
K. Raya et al. Chin.Phys.C46 013105(2022)

Starting from fully-dressed **quasiparticles**, at ζ_H → **Sea** and **Gluon** content unveils, as prescribed by **QCD**

$$\left\{\zeta^2\frac{d}{d\zeta^2}\int_0^1 dy\,\delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi}\int_x^1\frac{dy}{y}\begin{pmatrix} P_{qq}^{NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{S}(y,t;\zeta)\end{pmatrix} = 0$$

DGLAP ~~leading order~~ evolution equations



- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

DGLAP: All orders evolution

Assumption: define an **effective** charge such that

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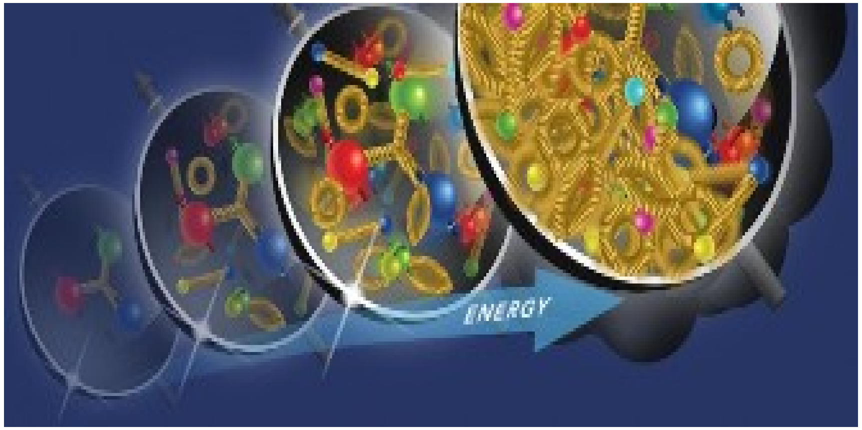
$$\left\{\zeta^2 \frac{d}{d\zeta^2}\mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi}\begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f\gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix}\right\}\begin{pmatrix} \langle x^n\rangle_{\rm NS}(\zeta) \\ \langle x^n\rangle_{\rm S}(\zeta) \\ \langle x^n\rangle_g(\zeta) \end{pmatrix} = 0$$

DGLAP ~~leading-order~~ evolution equations

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$



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DGLAP: All orders evolution

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PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\text{H}}(x) = q_{\text{H}}(x) + \bar{q}_{\text{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\text{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\text{H}}(y)$$

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Valence-quark PDF in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

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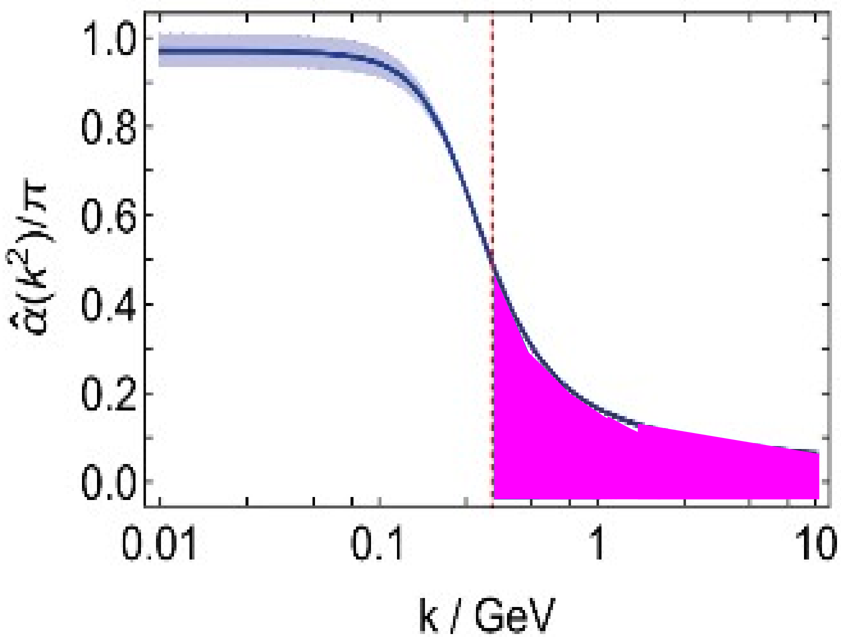
Moments' evolution is controlled by the integrated "strength" of the coupling beyond the hadron scale



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$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

The ratio of lightcone momentum fractions encodes the required information of the charge

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

Application 1: valence-quark PDF

$$\langle x^n \rangle_{q_H}^{\zeta} = \langle x^n \rangle_{q_H}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right) = \langle x^n \rangle_{q_H}^{\zeta_H} \left[\frac{\langle x \rangle_{q_H}^{\zeta}}{\langle x \rangle_{q_H}^{\zeta_H}} \right]^{\gamma_{qq}^n / \gamma_{qq}}$$

This ratio encodes the information of the charge

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve “all” the Mellin moments up and reconstruct the PDF.

DGLAP: All orders evolution

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This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x \rangle^{\zeta_H}_{u_\pi} = \langle x \rangle^{\zeta_H}_{d_{\bar{\pi}}} = \frac{1}{2}$$

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

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Capitalizing on the Mellin moments of asymptotically large order:

$$q(x;\zeta) \underset{x \to 1}{\sim} (1-x)^{\beta(\zeta)} \left(1 + \mathcal{O}(1-x)\right)$$

$$\beta(\zeta) = \beta(\zeta_H) + \frac{3}{2} \ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle}$$

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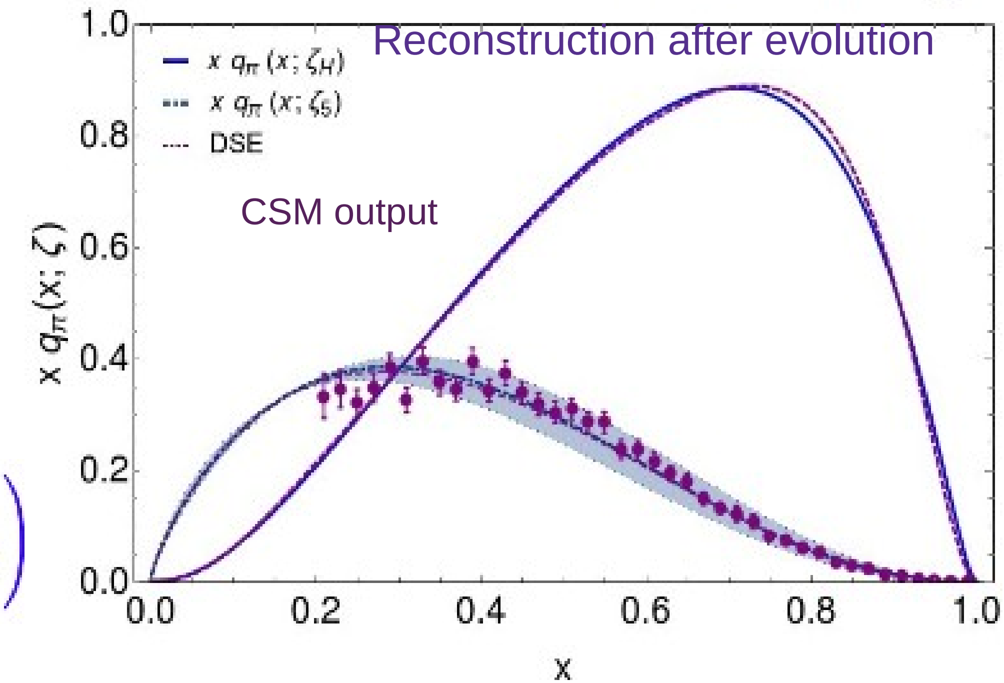

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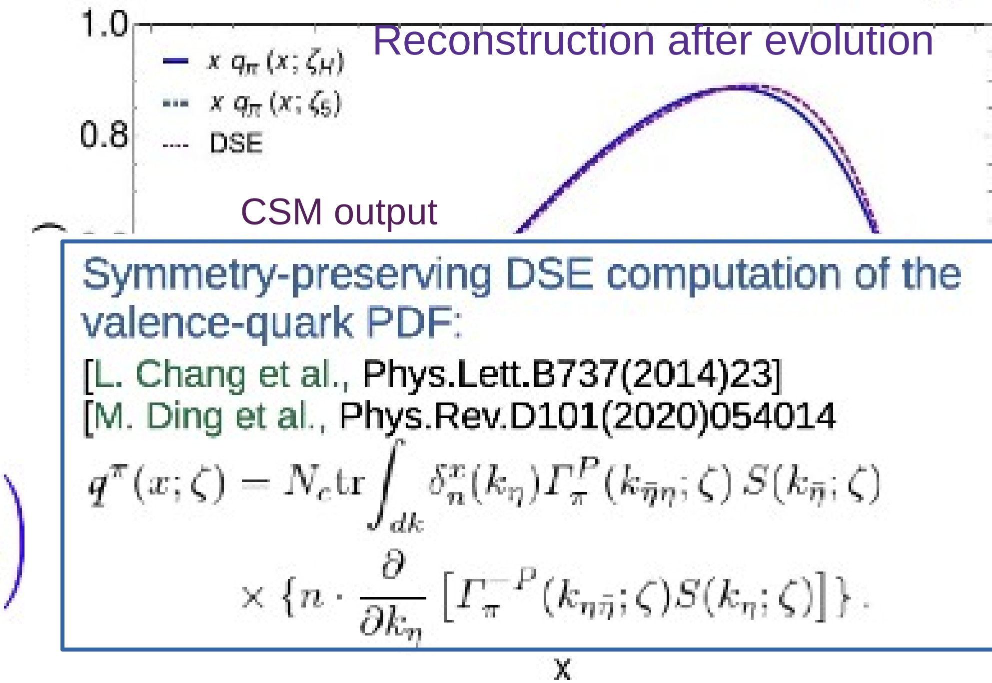

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014

$$q^\pi(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta^x_n(k_\eta) \Gamma^P_\pi(k_{\bar\eta\eta};\zeta)\, S(k_{\bar\eta};\zeta) \times \{ n \cdot \frac{\partial}{\partial k_\eta} [\Gamma^{-P}_\pi(k_{\eta\bar\eta};\zeta) S(k_\eta;\zeta)] \} .$$

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

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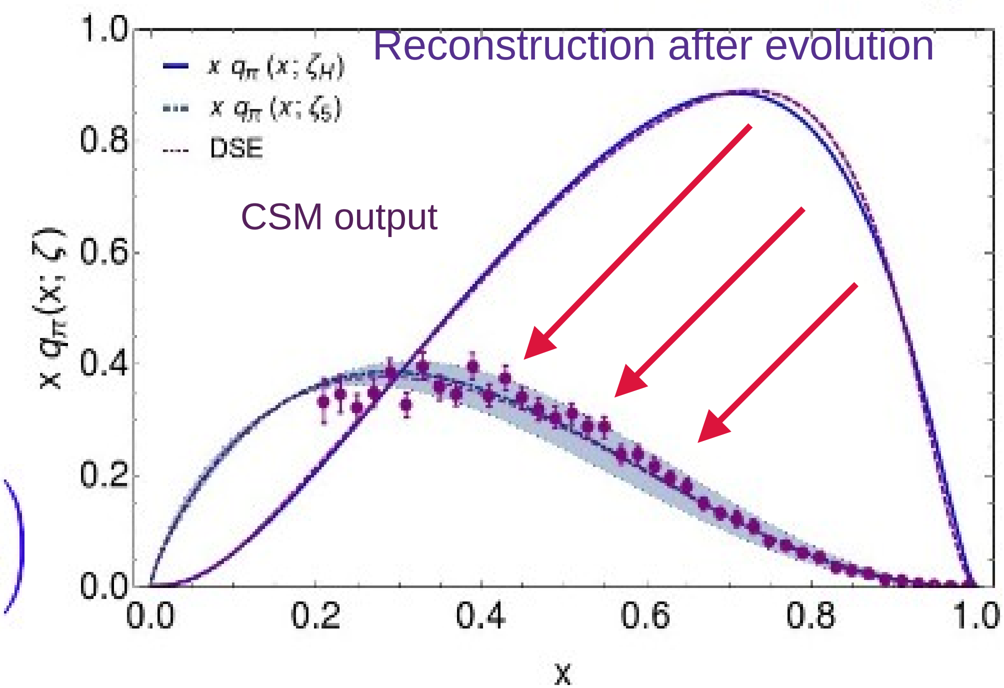

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

Application 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

DGLAP: All orders evolution

Z-F. Cui et al., Phys.Rev.D105 L091502 (2022)

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Reported lattice moments agree very well with the recursion formula

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

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Moments from global fits can be also compared to the estimated from recursion !

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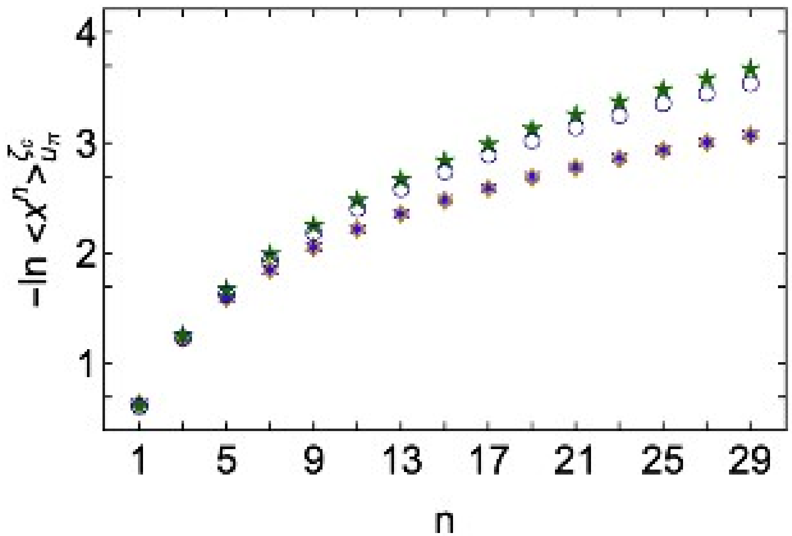
Moments computed from: P. Barry et al., PRL127(2021)232001



DGLAP: All orders evolution

Z-F. Cui et al., Phys.Rev.D105 L091502 (2022)

Application 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad q(x;\zeta_H) = 1$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

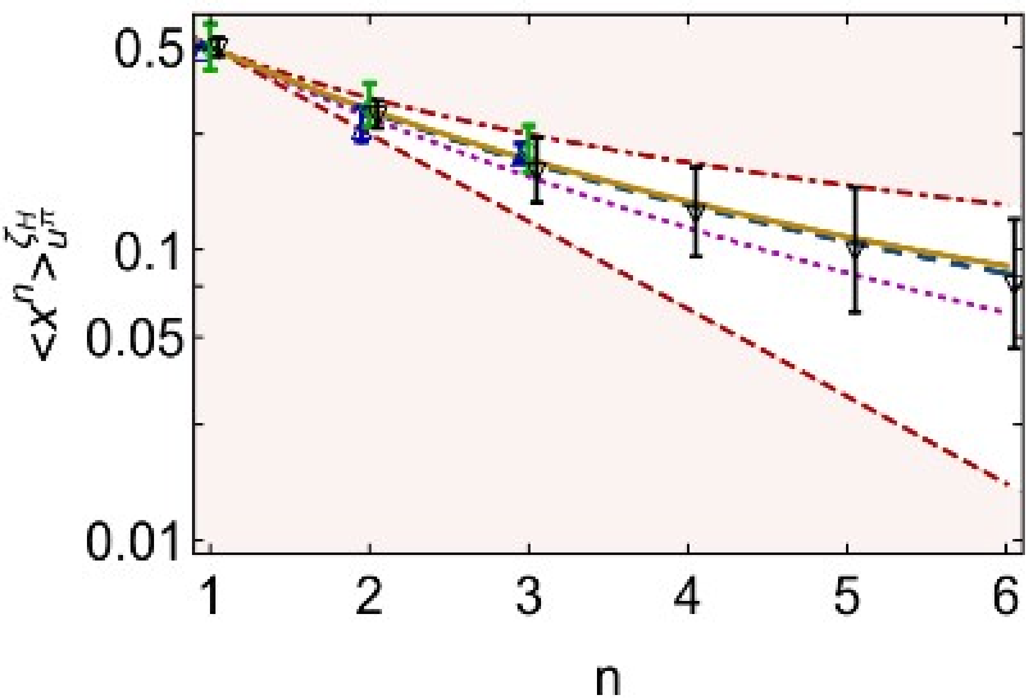
DGLAP: All orders evolution

Z-F. Cui et al., Phys.Rev.D105 L091502 (2022)

Application 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad q(x;\zeta_H) = 1$$



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n	Joo:2019bzr [61]	Sufian:2019bol [62]	Alexandrou:2021mmi [63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.

DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathsf{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_{q} = \theta(\zeta - M_q)$$

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Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + 2 n_f \mathcal{P}^{\zeta}_q \gamma^n_{ug} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

Full singlet and sea

$$\langle x^n \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x^n \rangle^{\zeta}_{\Sigma^q_H} \,, \quad \langle x^n \rangle^{\zeta}_{S_H} = \sum_q \langle x^n \rangle^{\zeta}_{S^q_H}$$

DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

Application 4: glue and sea from valence

$$M_q = \zeta_H, \; \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

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$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g}\left(S^n_- - S^n_+\right) \\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2}\mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_\pm/\gamma_{uu}}$$

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$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

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In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm \frac{\lambda^n_\mp - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

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DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

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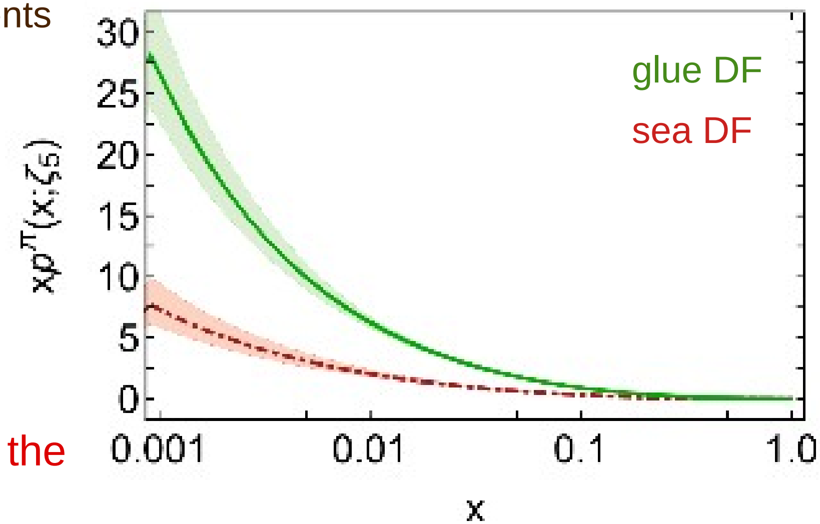
Compute all the moments and reconstruct:

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}} = \left[\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$



★ The only required input is the the momentum fraction at the probed empirical scale!!

DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

Application 4: glue and sea from valence

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n=1 case
$n_f = 4$

$$\langle x \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x \rangle^{\zeta}_{q_H} + \langle x \rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7}\left[S(\zeta_H, \zeta)\right]^{7/4}$$

$$\langle x \rangle^{\zeta}_{g_H} = \frac{4}{7}\left(1 - \left[S(\zeta_H, \zeta)\right]^{7/4}\right)$$

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DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

Application 4: glue and sea from valence

$$M_q = \zeta_H, \ \forall q$$

All quarks active

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$$\langle x \rangle^{\zeta}_{\Sigma_\pi} = \langle 2x \rangle^{\zeta}_{q_\pi} + \langle x \rangle^{\zeta}_{S_\pi} = \frac{3}{7} + \frac{4}{7} \left[S(\zeta_H, \zeta) \right]^{7/4}$$

$$\langle x \rangle^{\zeta}_{g_\pi} = \frac{4}{7} \left(1 - \left[S(\zeta_H, \zeta) \right]^{7/4} \right)$$

ζ_5	$\langle 2x \rangle^\pi_q$	$\langle x \rangle^\pi_g$	$\langle x \rangle^\pi_{\text{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

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Z-F. Cui et al., arXiv:2006.1465

R.S. Sufian et al., arXiv:2001.04960

DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

Application 4: glue and sea from valence

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All quarks active

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All n
$n_f = 4$

$$\Sigma_H(x) \underset{\zeta^2 \to \infty}{=} \frac{3}{7} \frac{\delta(x)}{x}$$

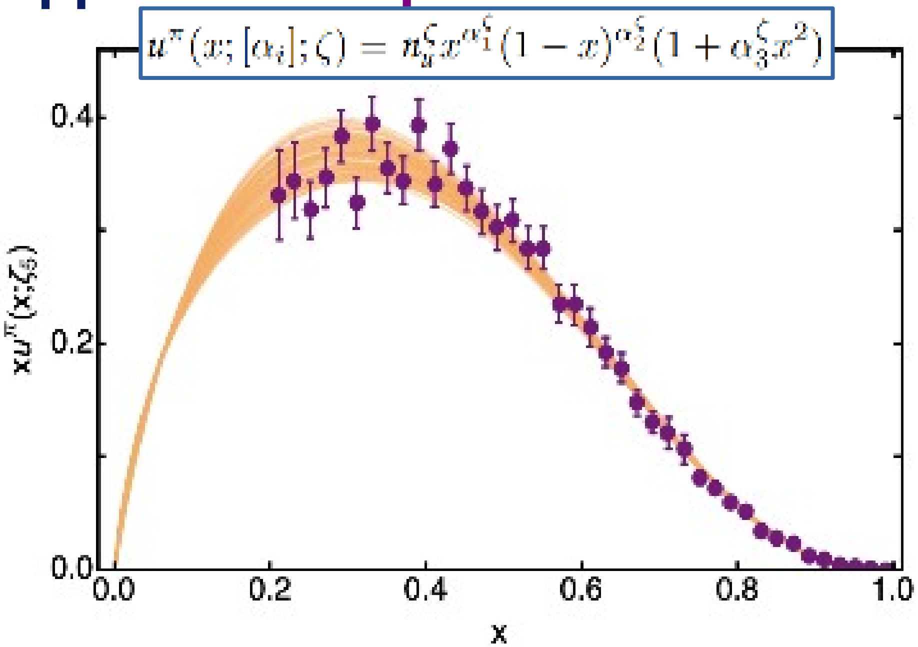
$$g_H(x) \underset{\zeta^2 \to \infty}{=} \frac{4}{7} \frac{\delta(x)}{x}$$

★ The only required input is the the momentum fraction at the probed empirical scale!!

G. Altarelli, Phys. Rep. 81, 1 (1982)

All-orders DGLAP: reverse engineering

Application 5: pion and kaon PDFs fully reconstructed from data



Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

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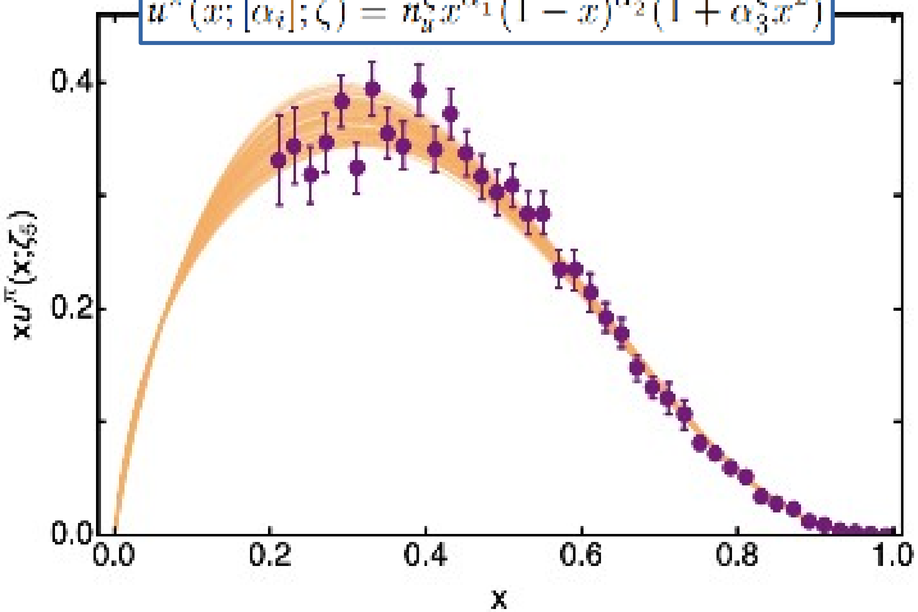
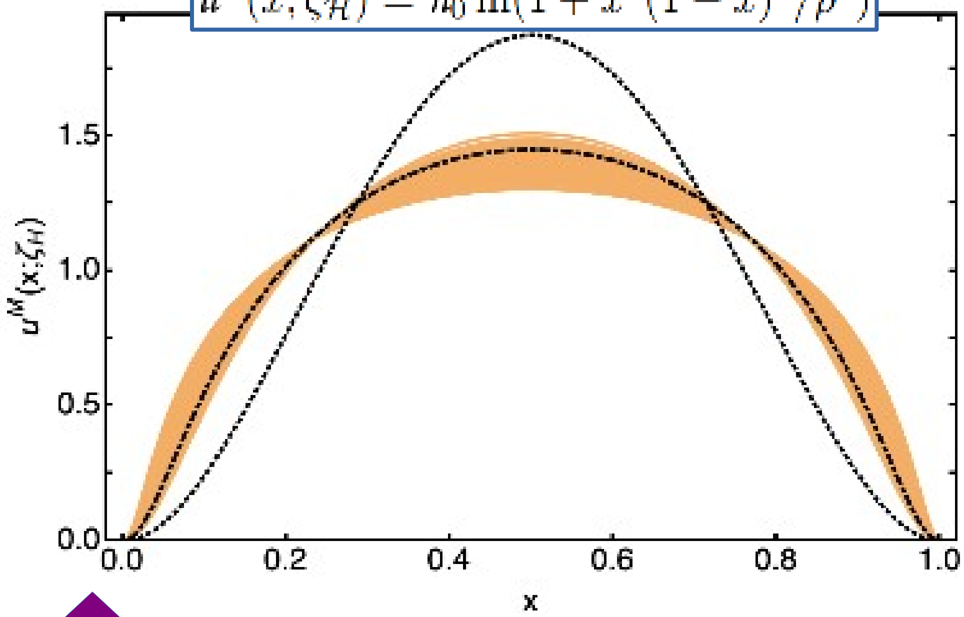
Application 5: pion and kaon PDFs fully reconstructed from data

$u^\pi(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$



$u^\pi(x;\zeta_\mathcal{H}) = n_0 \ln(1+x^2(1-x)^2/\rho^2)$



All-orders back evolution from empirical to hadronic scale

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

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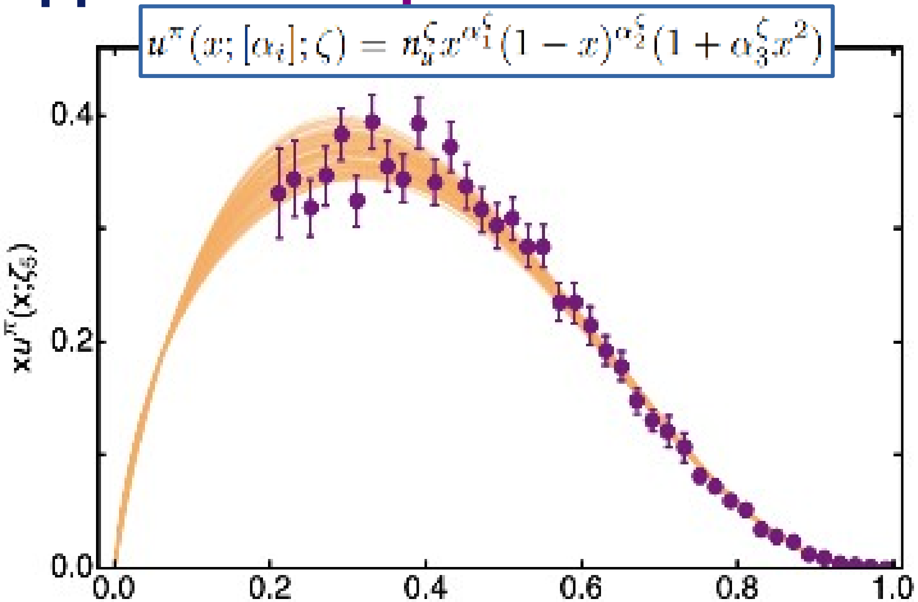

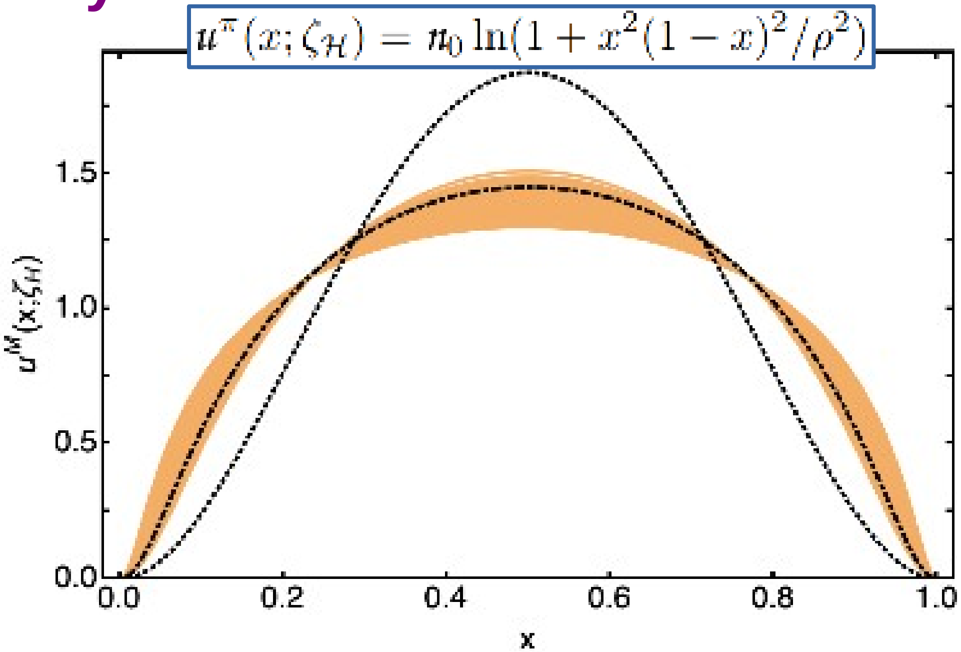

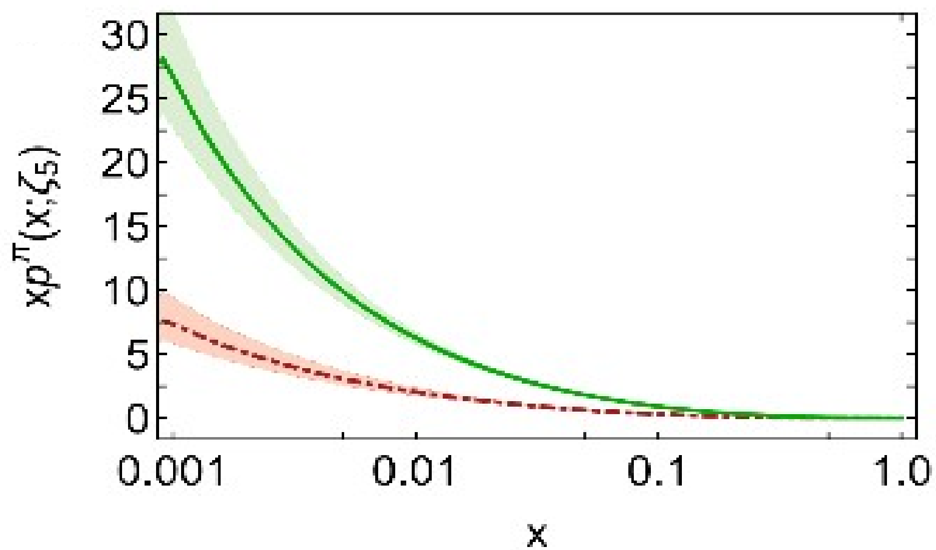

All-orders evolution from hadronic up to empirical scale, delivering glue and sea together with valence-quark PDFs

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

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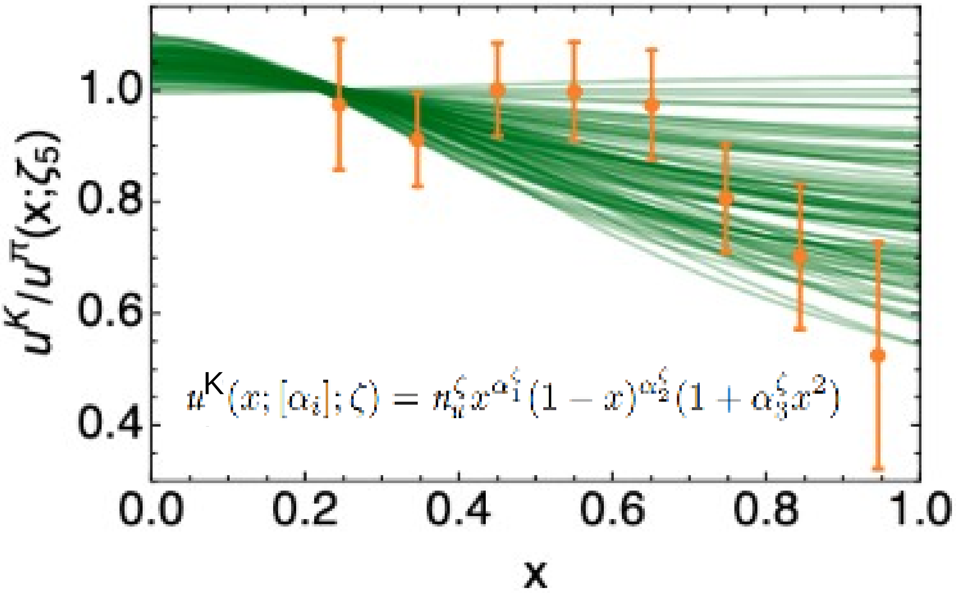
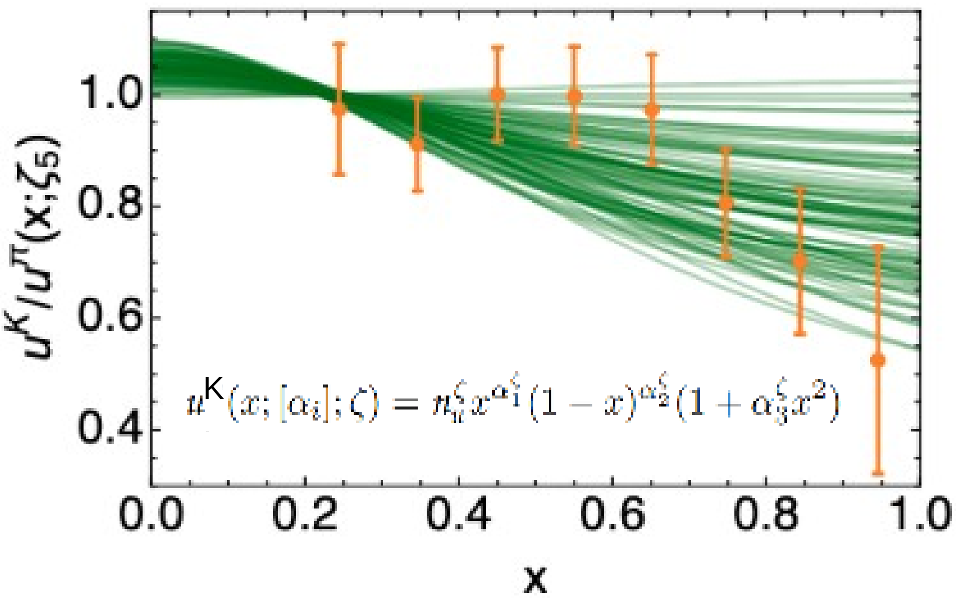


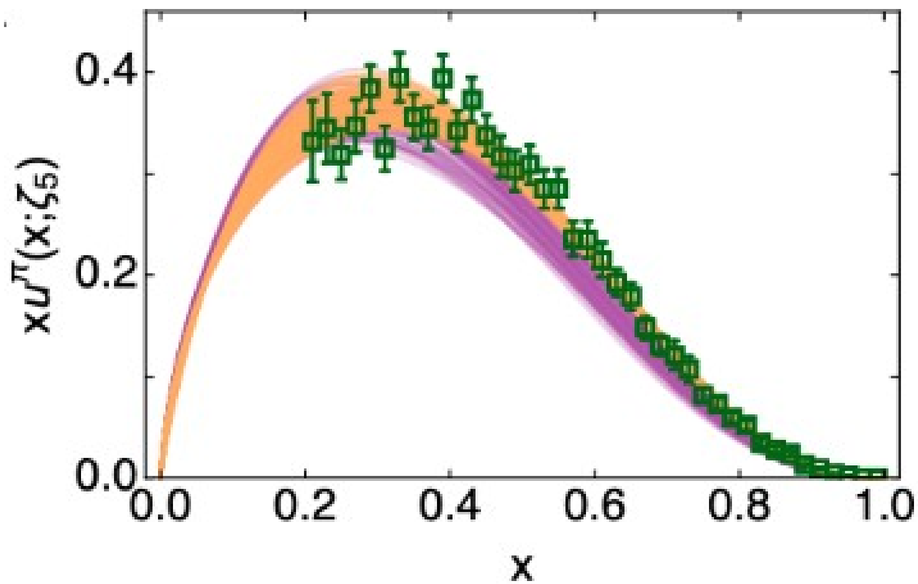
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

All-orders DGLAP: reverse engineering

Application 5: pion and kaon PDFs fully reconstructed from data





Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)
Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

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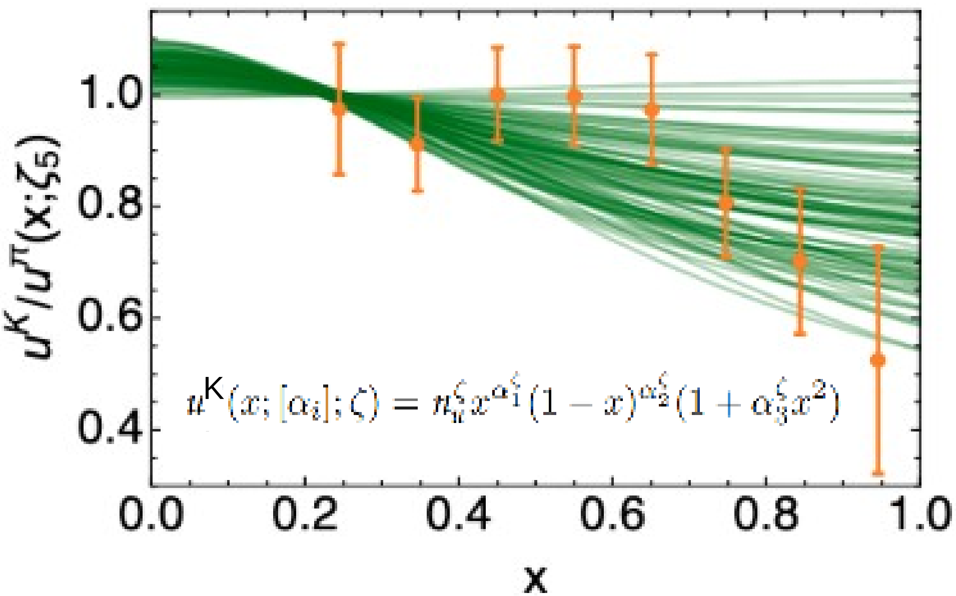

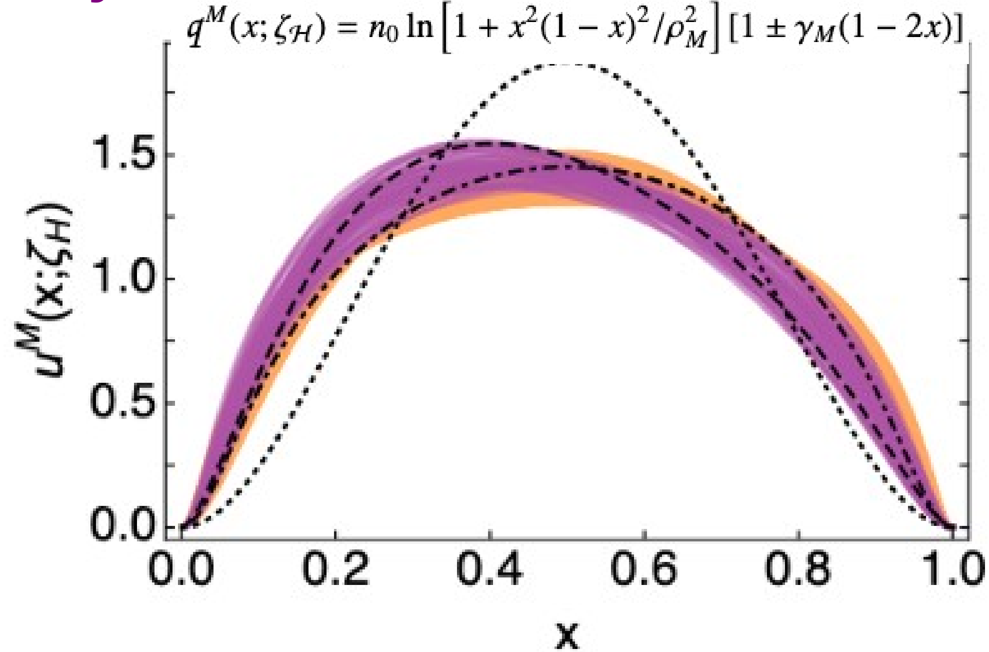

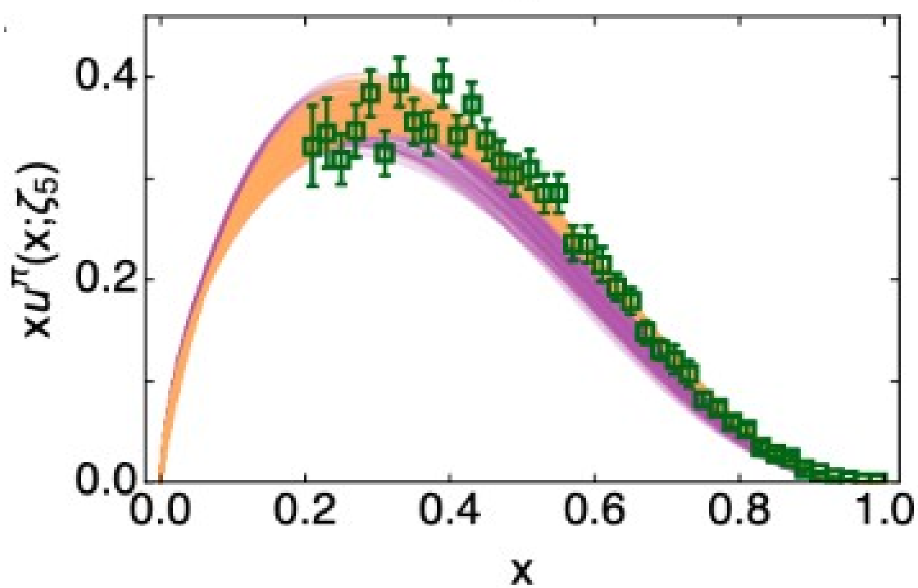

All-orders back evolution from empirical to hadronic scale

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

All-orders DGLAP: reverse engineering

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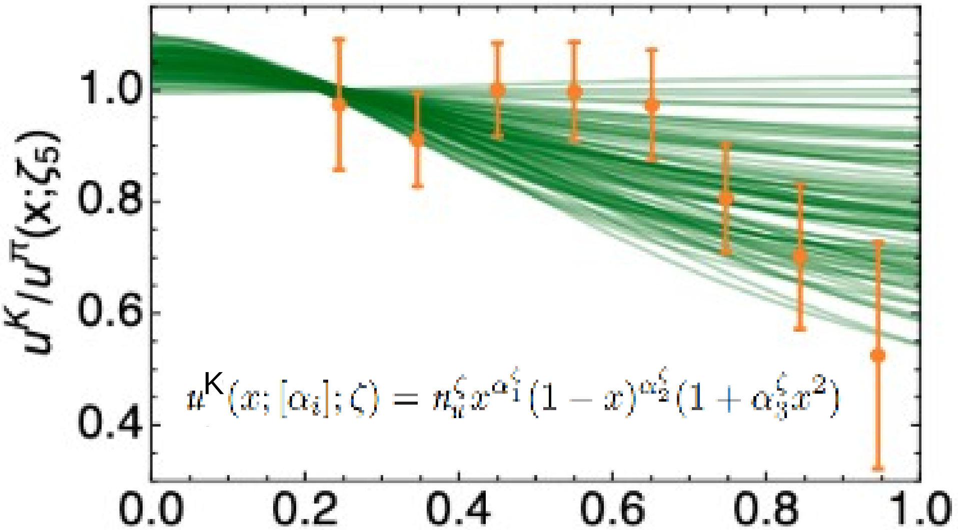

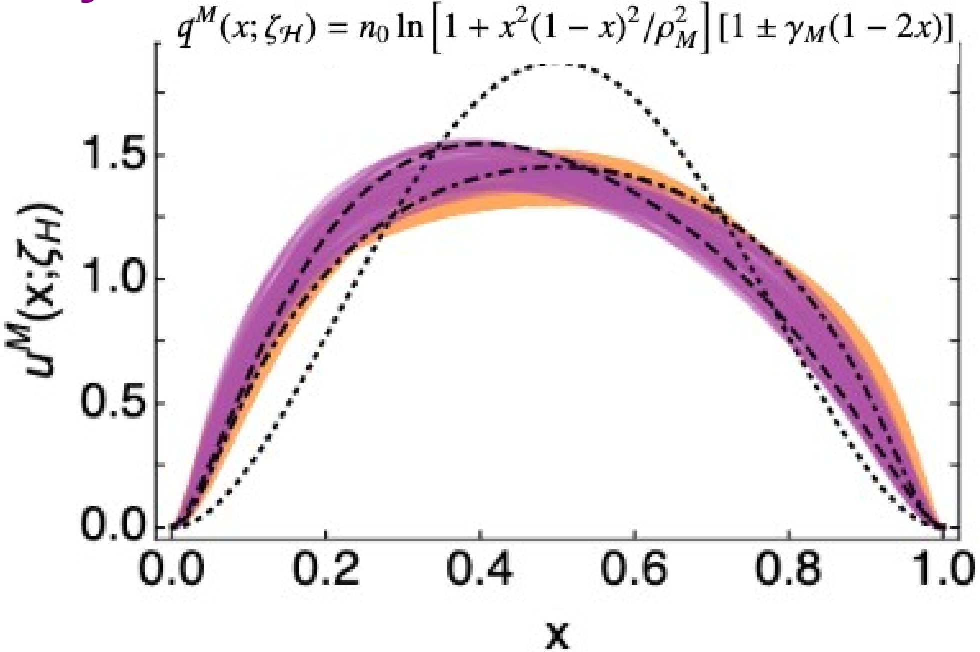

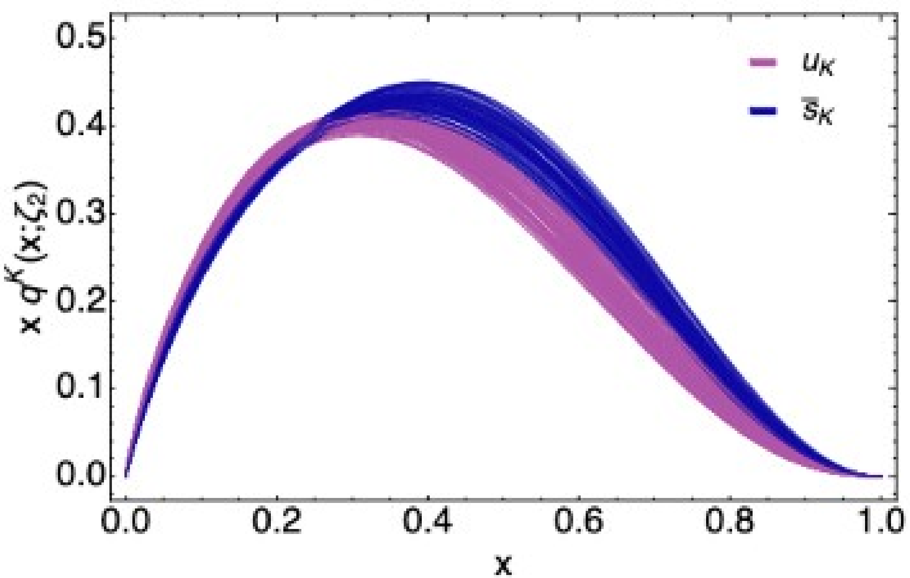

All-orders evolution from hadronic up to empirical scale, delivering valence-quark and -antiquark PDFs

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)
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All-orders DGLAP: reverse engineering

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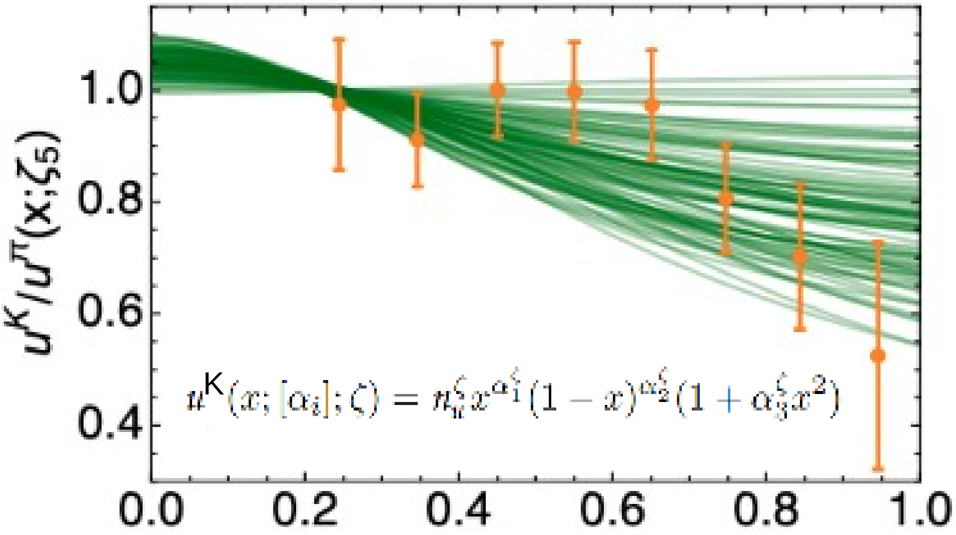

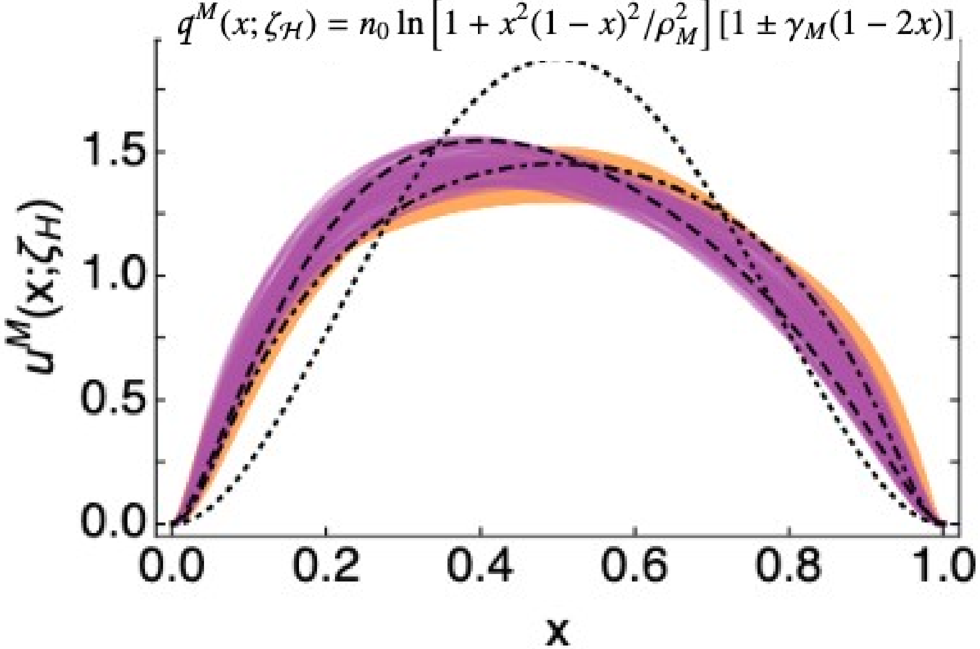

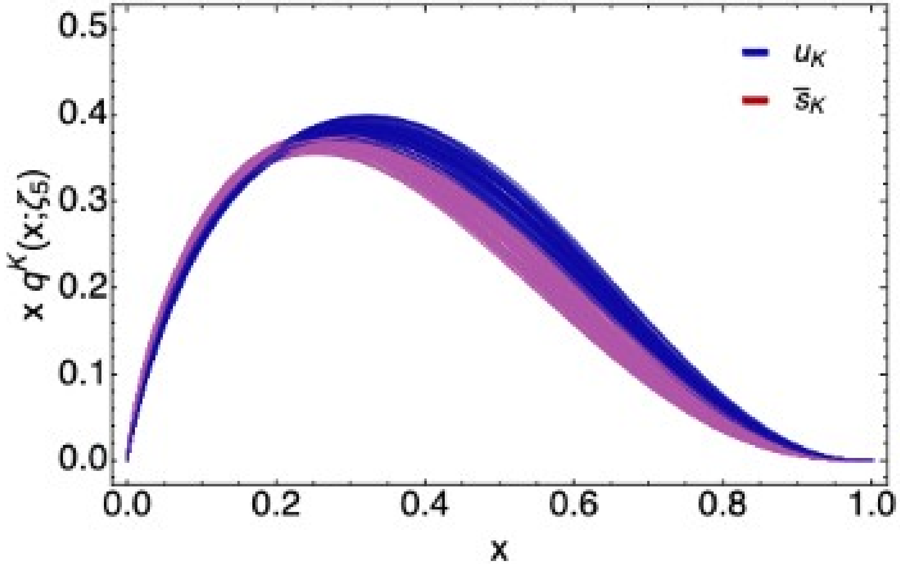

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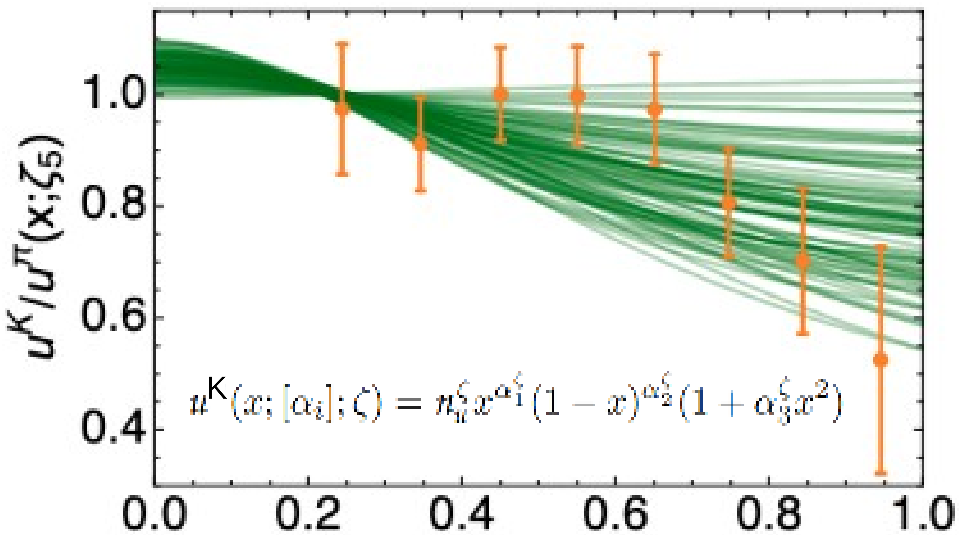
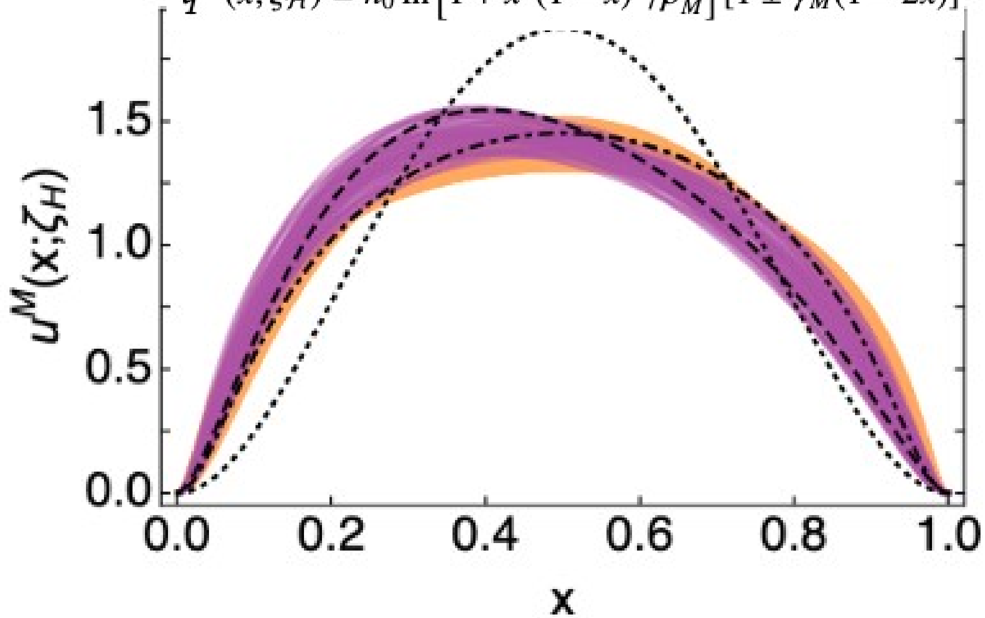
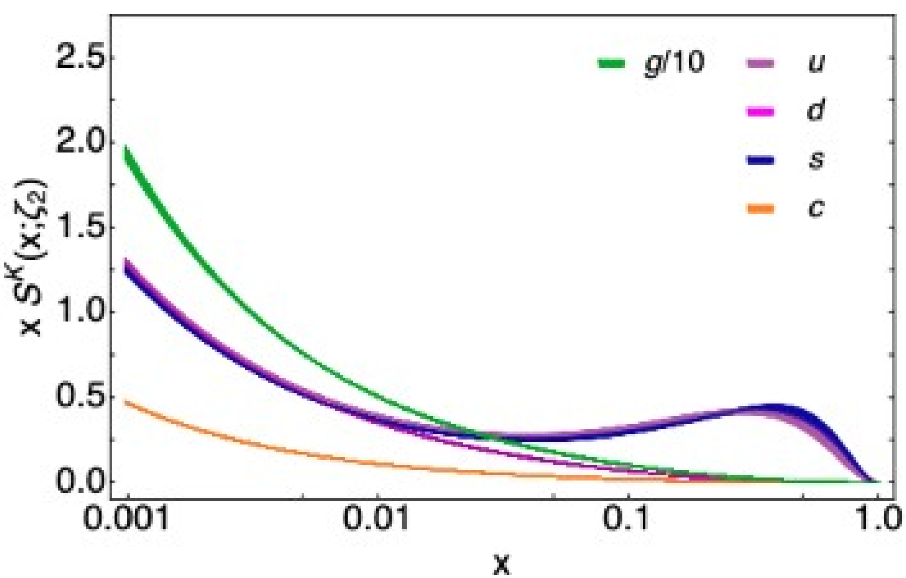


$$q^M(x;\zeta_{\mathcal{H}}) = n_0 \ln\left[1 + x^2(1-x)^2/\rho_M^2\right]\left[1 \pm \gamma_M(1-2x)\right]$$





All-orders evolution from hadronic up to empirical scale, delivering valence-quark and -antiquark PDFs, and, again, glue and sea PDFs.

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)
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All-orders DGLAP: reverse engineering

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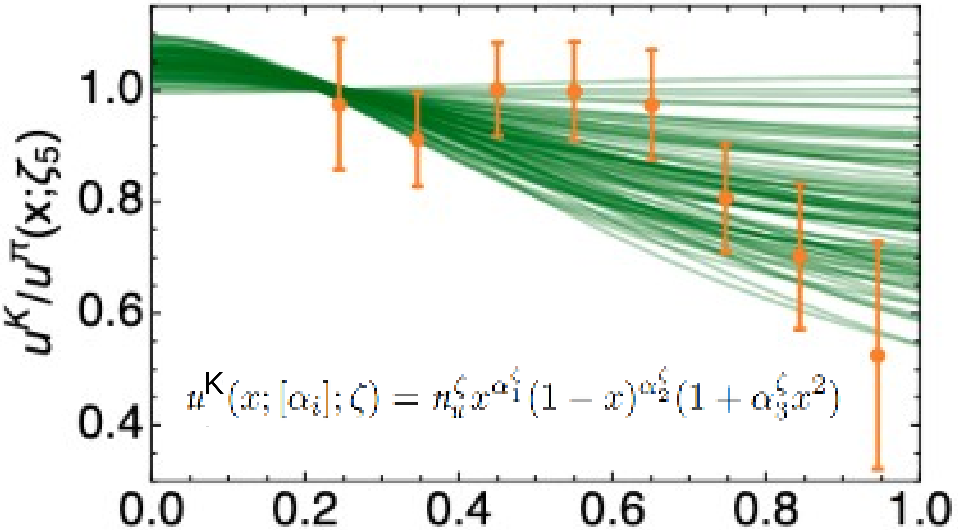

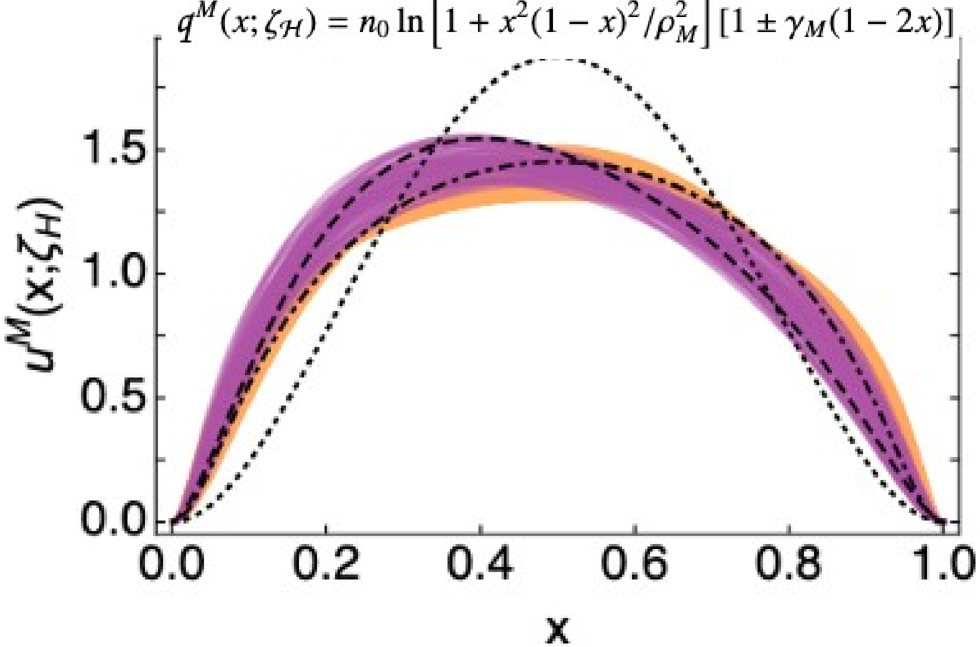

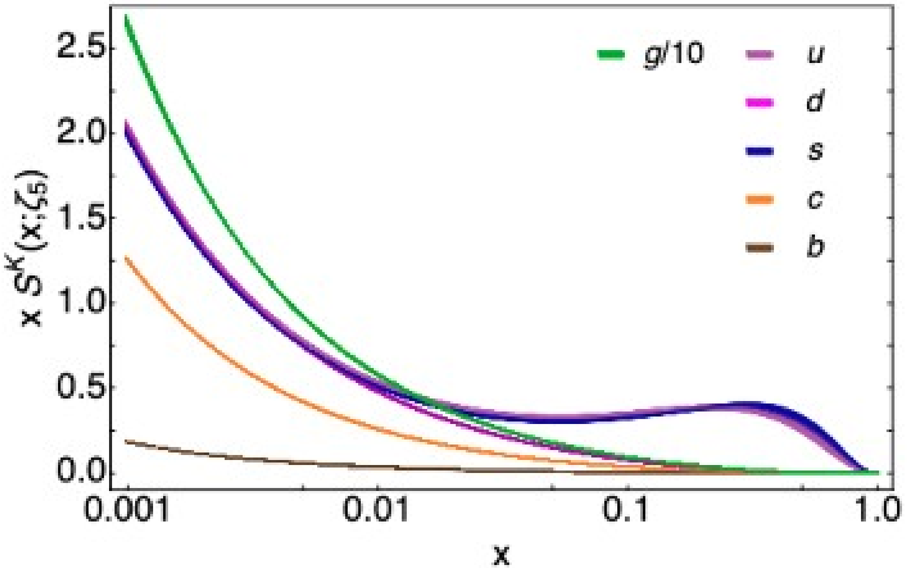

All-orders evolution from hadronic up to empirical scale, delivering valence-quark and -antiquark PDFs, and, again, glue and sea PDFs.

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Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

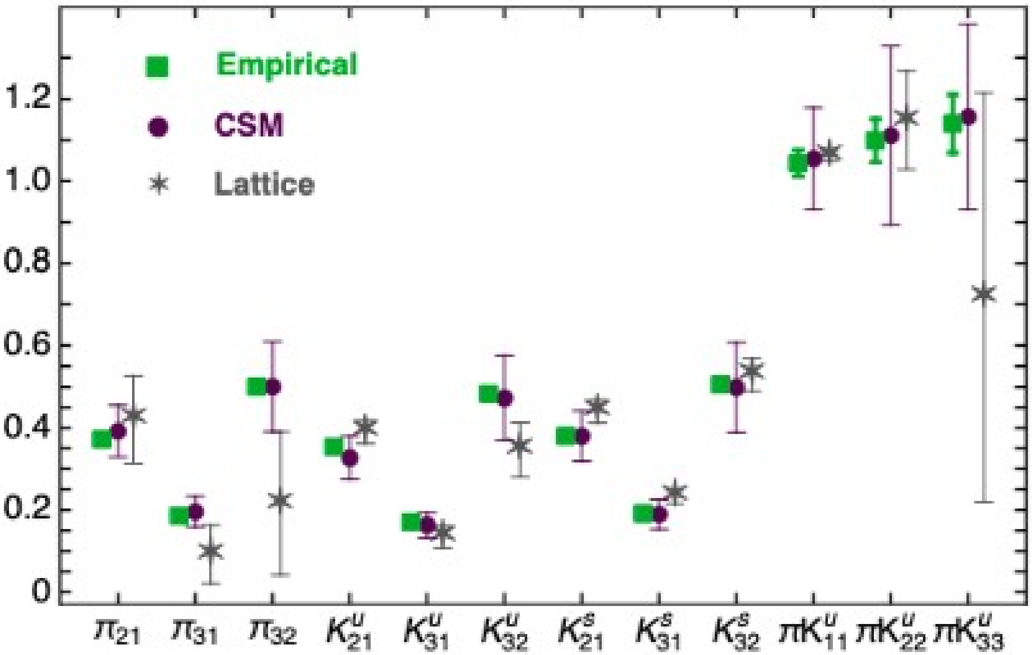
All-orders DGLAP: reverse engineering

Application 5: pion and kaon PDFs fully reconstructed from data

	$\langle x\rangle^{\zeta}_{q^K}$	$\langle x\rangle^{\zeta}_{S^K_q}$	$\langle x\rangle^{\zeta}_{g^K}$	$\langle x\rangle^{\zeta}_{q^\pi}$
ζ_2				
u	0.230(6)(10)	0.028(2)		0.241(5)(10)
d	0	0.028(2)		0.241(5)(10)
s	0.252(6)(11)	0.026(1)		
c	0	0.008(1)		
b	0	0		
g			0.428(18)	
ζ_5				
u	0.197(5)(9)	0.036(2)		0.207(4)(9)
d	0	0.036(2)		0.207(4)(9)
s	0.216(5)(9)	0.034(2)		
c	0	0.019(1)		
b	0	0.003(1)		
g			0.461(20)	

	empirical		[32, lQCD]	
M	π	K	π	K
l	0.538(15)	0.286(12)	0.499(55)	0.317(19)
s	0.026(01)	0.278(13)	0.036(15)	0.339(11)
c	0.008(01)	0.008(01)	0.013(16)	0.028(21)
q	0.572(15)	0.572(18)	0.575(79)	0.683(50)
g	0.428(18)	0.428(18)	0.402(53)	0.422(67)

[32] Alexandrou, et al., arXiv:2405.08529 [hep-lat]



$$\pi_{ij} = \langle x^i\rangle_{u^\pi_S}/\langle x^j\rangle_{u^\pi_S}$$

$$K^q_{ij} = \langle x^i\rangle_{q^K_S}/\langle x^j\rangle_{q^K_S}$$

$$\pi K^u_{ij} = \langle x^i\rangle_{u^\pi_S}/\langle x^j\rangle_{u^K_S}$$

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

CSM = Z-F Cui, et al., Eur. Phys. J. C80 (2020) 1064.
Lattice = C. Alexandrou, et al., Phys. Rev. D 103 (1) (2021) 014508; Phys. Rev. D 104 (5) (2021) 054504.

All-orders DGLAP: fragmentation functions

Application 6: pion and kaon FFs

c.f. Hui-Yu Xin's Talk on Thursday!!

- Solve cascade equations at $\zeta_H \Rightarrow$ complete FFs
- Use all-orders (AO) scheme to evolve solutions to scales relevant for measurements, *e.g.*, $\zeta = \zeta_2 := 2$ GeV
  - AO scheme is nonperturbative extension of DGLAP

    See José Rodríguez-Quintero's talk tomorrow

- Approach guarantees that particle and momentum sum rules are preserved

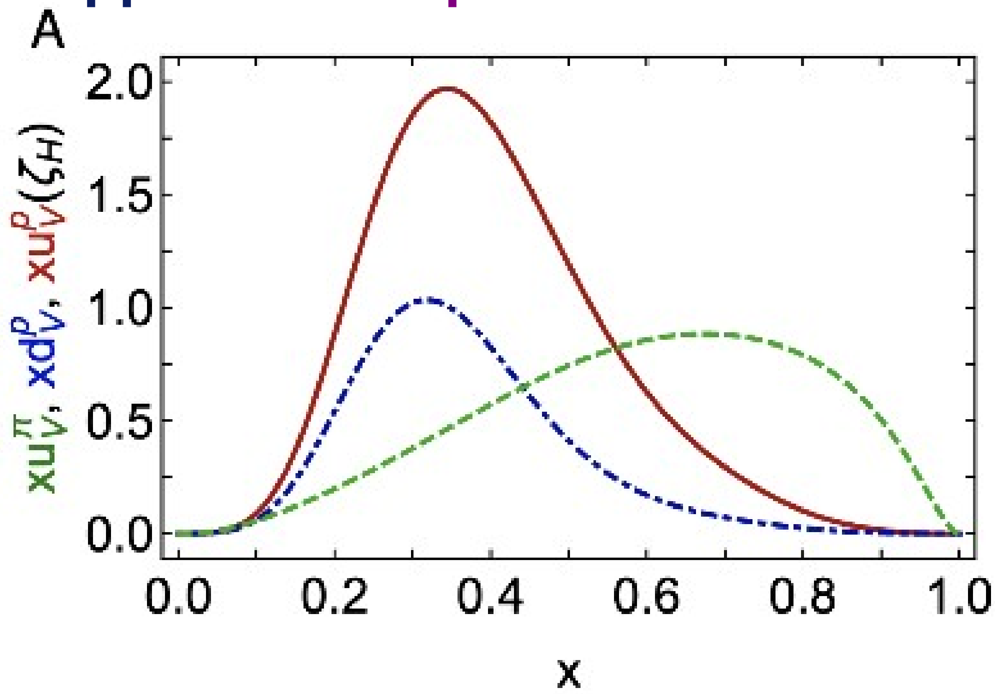
$$\sum_h \int_0^1 dz\, z D_p^h(z;\zeta) = 1$$

Table 2 SCI FF momentum fractions obtained from solutions of the cascade equations at the hadron scale and after evolution to $\zeta = \zeta_2 := 2\,\text{GeV}$, following the prescription described in Sect. 4. (No entry means the fraction is zero. $c \to q \to h$ contributions are negligible in all cases.)

h	$\pi^+ + \pi^0 + \pi^-$		K^+	
	$\zeta_{\mathcal{H}}$	ζ_2	$\zeta_{\mathcal{H}}$	ζ_2
$\langle z\rangle^h_{D}$ [illegible]	0.664	0.433	0.182	0.119
$\langle z\rangle^h_{D}$ [illegible]		0.115		0.032
$\langle z\rangle^h_{D}$ [illegible]		0.085		0.023
$\langle z\rangle^h_{D}$ [illegible]		0.031		0.009
$\langle z\rangle^h_{D}$ [illegible]		0.115		0.007
$\langle z\rangle^h_{D}$ [illegible]	0.664	0.443	0.042	0.028
$\langle z\rangle^h_{D}$ [illegible]		0.085		0.005
$\langle z\rangle^h_{D}$ [illegible]		0.031		0.002
$\langle z\rangle^h_{D}$ [illegible]		0.017		0.069
$\langle z\rangle^h_{D}$ [illegible]		0.017		0.069
$\langle z\rangle^h_{D}$ [illegible]	0.098	0.059	0.396	0.239
$\langle z\rangle^h_{D}$ [illegible]		0.005		0.019
$\langle z\rangle^{\zeta}_{D^h}$ [illegible]	0.083	0.083	0.023	0.023
$\langle z\rangle^{\zeta}_{D^h}$ [illegible]	0.083	0.083	0.005	0.005
$\langle z\rangle^{\zeta}_{D^h}$ [illegible]	0.012	0.012	0.050	0.050

All-orders DGLAP: proton case

Application 6: proton PDFs



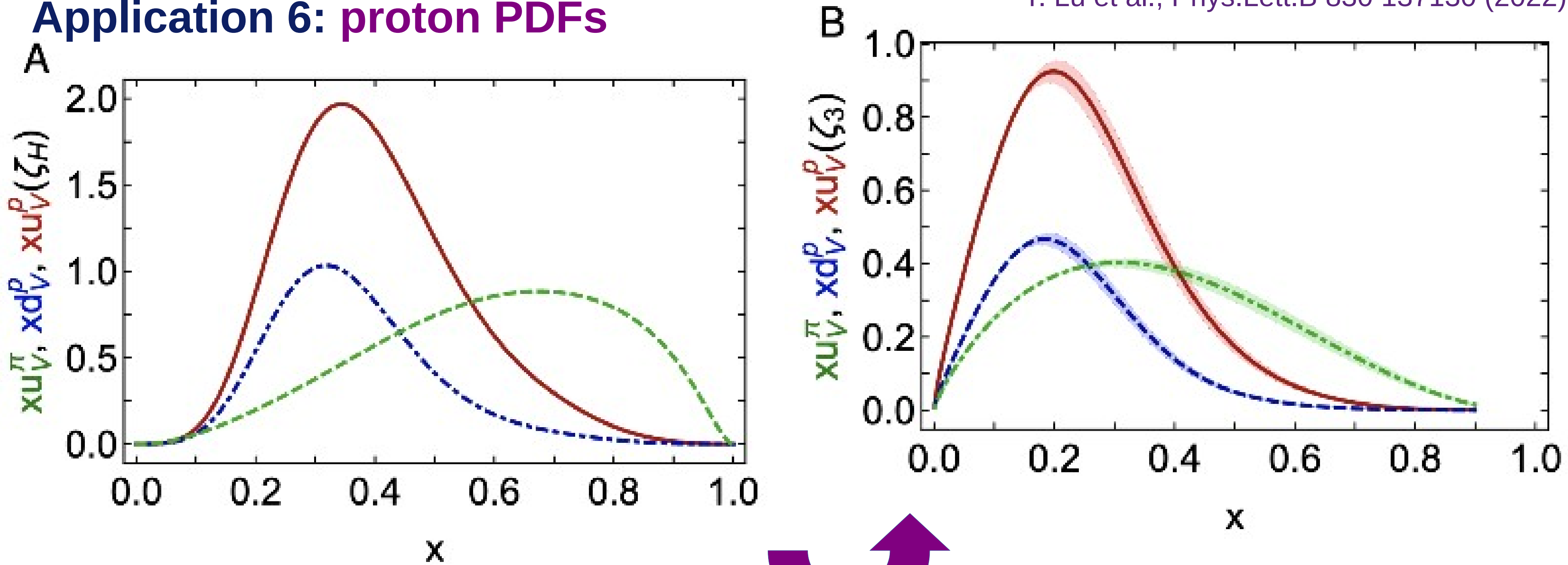
A symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach

[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]

All-orders DGLAP: proton case

Y. Lu et al., Phys.Lett.B 830 137130 (2022)

Application 6: proton PDFs



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All-orders evolution from hadronic up to empirical scale.

All-orders DGLAP: proton case

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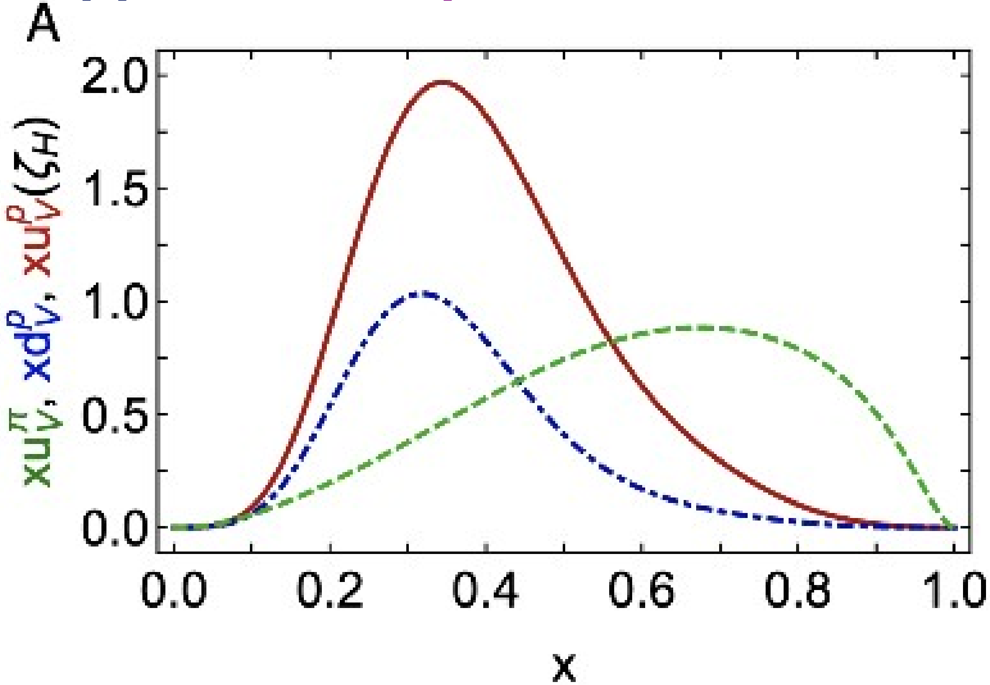

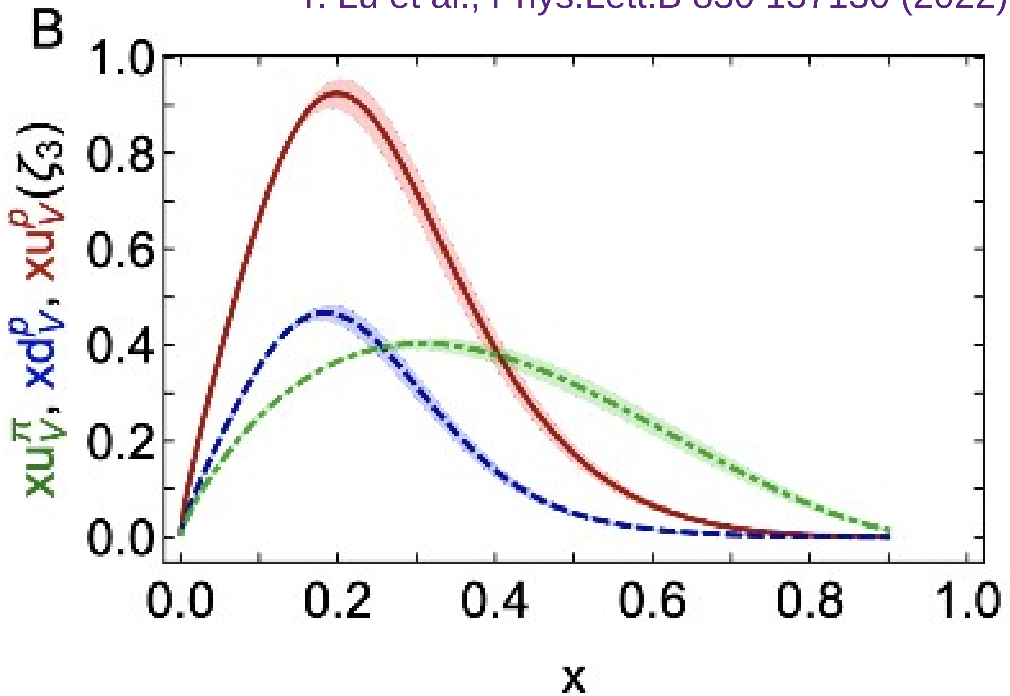

A symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach
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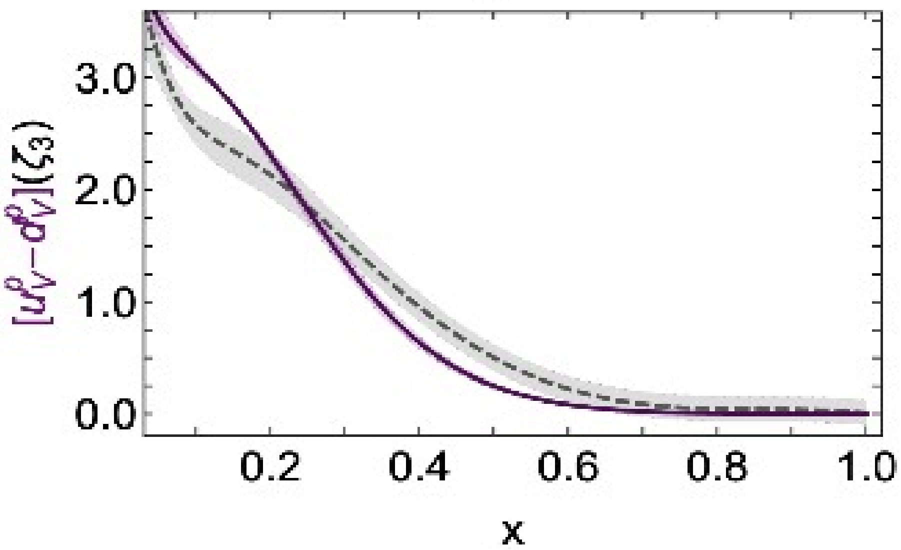
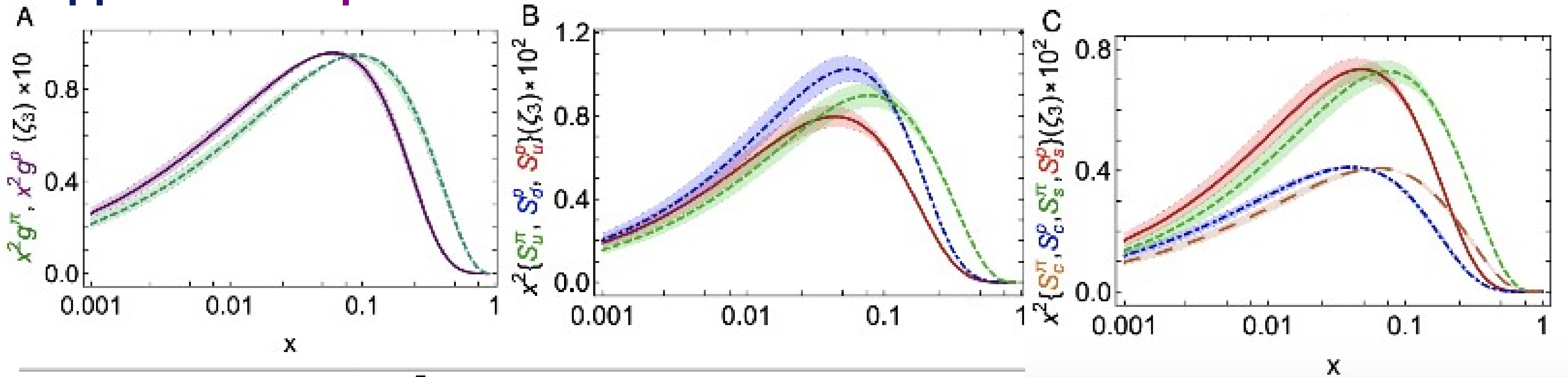
Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]



All-orders DGLAP: proton case

Application 6: proton PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)



pion	u^π	$\bar{d}^\pi$	g^π	S_π^u	S_π^d	S_π^s	S_π^c
$\langle x \rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2 \rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3 \rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x \rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2 \rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3 \rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x \rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2 \rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3 \rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x \rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2 \rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3 \rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)

All-orders DGLAP: proton case

Application 7: polarized PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

All-orders DGLAP: proton case

Application 7: polarized PDFs

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In general, at any momentum scale $\zeta \geq M_c$:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \, ,$$

$$\Delta G^{\zeta}_H = \langle x^0 \rangle^{\zeta}_{\tilde{g}^q_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \left\{ \frac{12}{29} \left([S(\zeta_H, M_s)]^{-87/32} - 1 \right) [S(M_s, M_c)]^{-81/32} [S(M_c, \zeta)]^{-75/32} \right.$$
$$\left. + \frac{4}{9} \left([S(M_s, M_c)]^{-81/32} - 1 \right) [S(M_c, \zeta)]^{-75/32} + \frac{12}{25} \left([S(M_c, \zeta)]^{-75/32} - 1 \right) \right\}$$

$$S(\zeta_H, \zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: proton case

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In general, at any momentum scale $\zeta_H \leq \zeta$, and neglecting the mass thresholds:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \,,$$

$$\Delta G^{\zeta}_{H} = \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = a^{\zeta}_0 \, \tfrac{12}{25} \left[[\langle x \rangle^{\zeta}_{\mathcal{V}_H}]^{-75/32} - 1 \right]$$

All-orders DGLAP: proton case

Application 7: polarized PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)

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- A [CT18]+ no thresholds
- B [Ya2022]+no thresholds
- C [Ya2022]+[Chen2022] +no thresholds
- D [Ya2022]+ [Chen2022] + thresholds

	A	B	C	D
a^{ζ}_{0p}	0.74(11)	0.74(11)	0.65(02)	0.65(02)
ΔG^{ζ}_{p}	2.27(30)	1.50(25)	1.33(15)	1.41(16)
$\tilde{a}^{\zeta}_{0p}$	0.20(11)	0.38(11)	0.33(04)	0.32(04)

Abelian anomaly corrected:

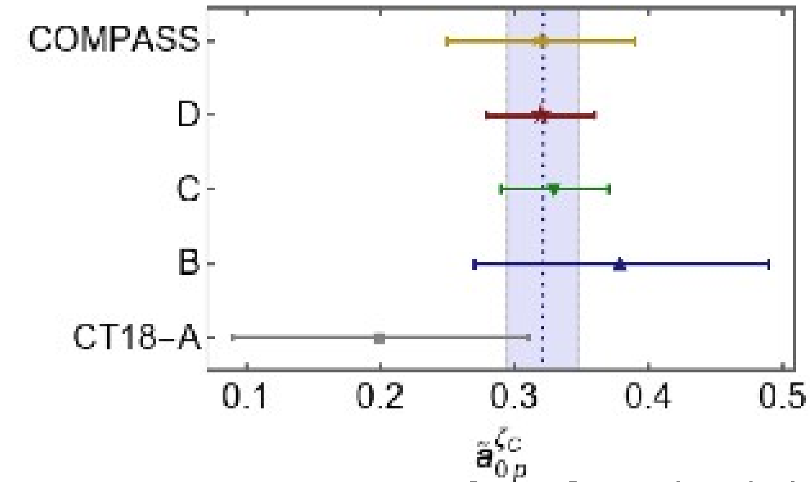
$$\tilde{a}^{\zeta}_{0p} = a^{\zeta}_{0p} - n_f \frac{\hat{\alpha}(\zeta)}{2\pi} \Delta G^{\zeta}_{p}$$

All-orders DGLAP: proton case

Application 7: polarized PDFs

Polarized PDFs DGLAP evolutions equations

Y. Lu et al., Phys.Lett.B 830 137130 (2022)



$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0$$

In general, at any momentum scale $\zeta_H \leq \zeta$, and neglecting the mass thresholds:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \,,$$

$$\Delta G^{\zeta}_H = \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = a^{\zeta}_0 \, \tfrac{12}{25} \left[[\langle x \rangle^{\zeta}_{\mathcal{V}_H}]^{-75/32} - 1 \right]$$

A [CT18]+ no thresholds

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C [Ya2022]+[Chen2022] +no thresholds

D [Ya2022]+ [Chen2022] + thresholds

	A	B	C	D
a^{ζ}_{0p}	0.74(11)	0.74(11)	0.65(02)	0.65(02)
ΔG^{ζ}_p	2.27(30)	1.50(25)	1.33(15)	1.41(16)
$\tilde{a}^{\zeta}_{0p}$	0.20(11)	0.38(11)	0.33(04)	0.32(04)

Abelian anomaly corrected:

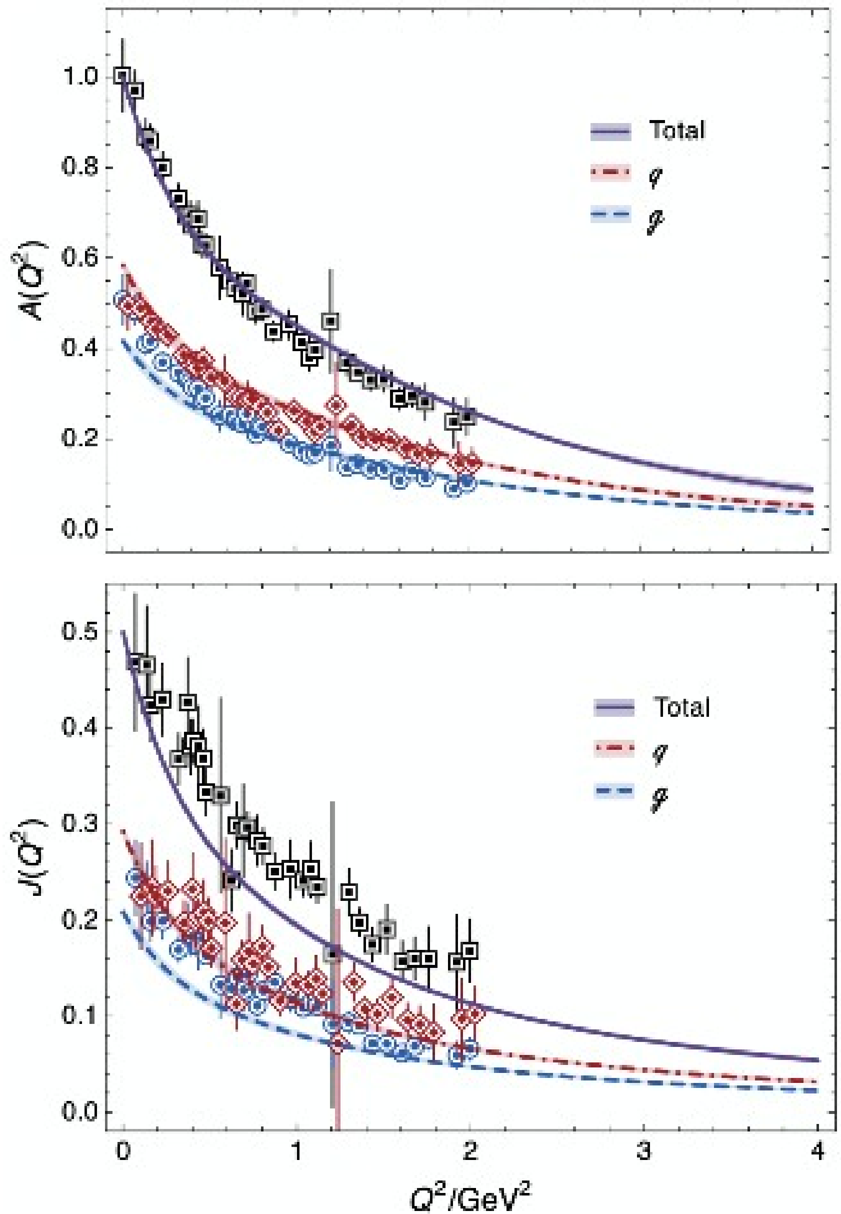
$$\tilde{a}^{\zeta}_{0p} = a^{\zeta}_{0p} - n_f \frac{\hat{\alpha}(\zeta)}{2\pi} \Delta G^{\zeta}_p$$

$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: proton case

Application 8: proton gravitational form factors

Z-Q. Yao et al., Eur.Phys.J.A 61 (2025) 5, 92

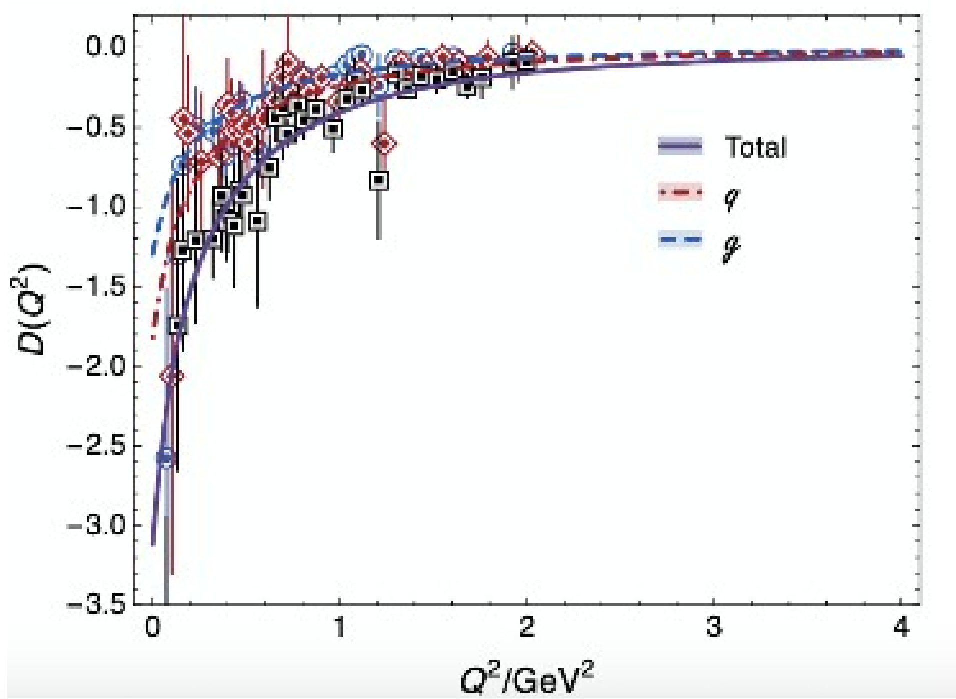


All-orders evolution implies:

$$F_p(t;\zeta) = \langle x \rangle_p^\zeta F(t)$$

$$F = A, J, D \text{ and } p = q, g, \Sigma_{\mathbf{p}}^q, \Sigma_{\mathbf{p}}.$$

Lattice data from
D.C. Hacket et al., Phys.Rev.Lett.132 (2025)251904



All-orders DGLAP: proton case

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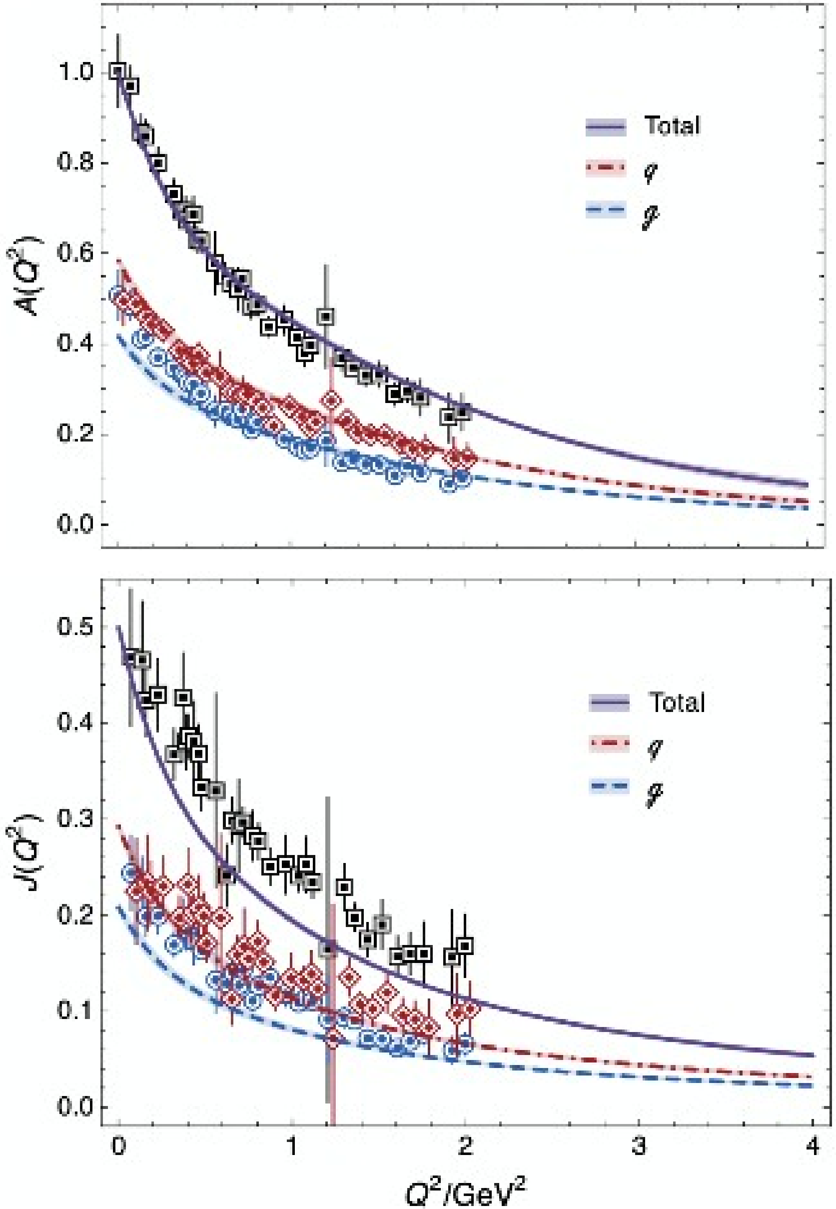

Z-Q. Yao et al., Eur.Phys.J.A 61 (2025) 5, 92

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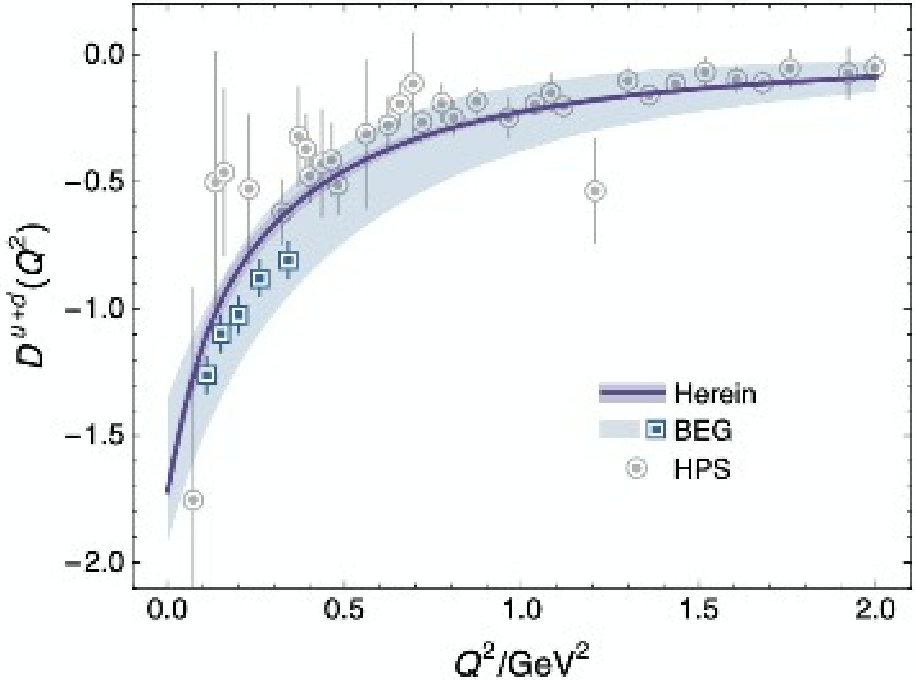
BEG: V. D. Burkardt et al., Nature 557 (7705) 396-399 (2018)



Summary

- Grunberg's notion of effective charge, reinterpreted as a non-perturbative renormalization scheme for the running coupling extends naturally its domain to the IR and, extracted from an observable, eliminates the Landau pole. The one based on the Bjorken sum rule is shown to connect to other IR coupling definitions, specially to the **PI effective** charge. They are shaped by the expression of the **EHM** and become approximately conformal below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.

- Capitalizing on the latter, two main ideas emerge:
  (i) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons;
  and (ii) the reliability of an **all-orders** evolution scheme, based on an effective charge defined to make exact one-loop DGLAP equations, describing the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale increases.

- The robustness of the approach based on **all-orders** evolution of **CSM** results from **hadronic** to experimental scale has been proved with its application to the pion, kaon and proton cases, obtaining results for PDFs, unpolarized and polarized, FFs and GFFs which are show to be consistent with **Lattice QCD** and experimental data.

To be continued...