UNIVERSITÀ DI PAVIA

SHARP

INFN
Istituto Nazionale di Fisica Nucleare

NNLO extraction of Unpolarized Di-hadron fragmentation functions

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Supervisor:

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Marco Radici

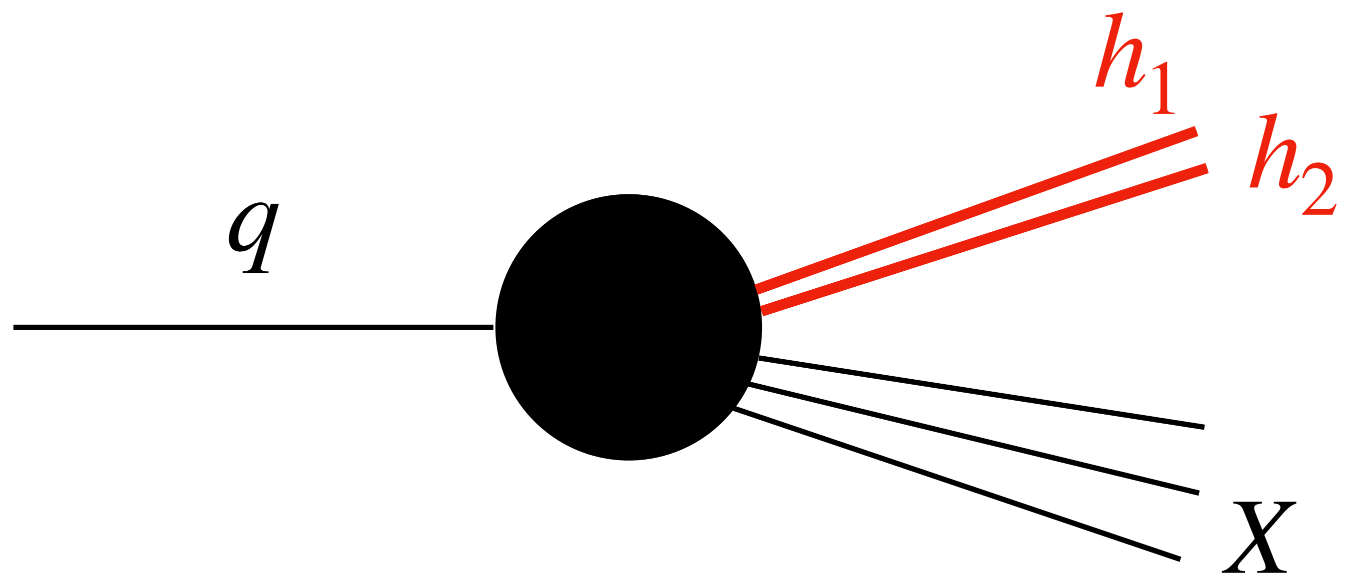
QCD evolution 26 El Escorial, Spain

This talk is supported by the SHARP Cost Action CA24159

MAP Collaboration
https://github.com/MapCollaboration

Di-Hadron Fragmentation Functions

Non perturbative $q \to h_1 h_2 X$



Di-Hadron Fragmentation Functions

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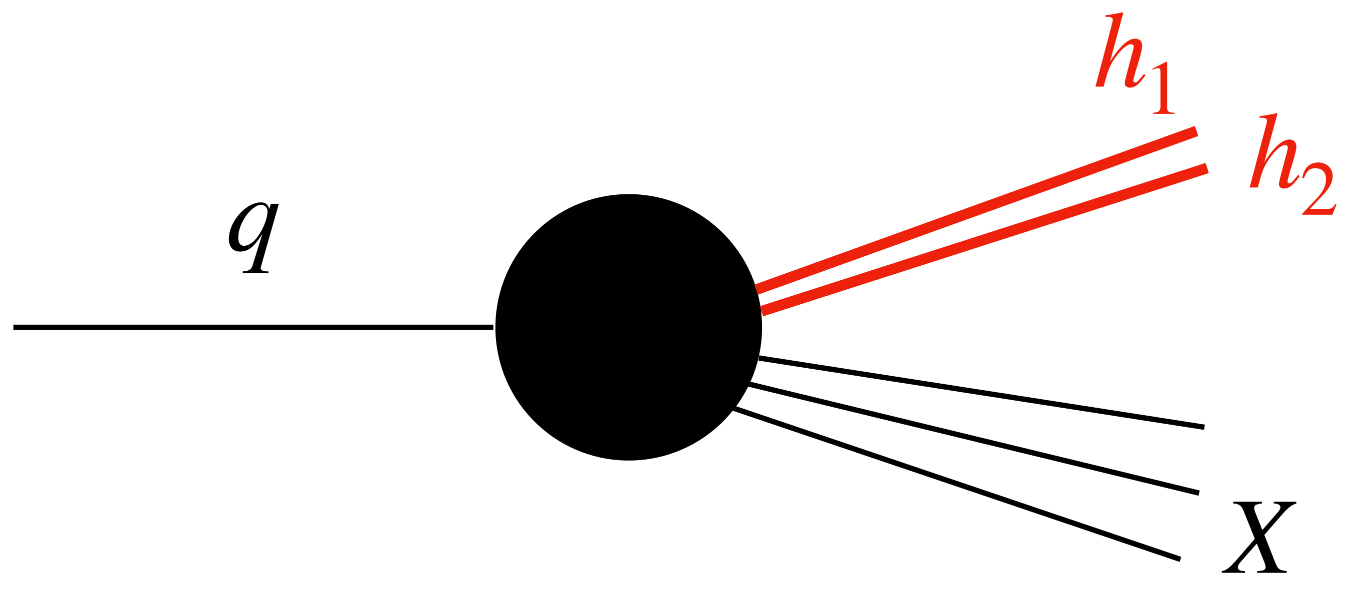

$$z_{1,2} = \frac{2E_{1,2}}{Q}$$

Di-Hadron Fragmentation Functions

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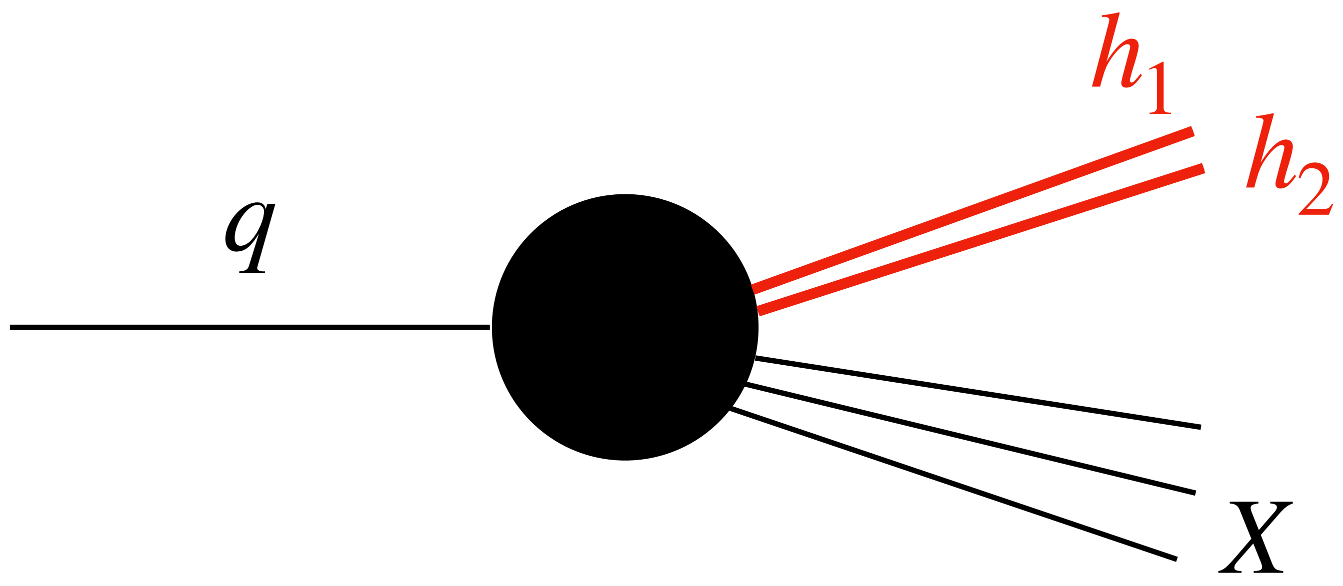
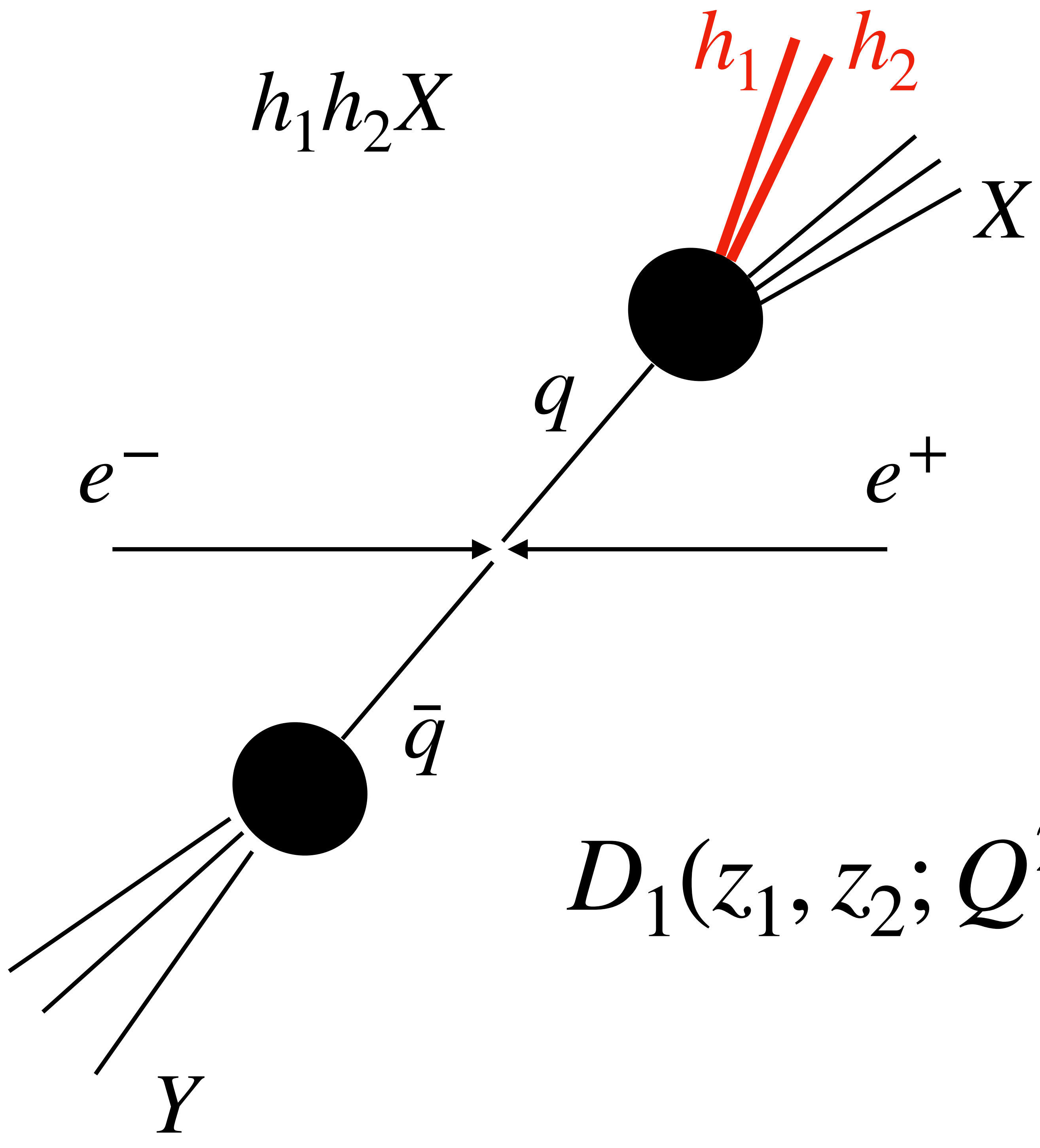


$$z_{1,2} = \frac{2E_{1,2}}{Q}$$

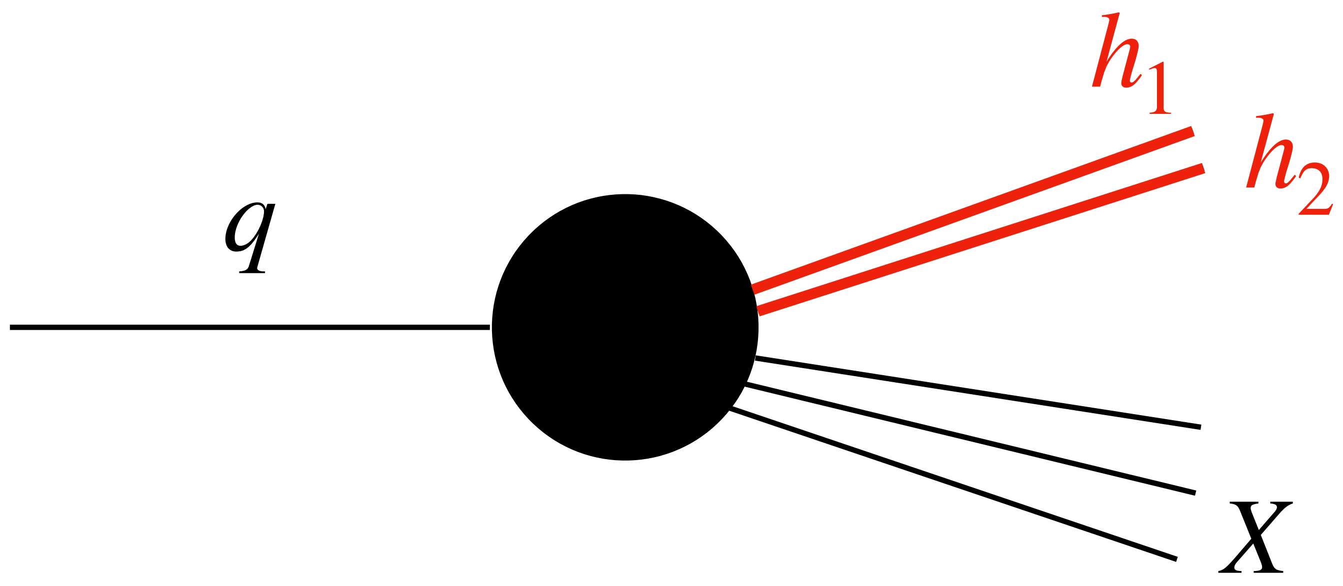
$e^+e^- \to$ $h_1 h_2 X$



$$D_1(z_1, z_2; Q^2)$$

Di-Hadron Fragmentation Functions

Non perturbative $q \to h_1 h_2 X$



$$z_{1,2} = \frac{2E_{1,2}}{Q}$$

$e^+e^- \to$

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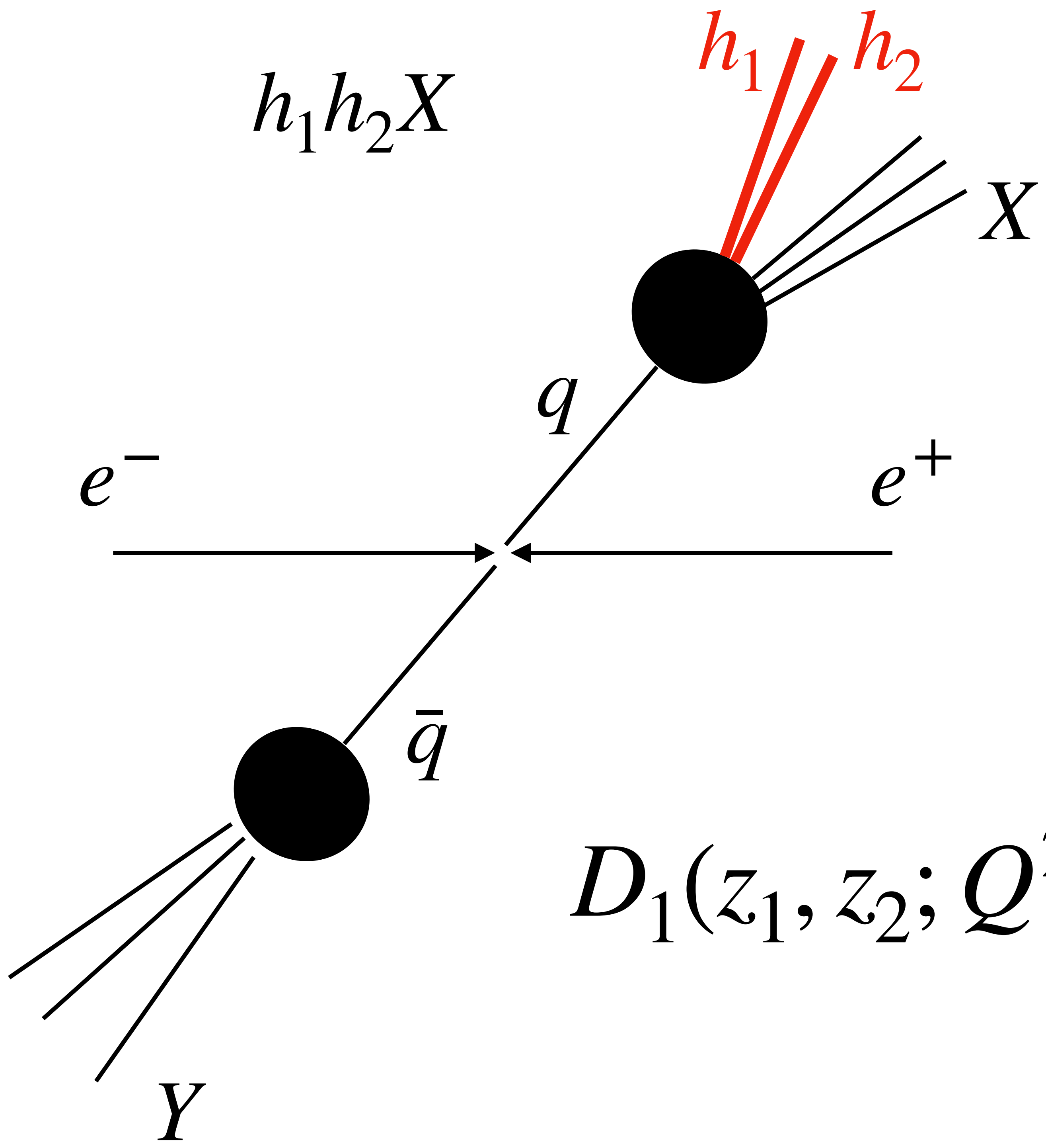
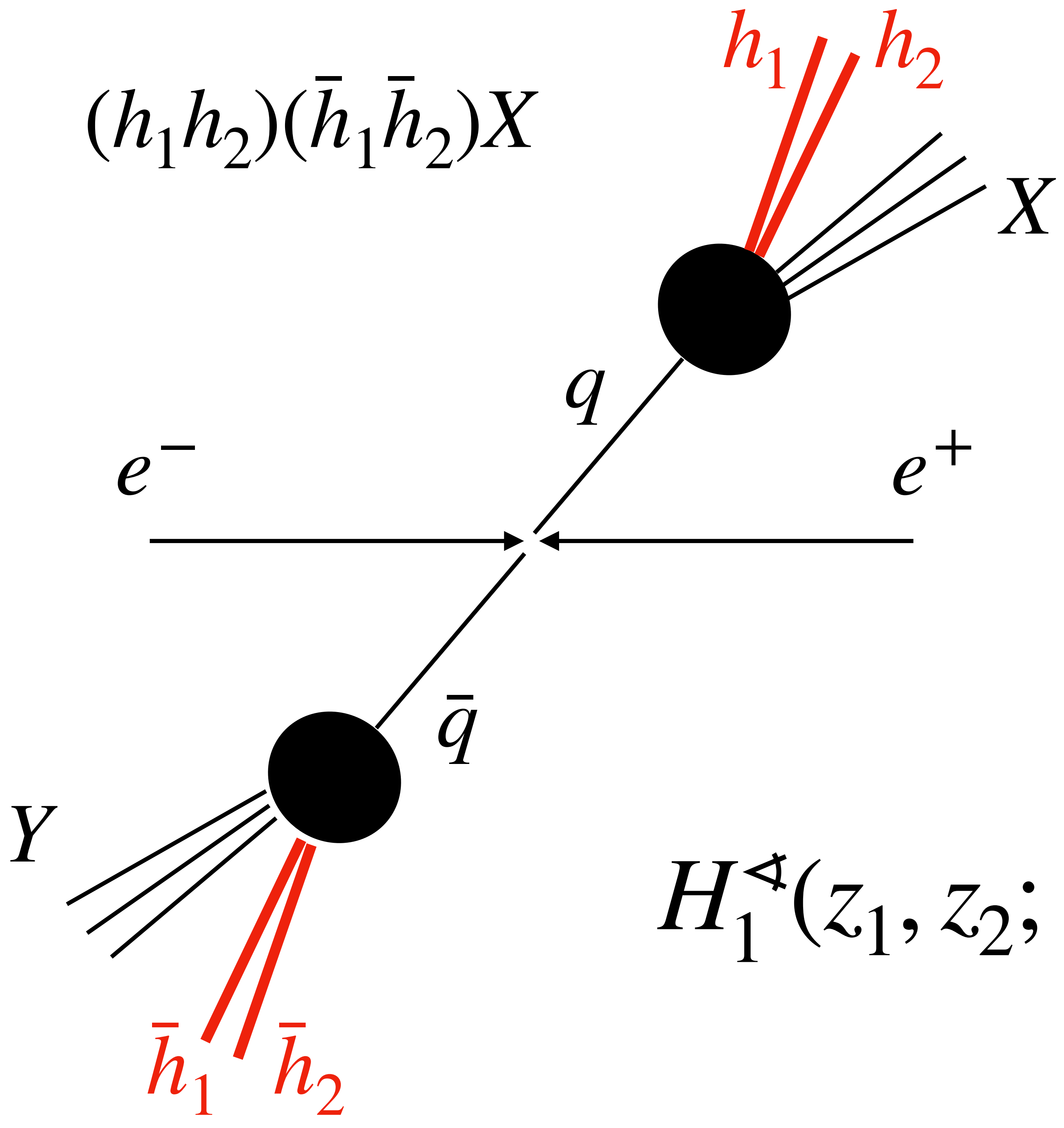


$$D_1(z_1, z_2; Q^2)$$

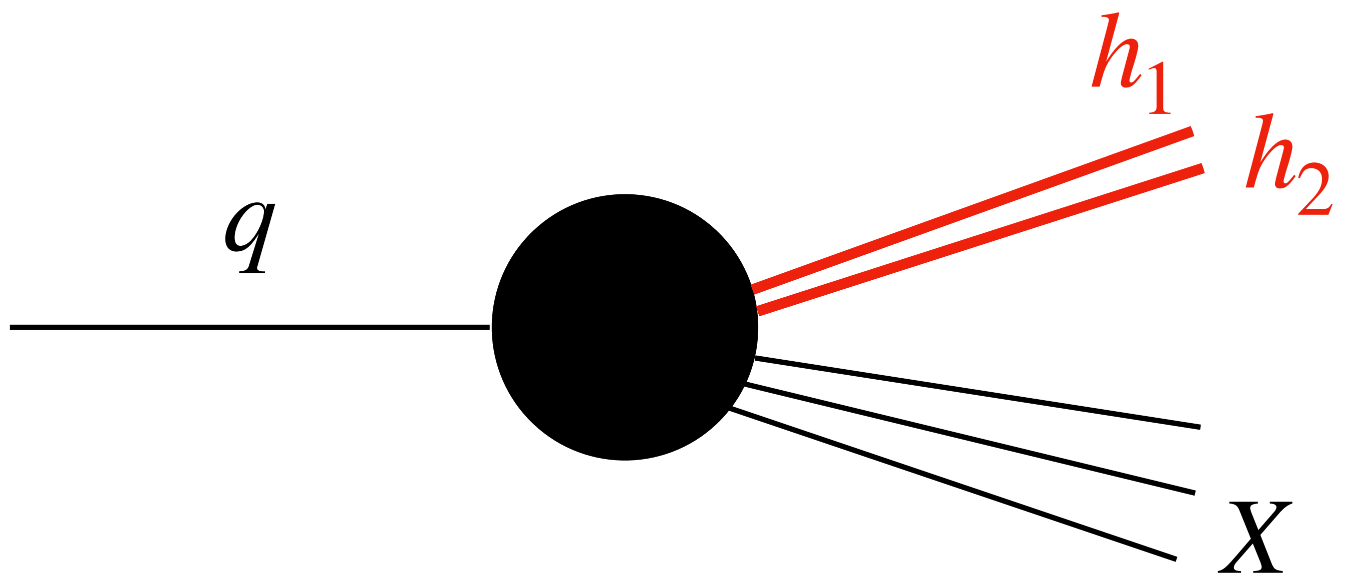
$(h_1 h_2)(\bar{h}_1 \bar{h}_2) X$



$$H_1^{\sphericalangle}(z_1, z_2; Q^2)$$

Extended Di-Hadron Fragmentation Functions

Non perturbative $q \to h_1 h_2 X$



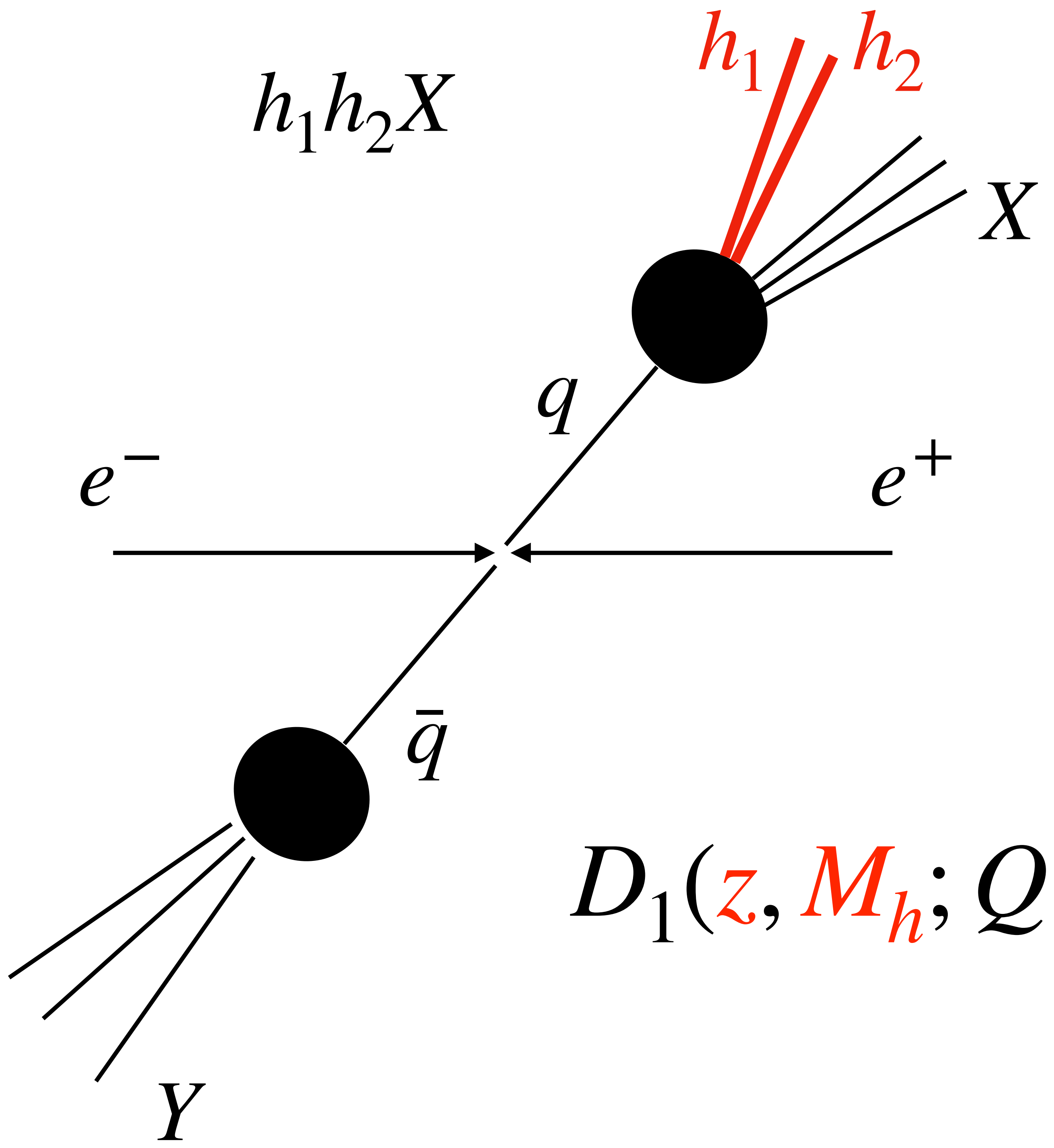
$z = z_1 + z_2$

$P_h = P_{h,1} + P_{h,2}$

$M_h = P_h^2$

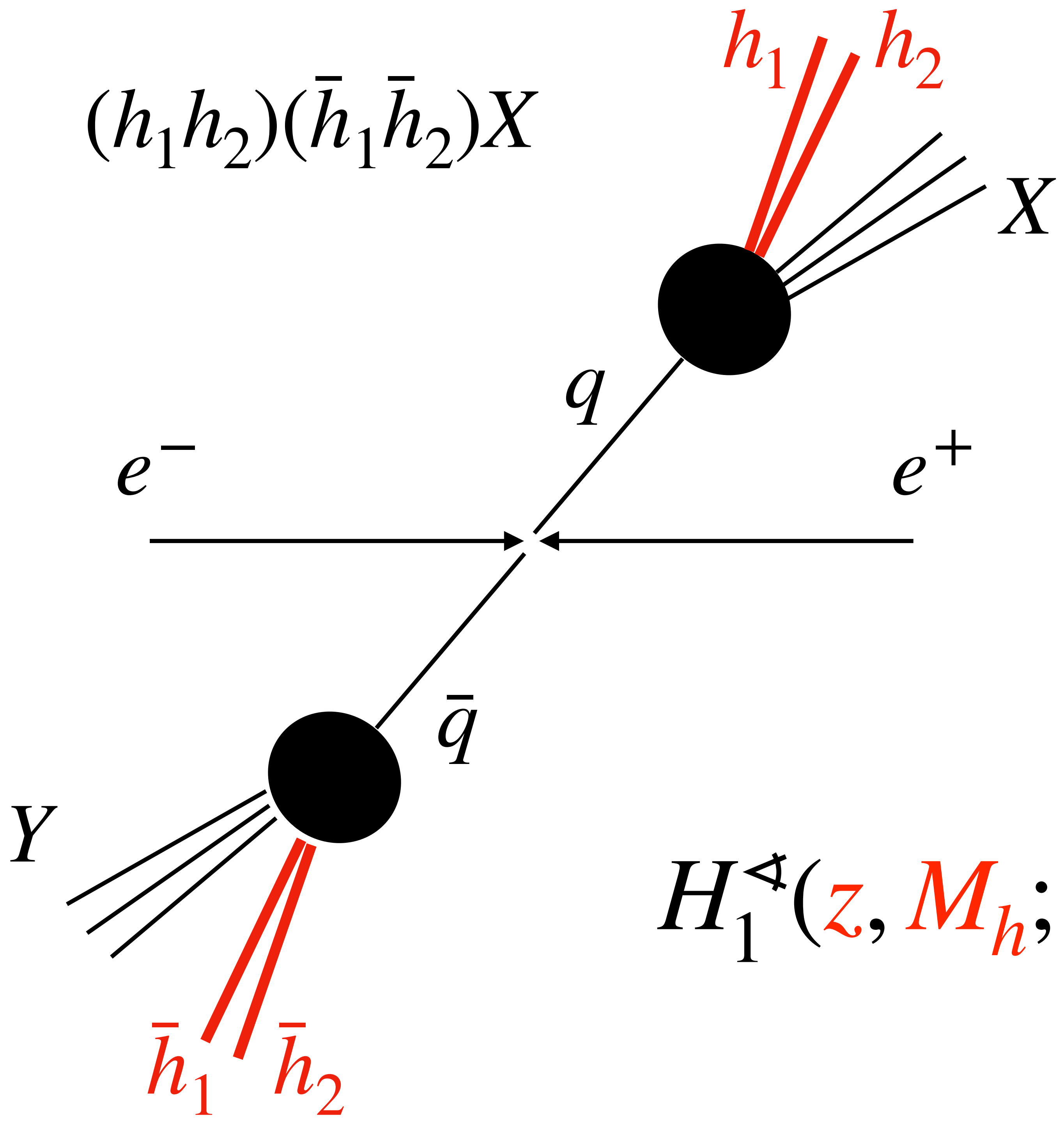
$e^+e^- \to$

$h_1 h_2 X$



$D_1(z, M_h; Q^2)$

$(h_1 h_2)(\bar{h}_1 \bar{h}_2)X$



$H_1^{\sphericalangle}(z, M_h; Q^2)$

Alternative method to extract trasversity PDF

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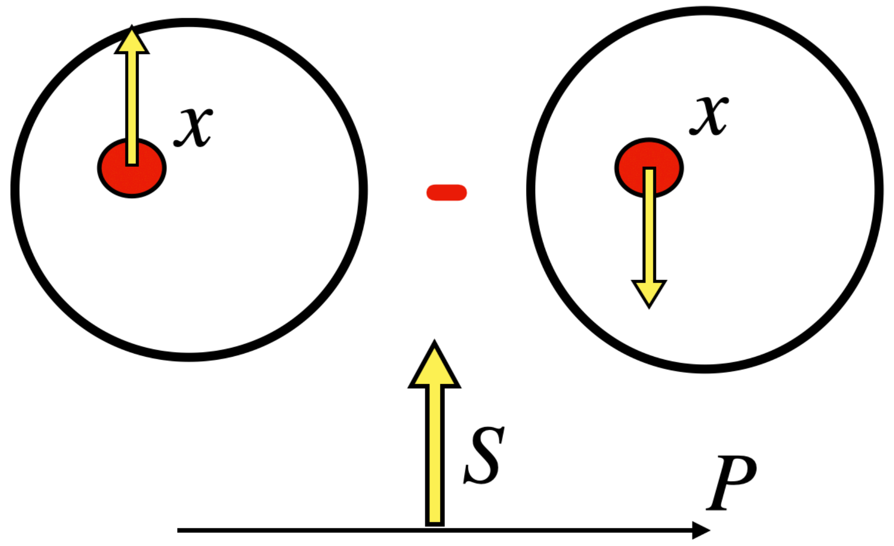
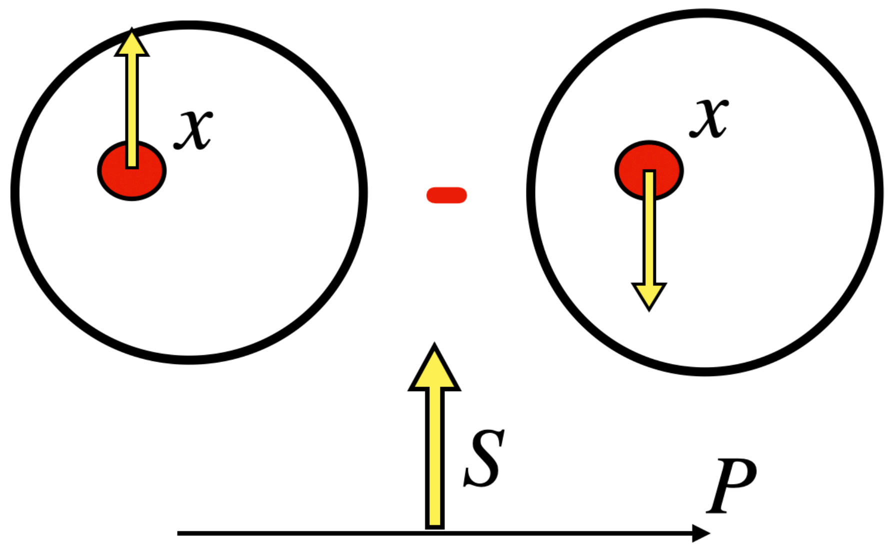


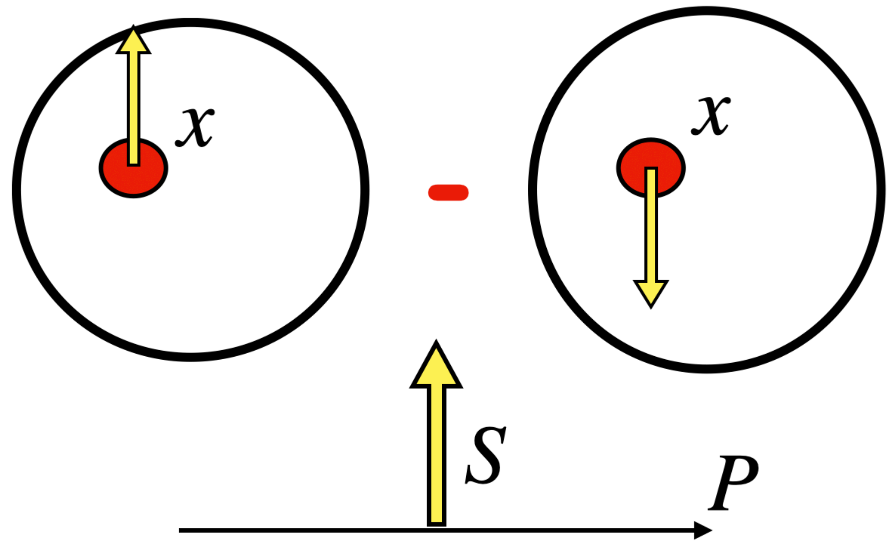
Chiral-odd function

Alternative method to extract trasversity PDF



Chiral-odd function → **It needs to couple to another chiral-odd function**

Alternative method to extract trasversity PDF



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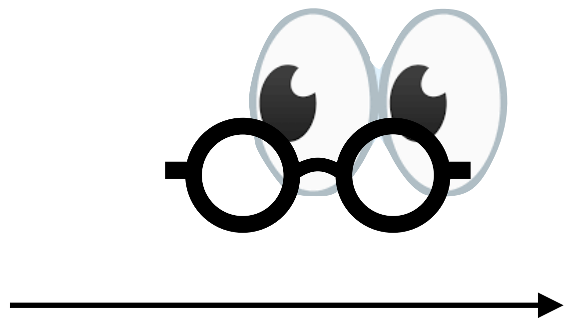
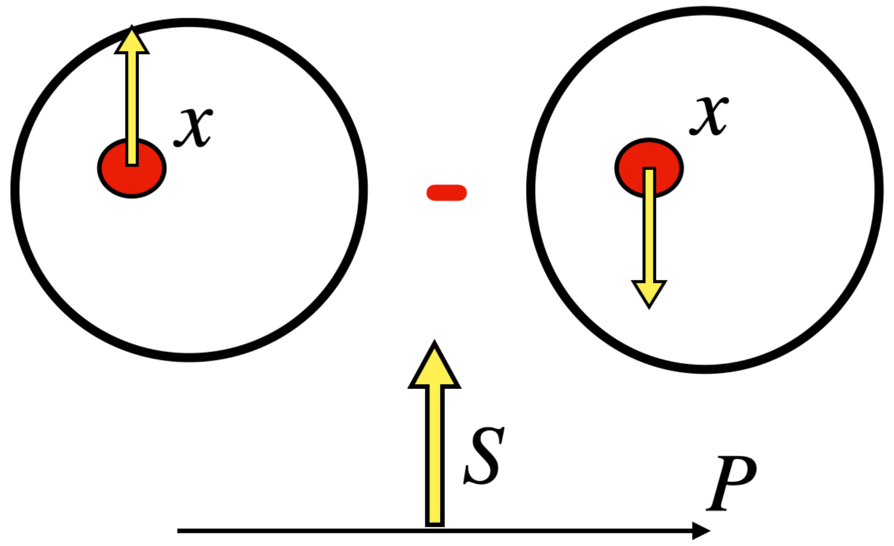


It needs to couple to another chiral-odd function

Collins effect

Alternative method to extract trasversity PDF

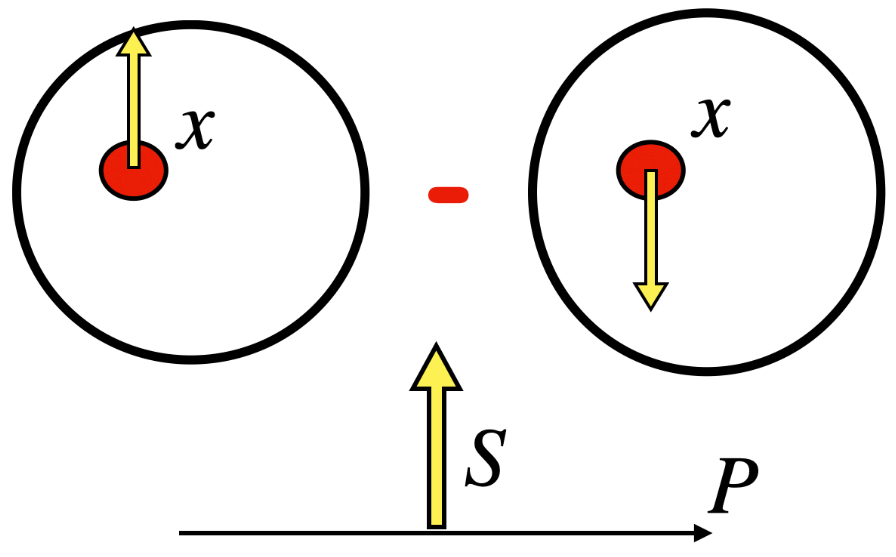


Chiral-odd function → **It needs to couple to another chiral-odd function**

Collins effect **Ex.: SIDIS**

$$h_1^q(x_B, k_T) \otimes H_1^{\perp,q}(z, p_\perp)$$

Alternative method to extract trasversity PDF



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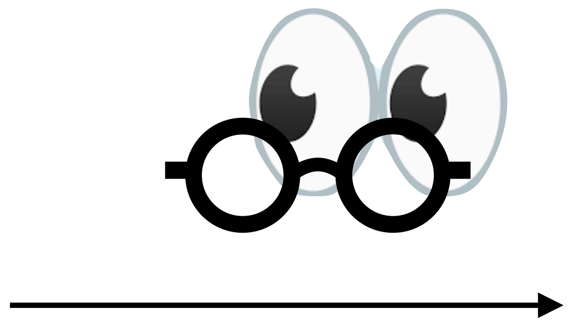


It needs to couple to another chiral-odd function

Collins effect

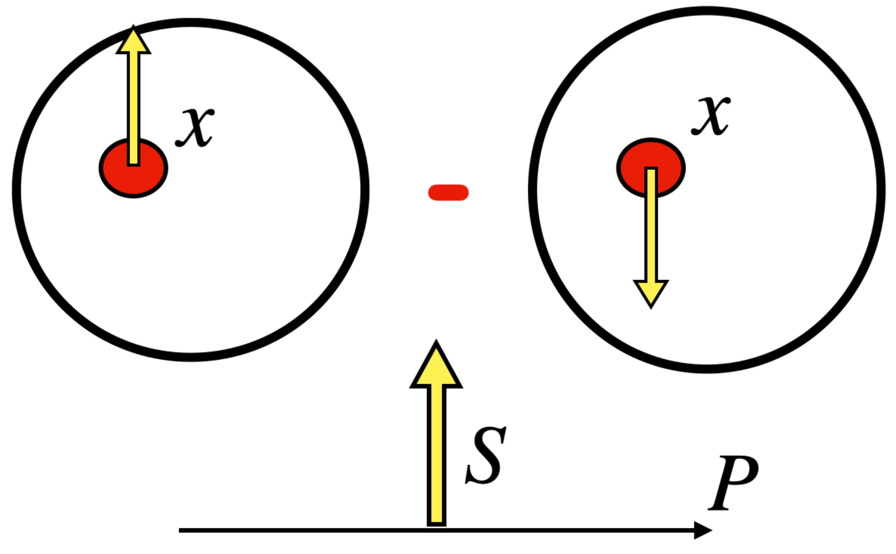
Ex.: SIDIS

$$h_1^q(x_B, k_T) \otimes H_1^{\perp,q}(z, p_\perp)$$

$\otimes$ convolution over $k_T, p_\perp$

Evolution in the framework of TMD-factorization

Alternative method to extract trasversity PDF



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Collins effect | Ex.: SIDIS | Di-hadron

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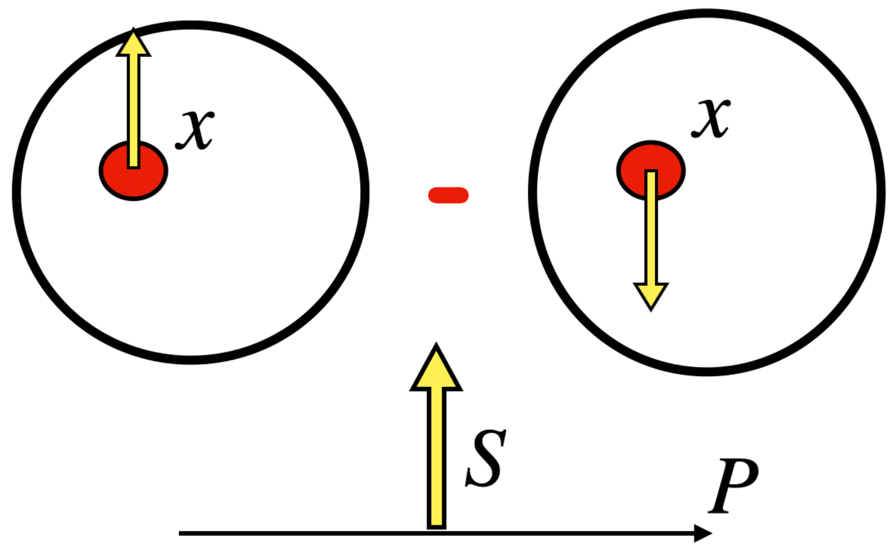

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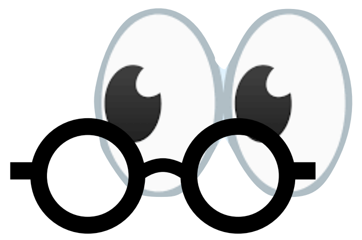

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Ex.: SIDIS

Di-hadron

$$h_1^q(x_B, k_T) \otimes H_1^{\perp,q}(z, p_\perp)$$

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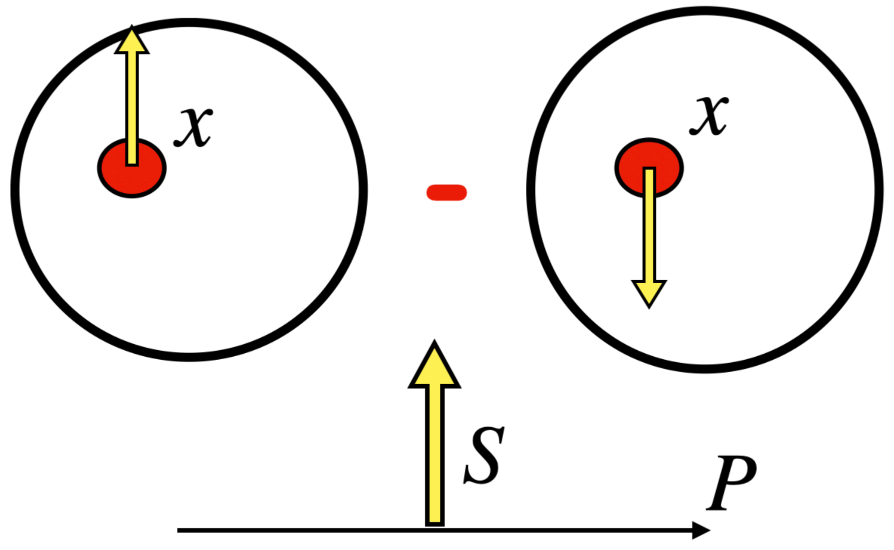
Evolution in the framework of TMD-factorization

$$h_1^q(x_B) \cdot H_1^{\sphericalangle,q}(z, M_h)$$

Alternative method to extract trasversity PDF



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Collins effect | Ex.: SIDIS | Di-hadron

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Evolution in the framework of TMD-factorization

Di-hadron

$$h_1^q(x_B) \cdot H_1^{\sphericalangle,q}(z, M_h)$$

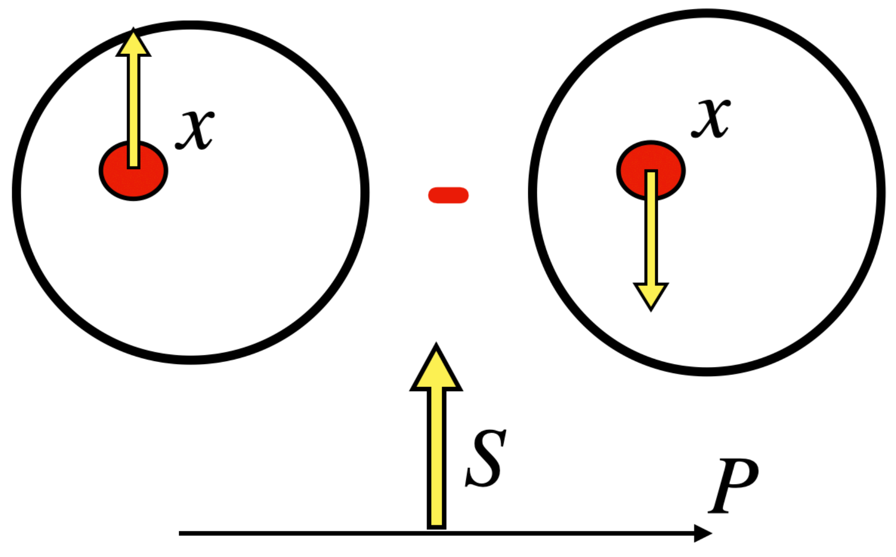
Simple products

Collinear Framework

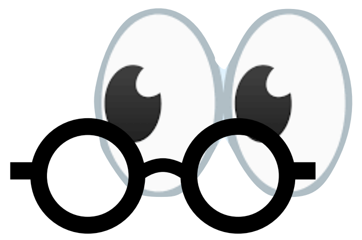
DGLAP eq. If $M_h \ll Q^2$

$h_1^q H_1^{\sphericalangle,q}$ **also in** $pp^\uparrow$ **collisions**

Alternative method to extract trasversity PDF



Chiral-odd function



It needs to couple to another chiral-odd function

Collins effect | **Ex.: SIDIS** | **Di-hadron**

$$h_1^q(x_B, k_T) \otimes H_1^{\perp,q}(z, p_\perp)$$

$\otimes$ convolution over $k_T, p_\perp$

Evolution in the framework of TMD-factorization

$$A_{UT}^{hh} \sim \frac{\sum_q e_q^2 \cdot h_1^q(x_B) \cdot H_1^{\sphericalangle,q}(z, M_h)}{\sum_q e_q^2 f_1^q(x_B) \cdot D_1^q(z, M_h)}$$

Simple products

Collinear Framework
DGLAP eq. If $M_h \ll Q^2$

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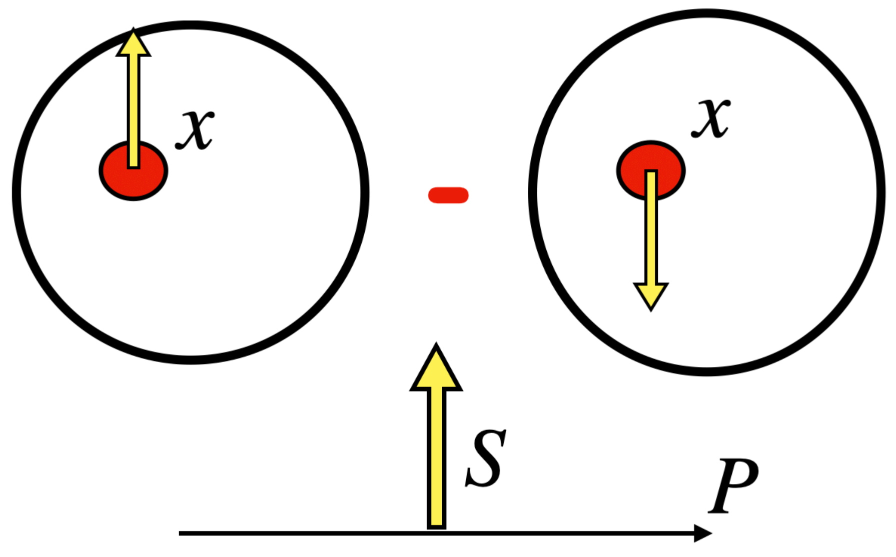

Chiral-odd function → It needs to couple to another chiral-odd function

Ex.: SIDIS

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Evolution in the framework of TMD-factorization

Di-hadron LO

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D_1 extraction

Previous extractions

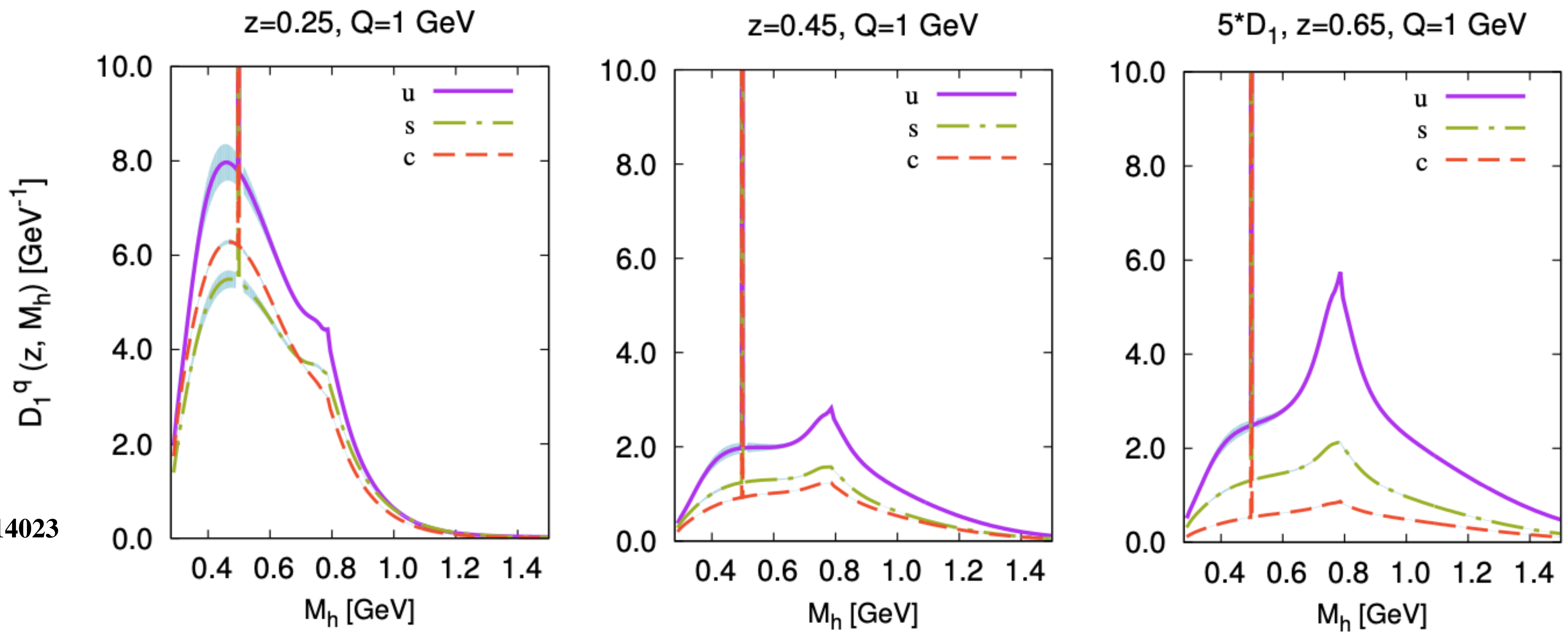
2012 extraction by Pavia group

Fit of MonteCarlo simulation

79 free parameters

LO

A.Bacchetta, M.Radici, A.Bianconi, A.Courtoy, Phys. Rev. D 85, (2012) 114023



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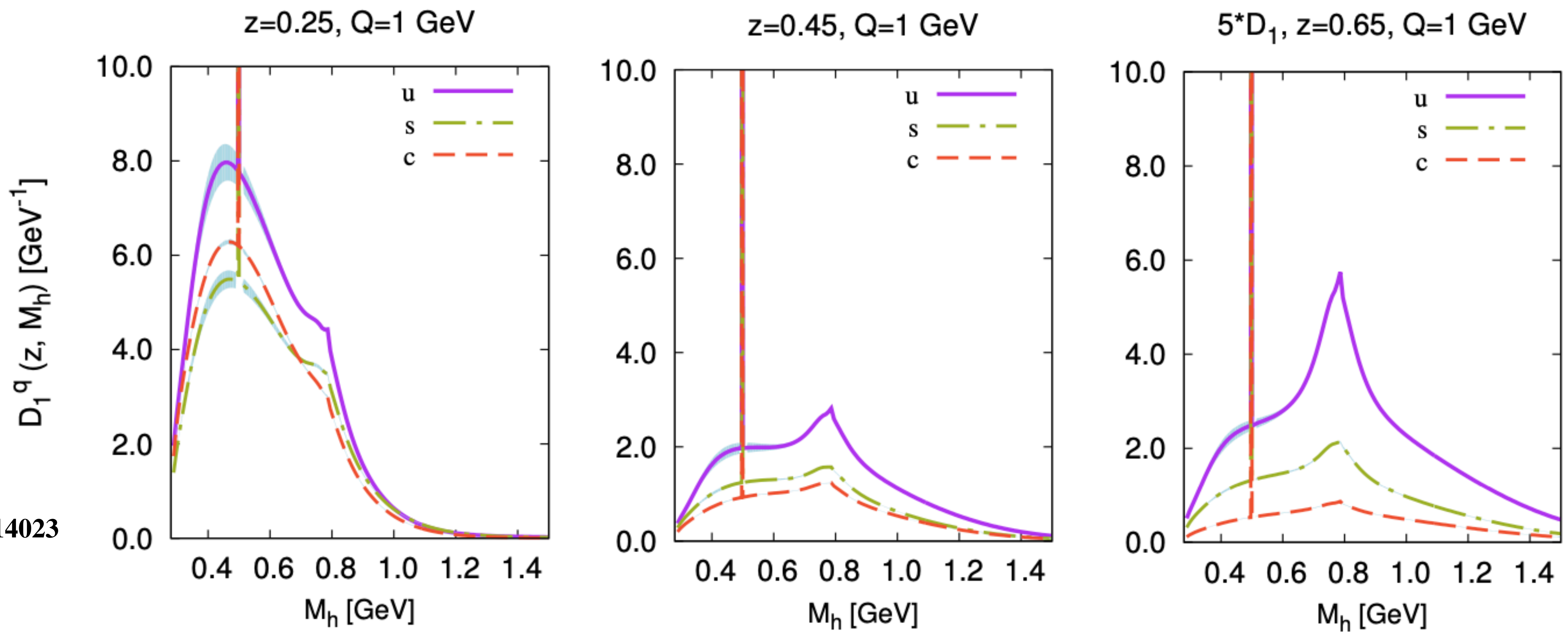


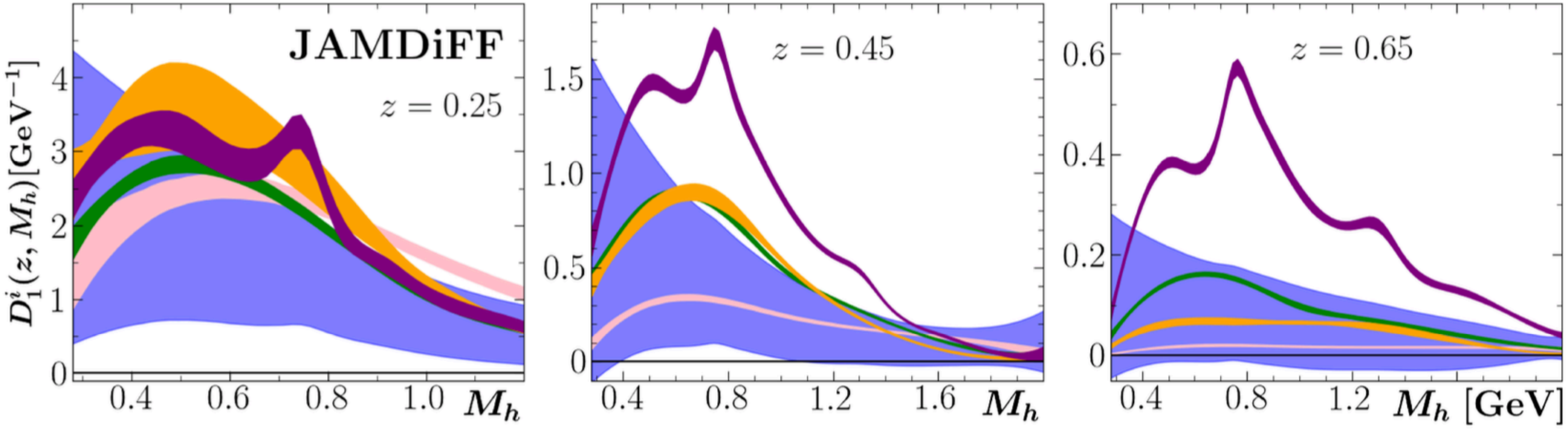
Latest extraction by JAM (2024)

2017 BELLE data at $\sqrt{S} = 10.58$ GeV

+ MonteCarlo simulation

195 free parameters

LO



N.Sato et al, Phys. Rev. D 109, (2024) 034024

D_1 extraction, GOALS

- **2017 BELLE data of $e^+e^- \rightarrow \pi^+\pi^- X$ at $\sqrt{S} = 10.58$ GeV**

D_1 extraction, GOALS

- **2017 BELLE data of $e^+e^- \to \pi^+\pi^- X$ at $\sqrt{S} = 10.58$ GeV**
- **Use of Monte-Carlo simulation for flavor separation only**

D_1 extraction, GOALS

- 2017 BELLE data of $e^+e^- \to \pi^+\pi^- X$ at $\sqrt{S} = 10.58$ GeV
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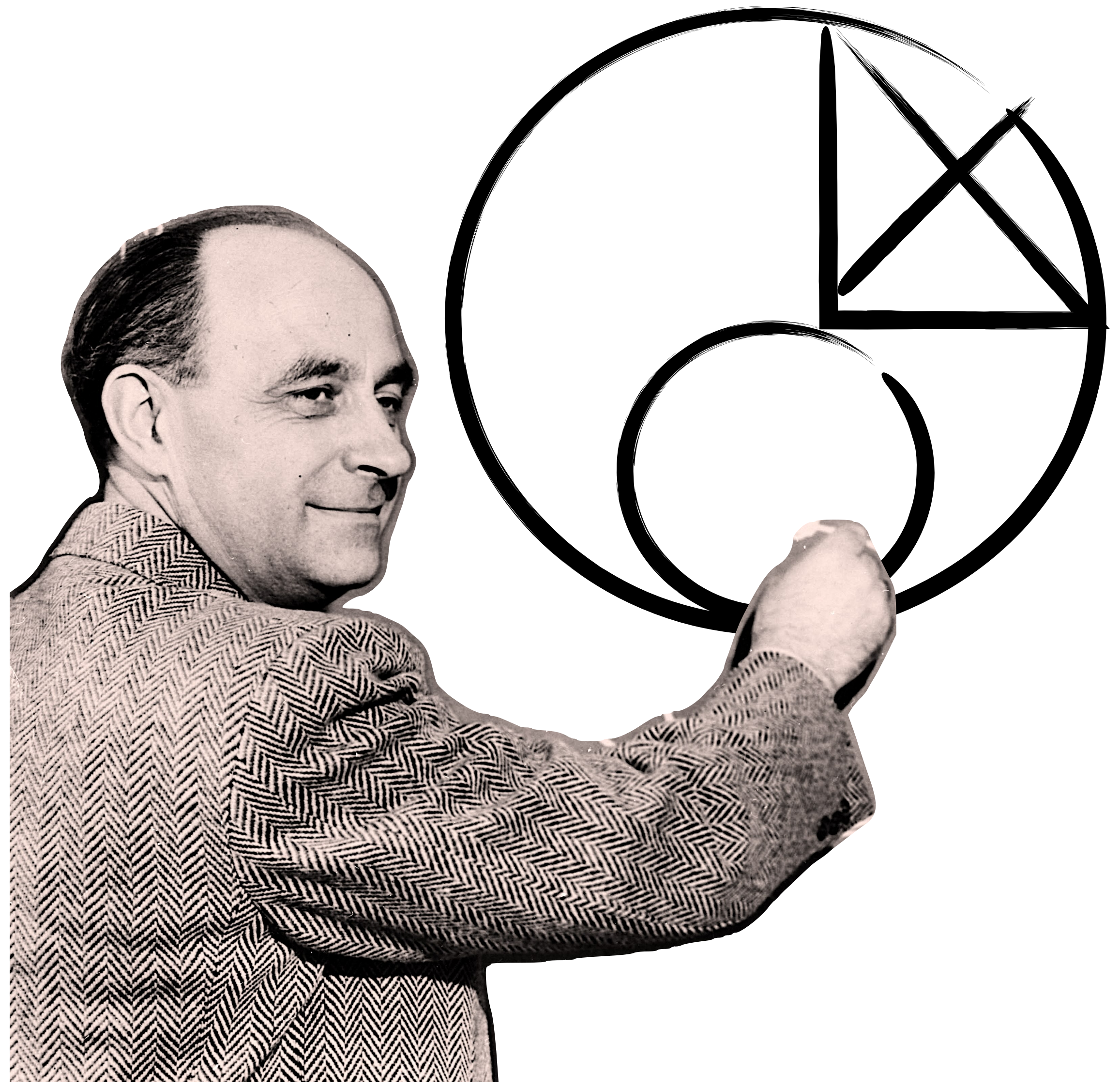
- Push the perturbative accuracy up to NNLO

D_1 extraction, GOALS

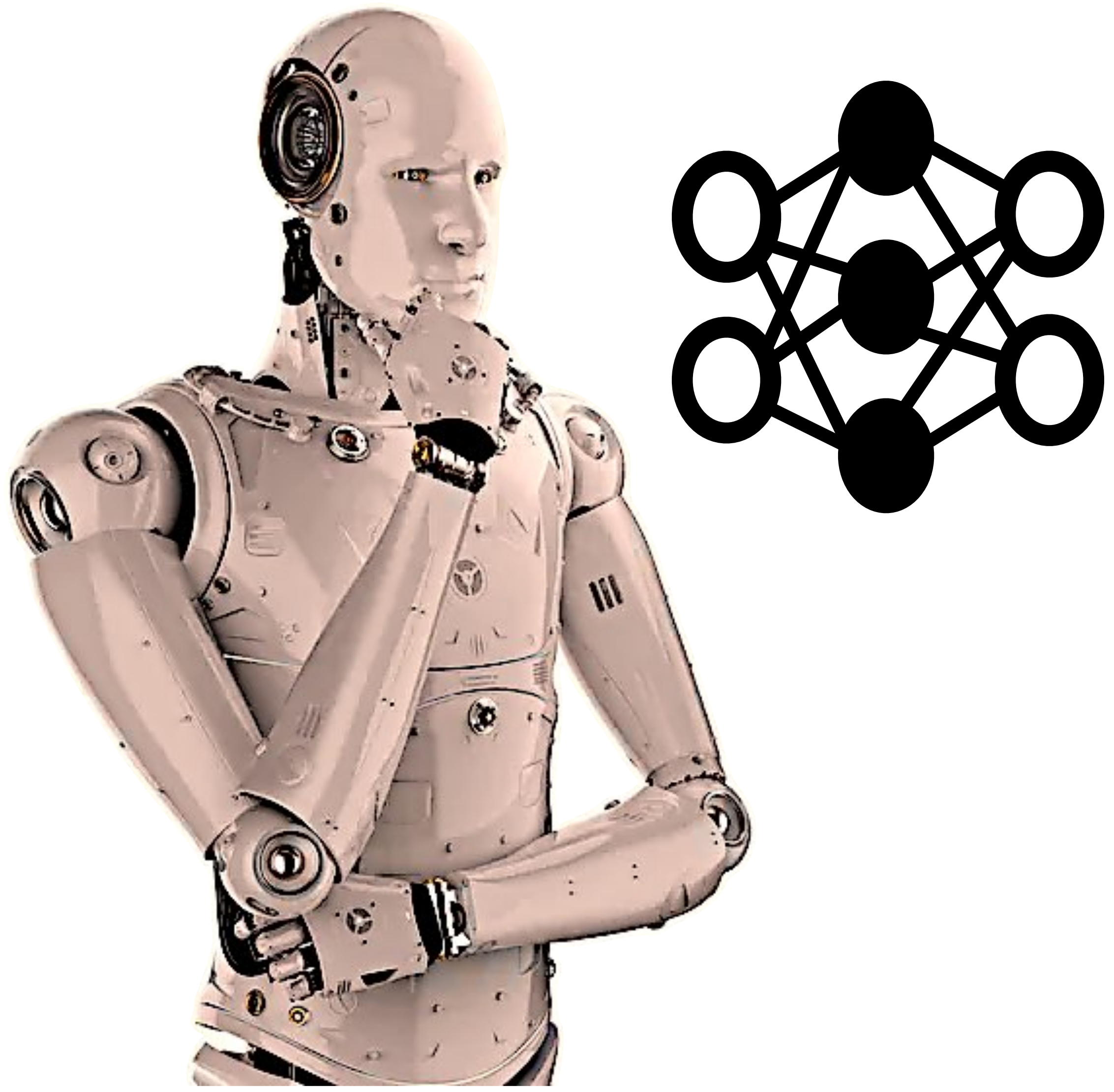
- 2017 BELLE data of $e^+e^- \to \pi^+\pi^- X$ at $\sqrt{S} = 10.58$ GeV
- Use of Monte-Carlo simulation for flavor separation only
- Push the perturbative accuracy up to NNLO
- Explore a Neural Network parameterisation

PHYSICS INFORMED



71 par

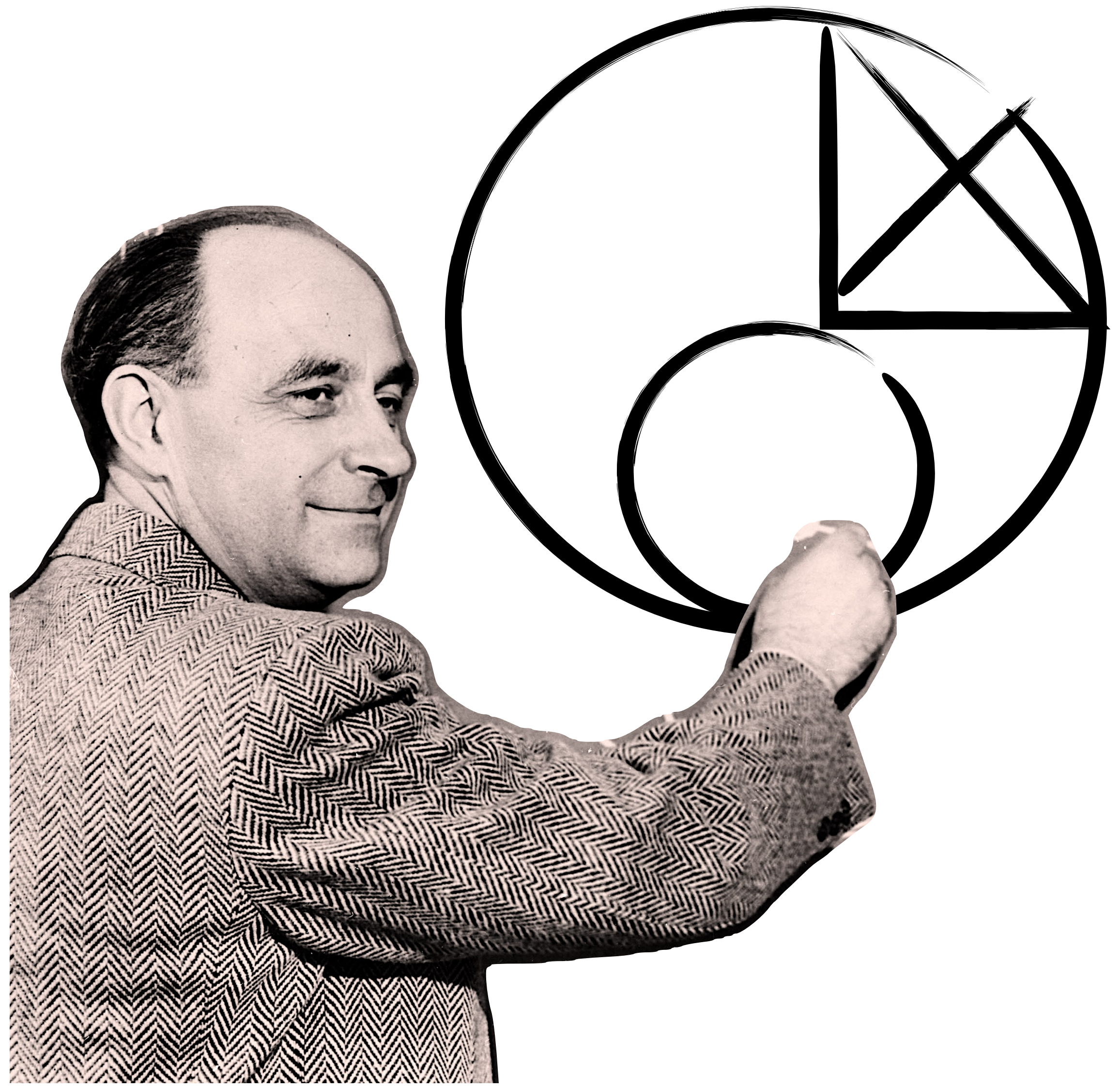
NEURAL NETWORK



NN architecture

[2,25,5]

PHYSICS INFORMED



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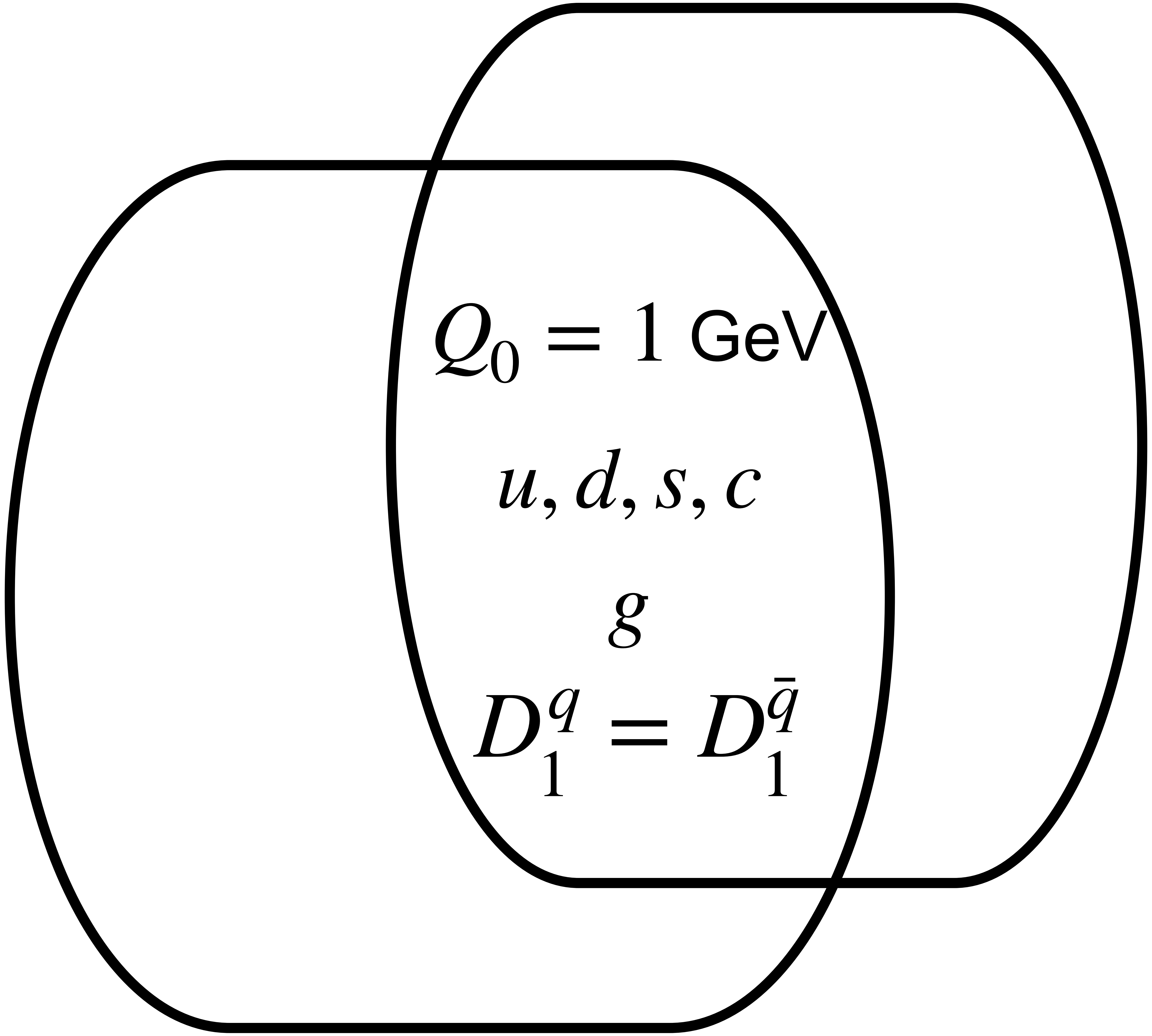
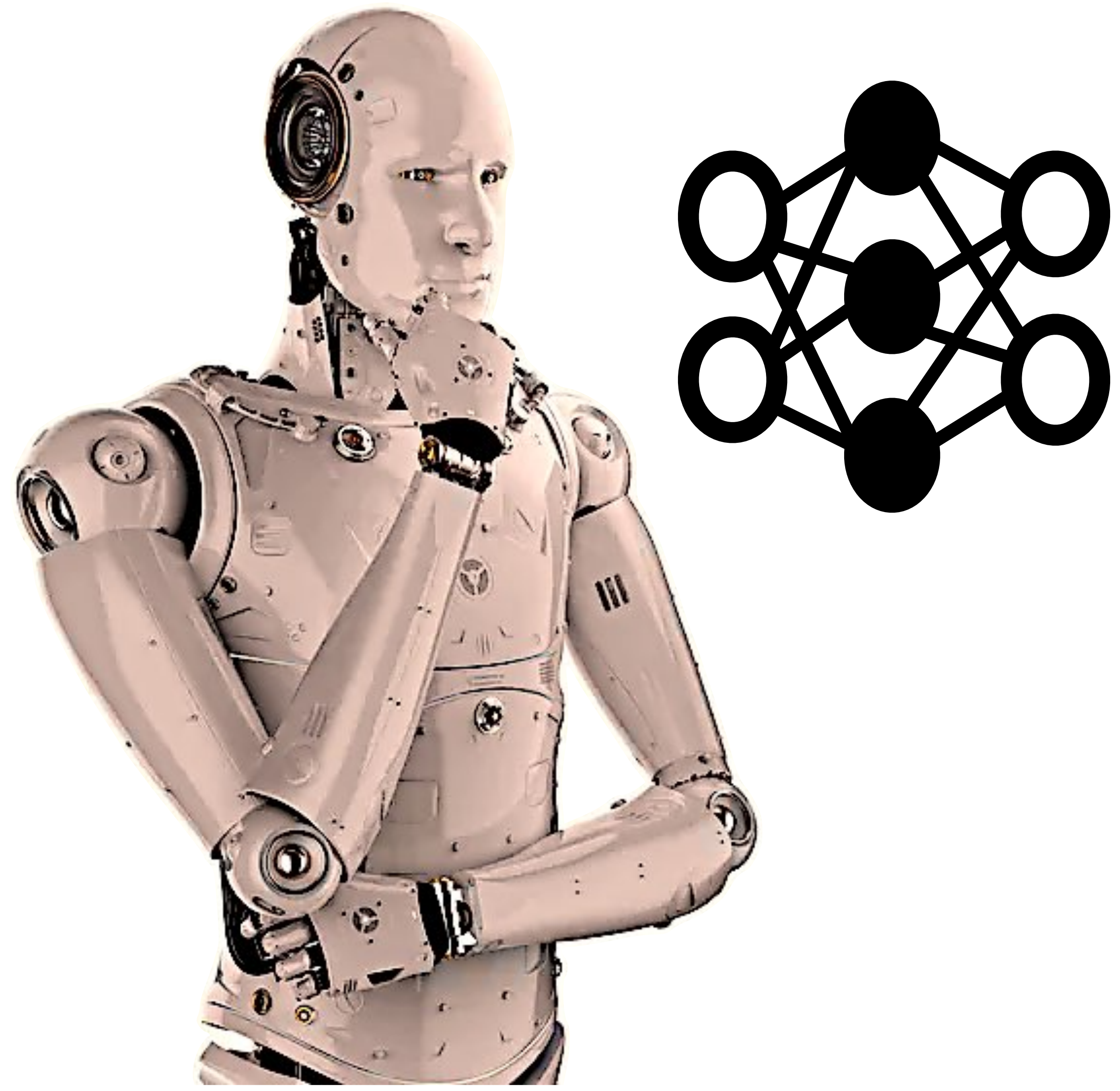


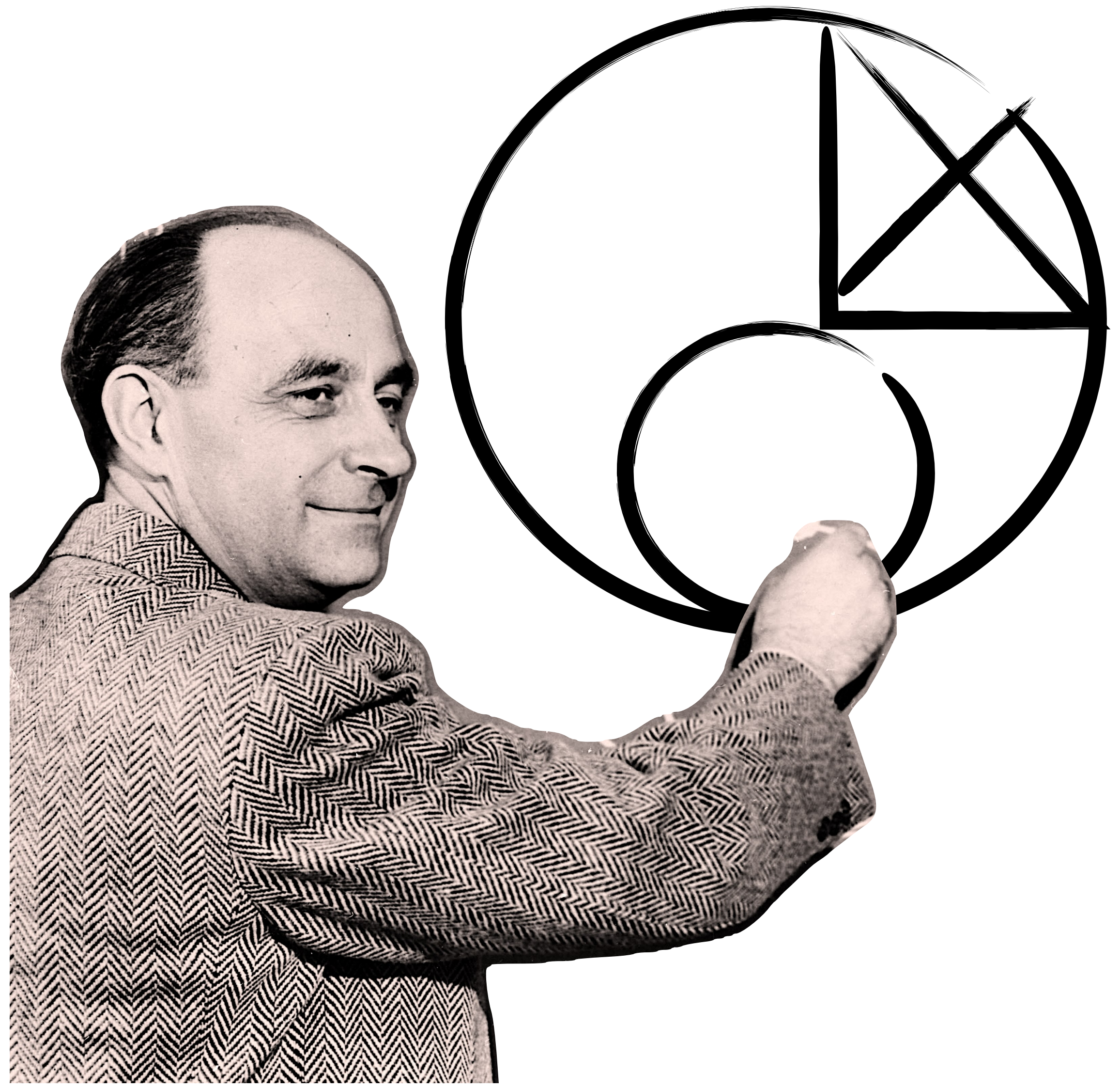
NEURAL NETWORK



NN architecture

[2,25,5]

PHYSICS INFORMED



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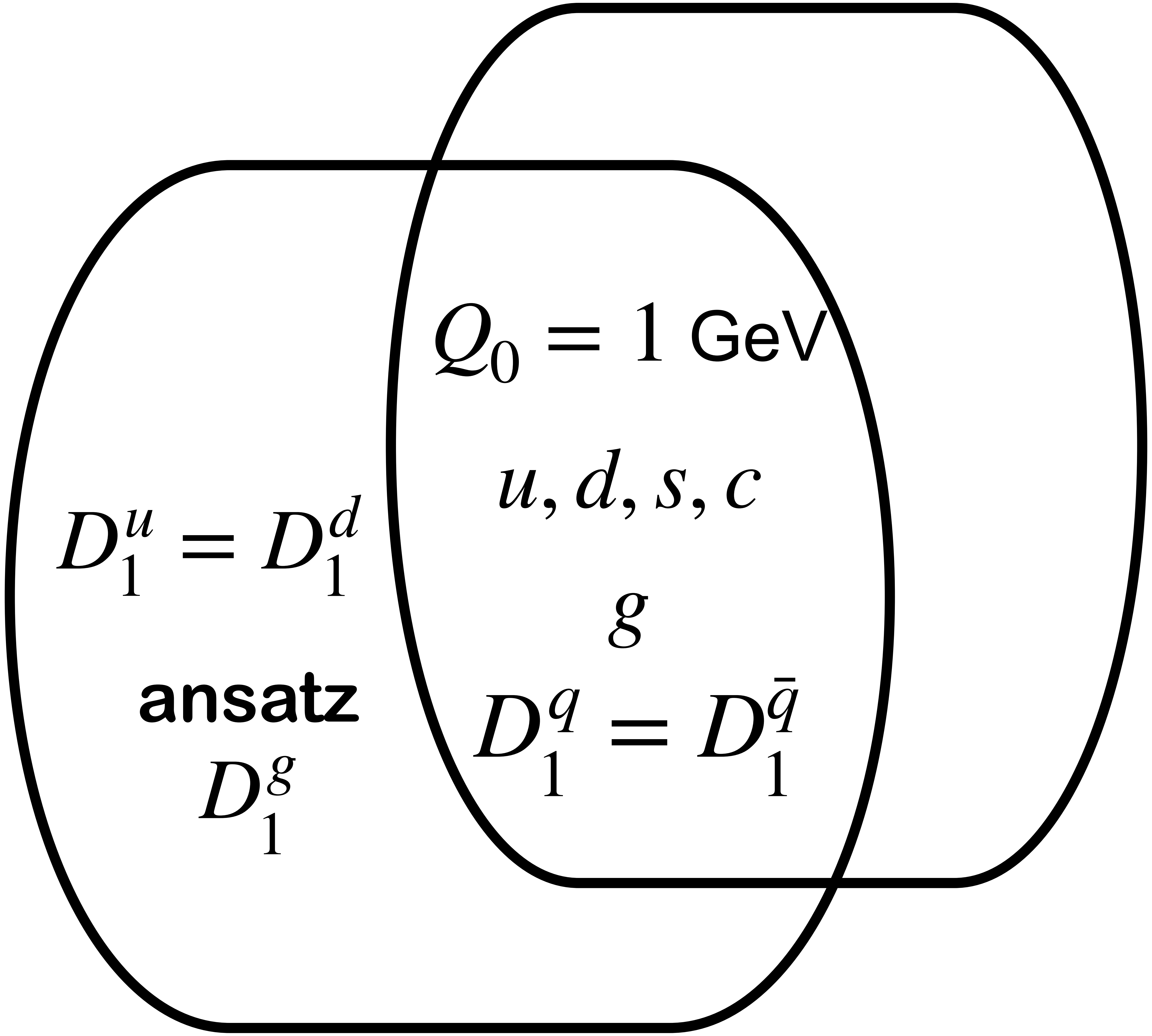
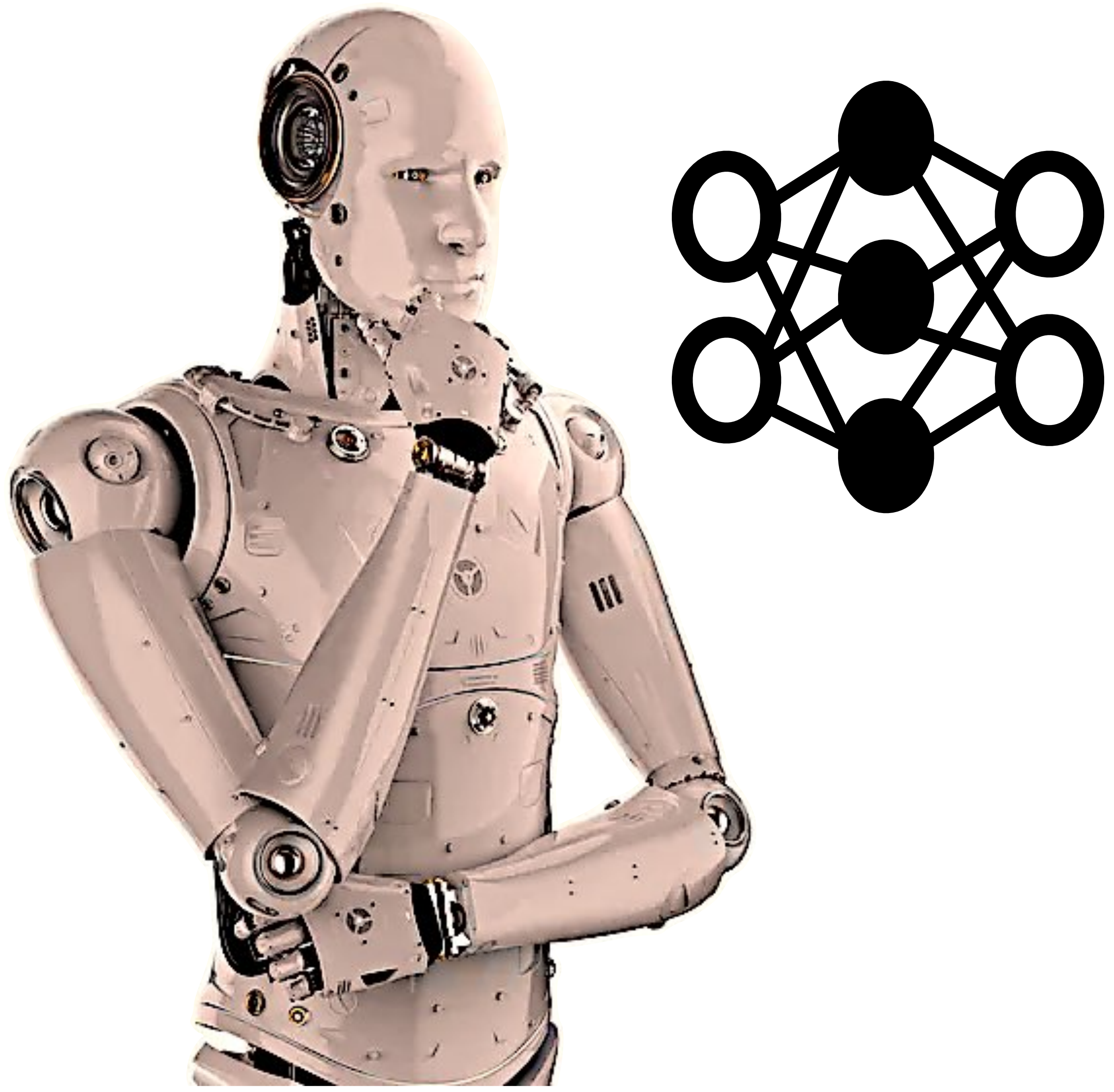


NEURAL NETWORK

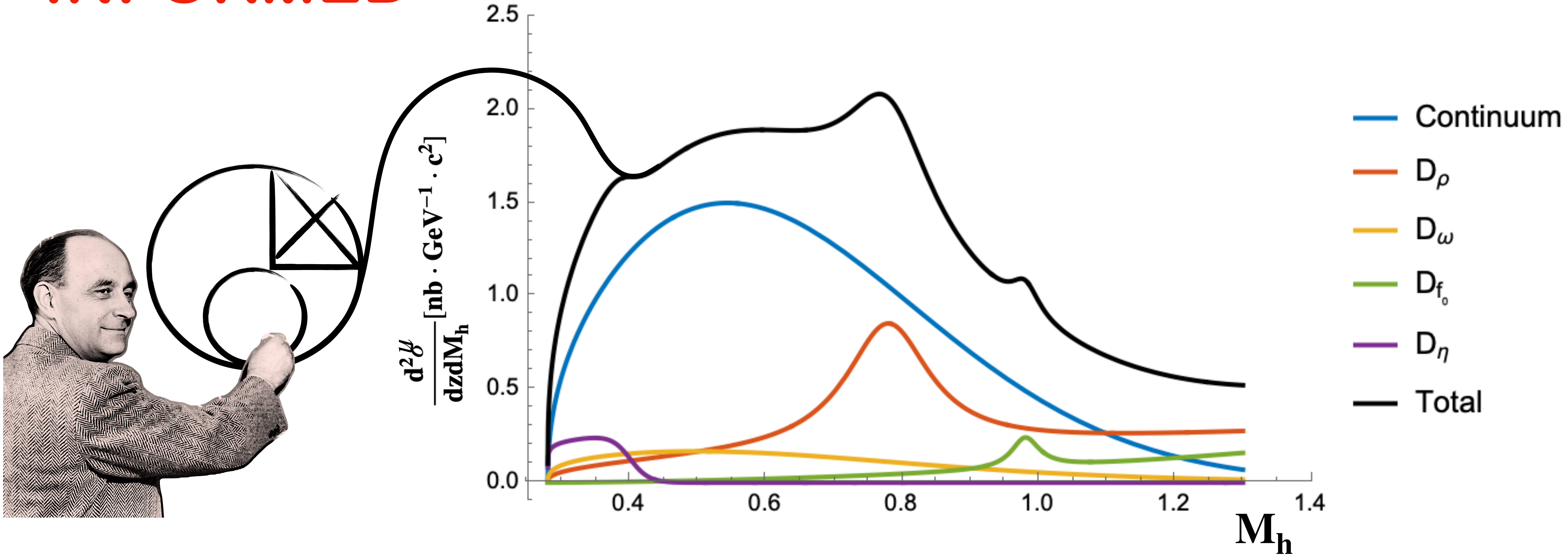


NN architecture

[2,25,5]

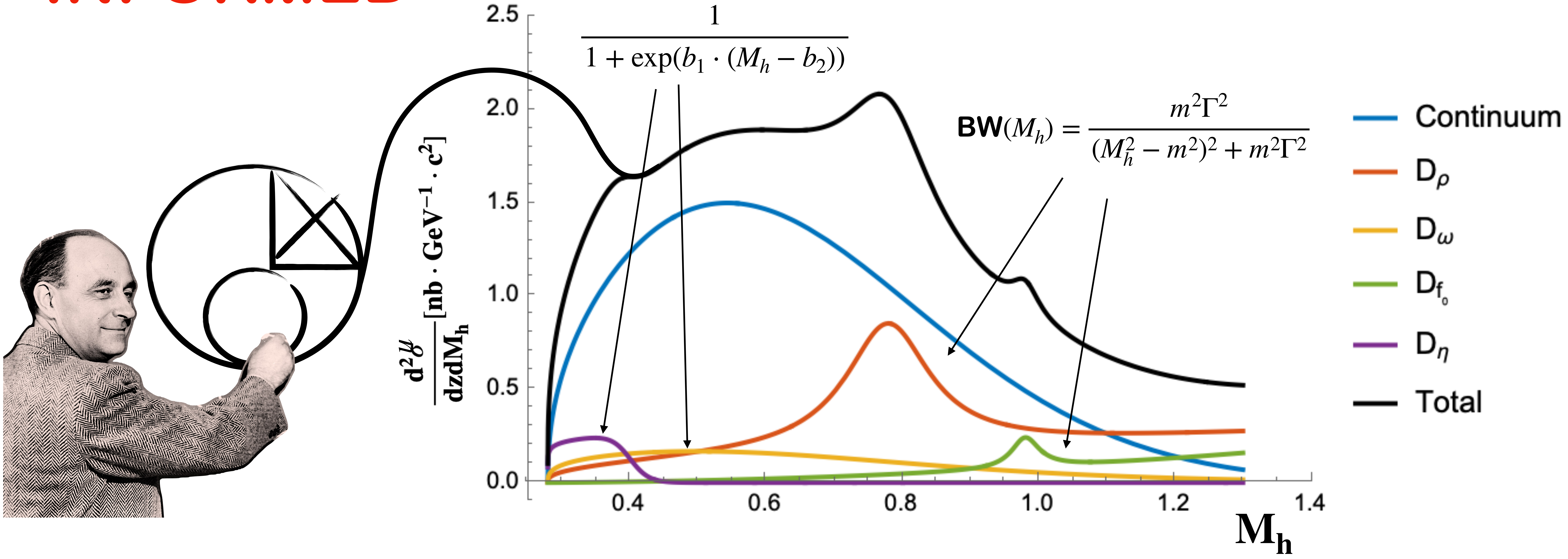
PHYSICS INFORMED

Example for the up quark parameterisation



PHYSICS INFORMED

Example for the up quark parameterisation



$$\mathbf{R}(M_h) = \frac{1}{2}\sqrt{M_h^2 - 4\,m_\pi^2}$$

$$z^\alpha(1-z)^\beta$$

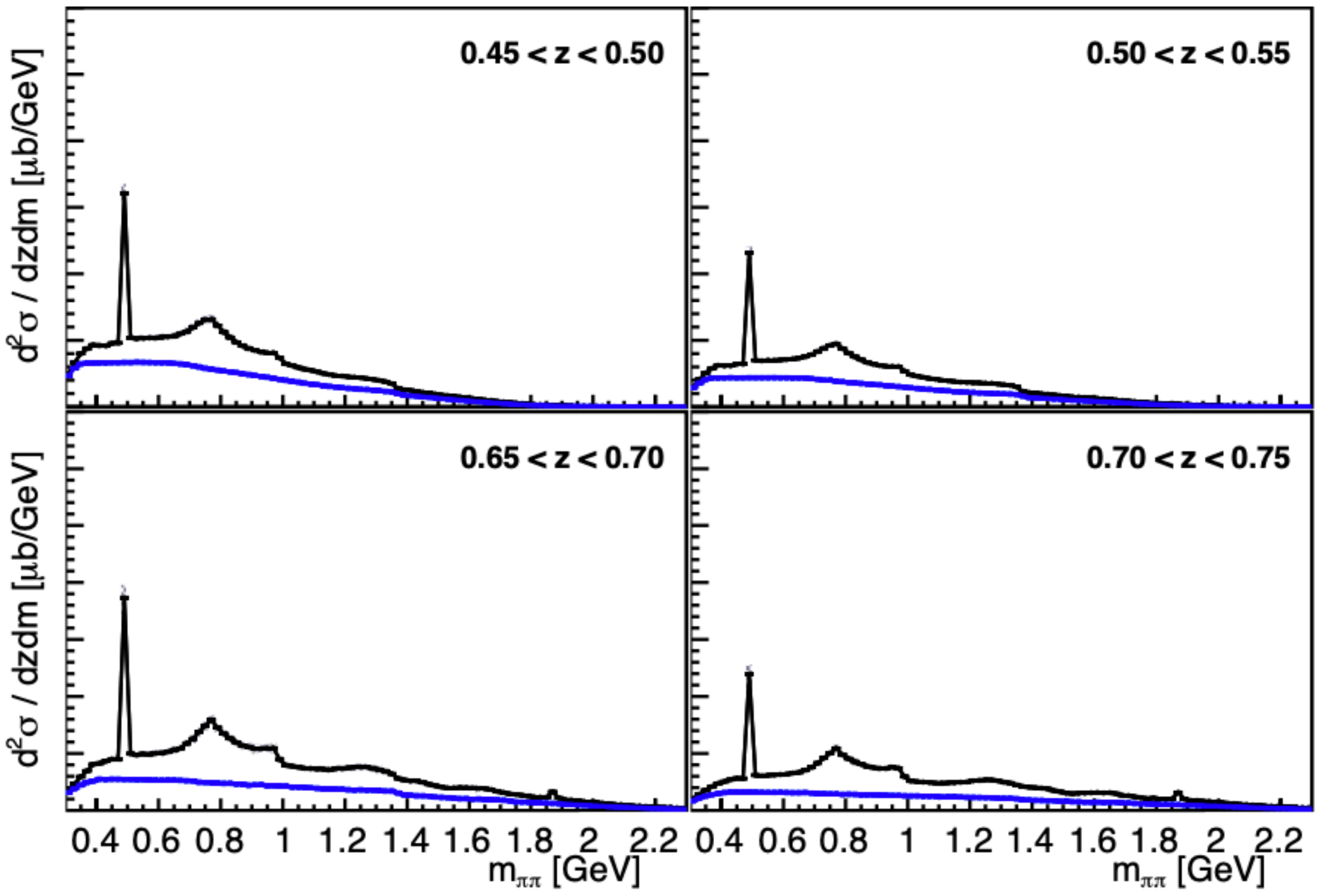
$$\mathbf{P}(a_1, a_2, a_3, a_4, a_5; z) = \frac{a_1}{z} + a_2 + a_3 \cdot z + a_4 \cdot z^2 + a_5 \cdot z^3$$

Flavour analysis

Flavour analysis

Physical review D 96 (2017)
R.Siedl et al

2017 BELLE data of $e^+e^- \to \pi^+\pi^- X$
at $\sqrt{S} = 10.58$ **GeV**



Flavour analysis

Physical review D 96 (2017)
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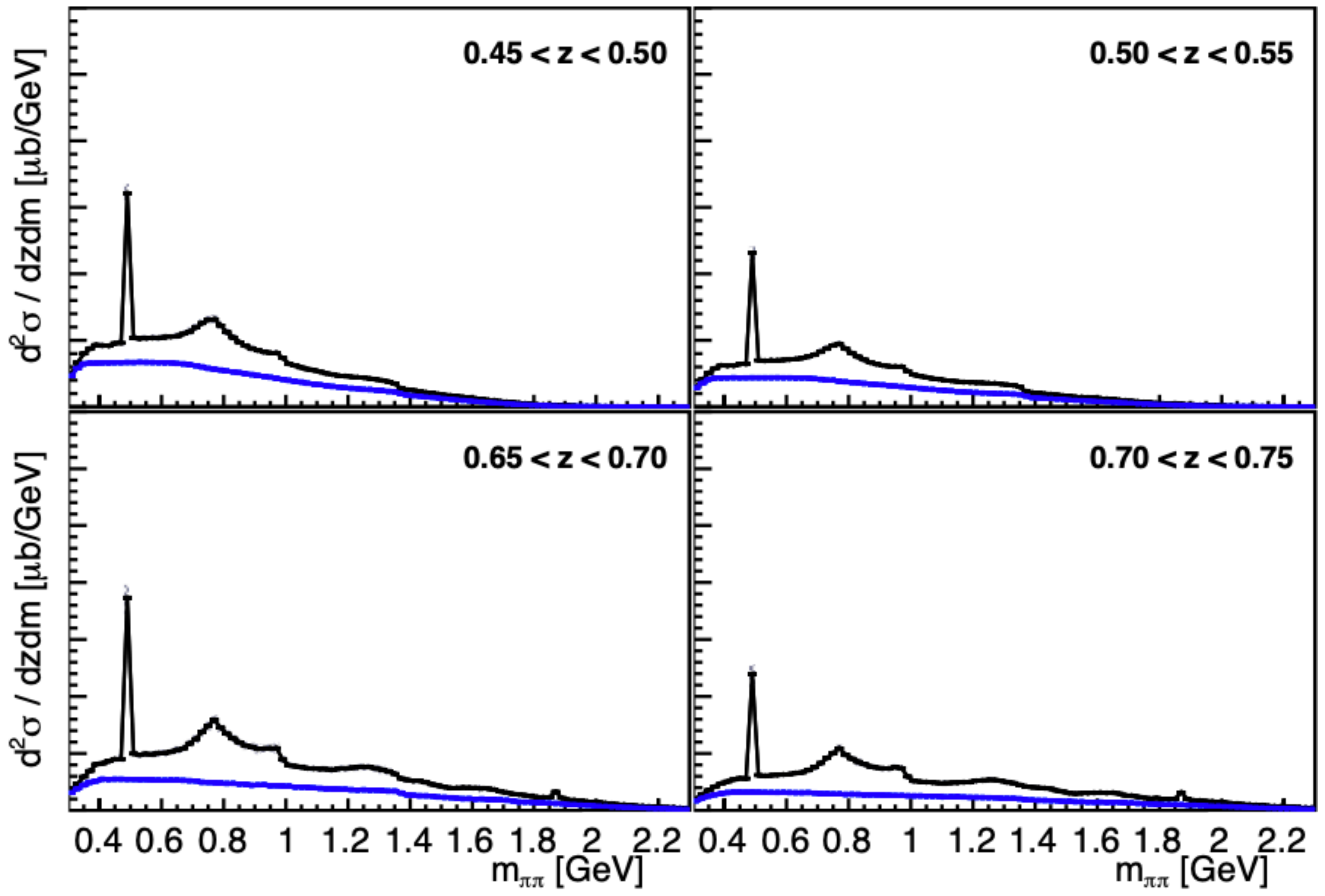
$$\frac{d\sigma}{dzdM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2}\sum_q e_q^2 D_1^q(z, M_h, Q) =$$



Flavour analysis

$$\frac{d\sigma}{dz dM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2} \sum_q e_q^2 D_1^q(z, M_h, Q)$$

Sum over $q, \bar{q}$

Flavour analysis

$$\frac{d\sigma}{dz dM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2} \sum_q e_q^2 D_1^q(z, M_h, Q) = \sum_q \frac{d\sigma^{\mathbf{q}}}{dz dM_h dQ^2}$$

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Flavour analysis

$$\frac{d\sigma}{dz dM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2}\sum_q e_q^2 D_1^q(z, M_h, Q) = \sum_q \frac{d\sigma^{\mathbf{q}}}{dz dM_h dQ^2}$$

Sum over $q, \bar{q}$

$$= 2 \cdot \left(\frac{4\pi\alpha^2}{Q^2} e_{\mathbf{u}}^2 D_1^{\mathbf{u}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{d}}^2 D_1^{\mathbf{d}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{s}}^2 D_1^{\mathbf{s}}(z, M_h, Q) + \ldots \right)$$

Flavour analysis

$$\frac{d\sigma}{dz dM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2}\sum_q e_q^2 D_1^q(z, M_h, Q) = \sum_q \frac{d\sigma^{\mathbf{q}}}{dz dM_h dQ^2}$$

Sum over $q, \bar{q}$

$q, \bar{q}$

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Flavour analysis

$$\frac{d\sigma}{dz dM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2} \sum_q e_q^2 D_1^q(z, M_h, Q) = \sum_q \frac{d\sigma^{\mathbf{q}}}{dz dM_h dQ^2}$$

Sum over $q, \bar{q}$

$$= 2 \cdot \left(\frac{4\pi\alpha^2}{Q^2} e_{\mathbf{u}}^2 D_1^{\mathbf{u}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{d}}^2 D_1^{\mathbf{d}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{s}}^2 D_1^{\mathbf{s}}(z, M_h, Q) + \ldots \right)$$

$$= 2 \cdot \frac{d\sigma}{dz dM_h dQ^2} \cdot \left(R^{\mathbf{u}}(z, M_h, Q) + R^{\mathbf{d}}(z, M_h, Q) + R^{\mathbf{s}}(z, M_h, Q) + R^{\mathbf{c}}(z, M_h, Q) \right)$$

Flavour analysis

$$\frac{d\sigma}{dzdM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2}\sum_q e_q^2 D_1^q(z, M_h, Q) = \sum_q \frac{d\sigma^{\mathbf{q}}}{dzdM_h dQ^2}$$

Sum over $q, \bar{q}$

$$= 2 \cdot \left(\frac{4\pi\alpha^2}{Q^2} e_{\mathbf{u}}^2 D_1^{\mathbf{u}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{d}}^2 D_1^{\mathbf{d}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{s}}^2 D_1^{\mathbf{s}}(z, M_h, Q) + \ldots \right)$$

$$= 2 \cdot \frac{d\sigma^{exp}}{dzdM_h dQ^2} \cdot \left(R_{MC}^{\mathbf{u}}(z, M_h, Q) + R_{MC}^{\mathbf{d}}(z, M_h, Q) + R_{MC}^{\mathbf{s}}(z, M_h, Q) + R_{MC}^{\mathbf{c}}(z, M_h, Q) \right)$$

$$R^{\mathbf{q}}(z, M_h, Q) = \frac{d\sigma^{\mathbf{q}}}{d\sigma} = \frac{e_{\mathbf{q}}^2 D_1^{\mathbf{q}}(z, Mh, Q)}{\sum_p e_p^2 D_1^p(z, Mh, Q)}$$

accessible from Monte Carlo simulations

Flavour analysis

$$\frac{d\sigma}{dzdM_h dQ^2} = \frac{4\pi\alpha^2}{Q^2}\sum_q e_q^2 D_1^q(z, M_h, Q) = \sum_q \frac{d\sigma^{\mathbf{q}}}{dzdM_h dQ^2}$$

Sum over $q, \bar{q}$

$$= 2\cdot\left(\frac{4\pi\alpha^2}{Q^2} e_{\mathbf{u}}^2 D_1^{\mathbf{u}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{d}}^2 D_1^{\mathbf{d}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2} e_{\mathbf{s}}^2 D_1^{\mathbf{s}}(z, M_h, Q) + \ldots\right)$$

$$= 2\cdot\frac{d\sigma^{exp}}{dzdM_h dQ^2}\cdot\left(R_{MC}^{\mathbf{u}}(z, M_h, Q) + R_{MC}^{\mathbf{d}}(z, M_h, Q) + R_{MC}^{\mathbf{s}}(z, M_h, Q) + R_{MC}^{\mathbf{c}}(z, M_h, Q)\right)$$

$$R^{\mathbf{q}}(z, M_h, Q) = \frac{d\sigma^{\mathbf{q}}}{d\sigma} = \frac{e_{\mathbf{q}}^2 D_1^{\mathbf{q}}(z, Mh, Q)}{\sum_p e_p^2 D_1^p(z, Mh, Q)}$$

accessible from Monte Carlo simulations

$$\mathscr{F}^{\mathbf{q}}(z, M_h; Q^2) = 2\cdot\frac{d\sigma^{exp}}{dzdM_h dQ^2}\cdot R_{MC}^{\mathbf{q}}(z, M_h, Q)$$

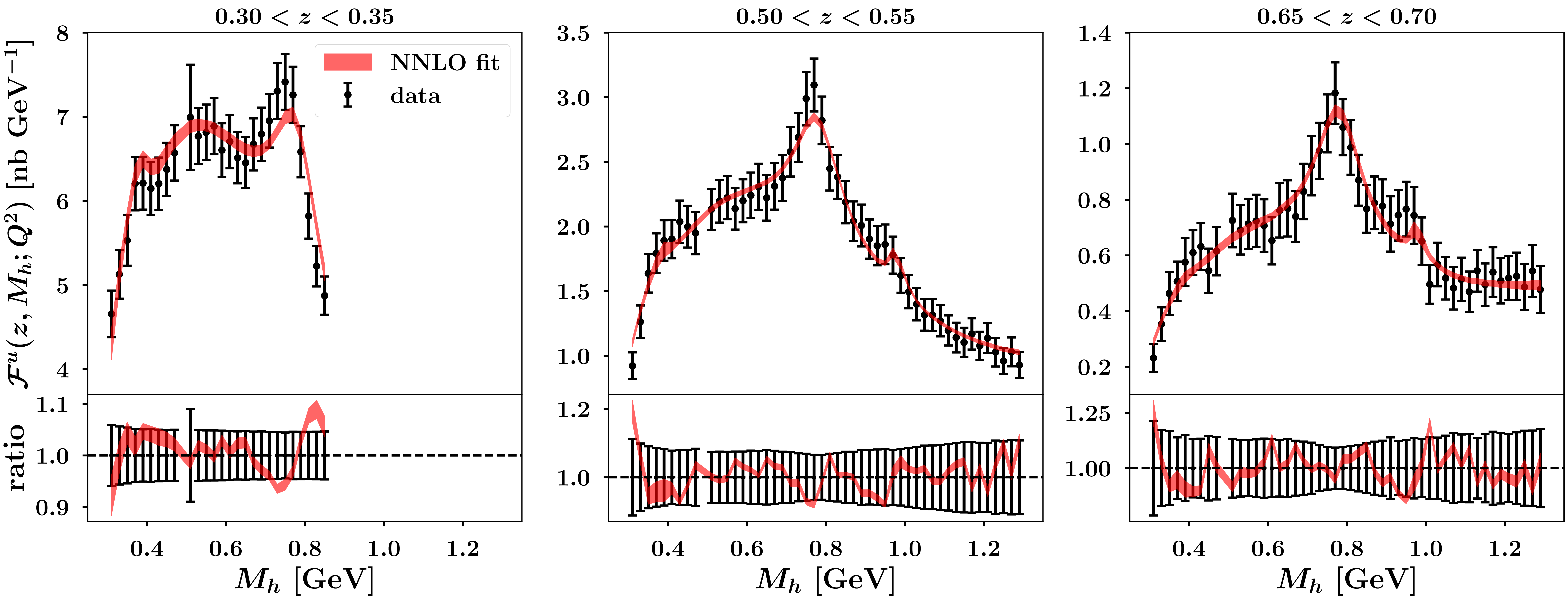
$$\mathscr{F}^q$$

$$q = u, d, s, c$$

RESULTS

V. Mahaut, L. Polano et al., JHEP02 (2026) 051

Predictions: Physics informed



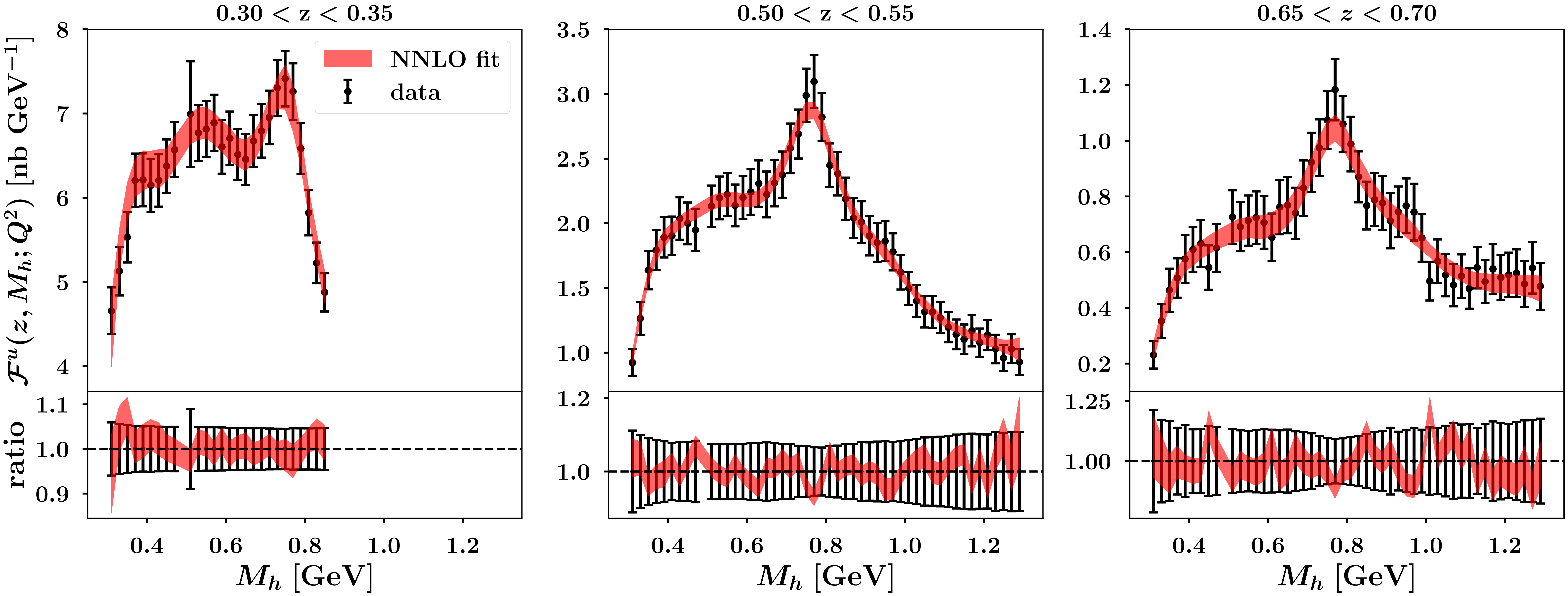
NNLO

100 replicas

$\chi^2_u = 0.380$ $\chi^2_s = 0.876$ $\chi^2_{tot} = 0.687$

$\chi^2_d = 0.613$ $\chi^2_c = 0.879$

Predictions: Neural Network



NNLO

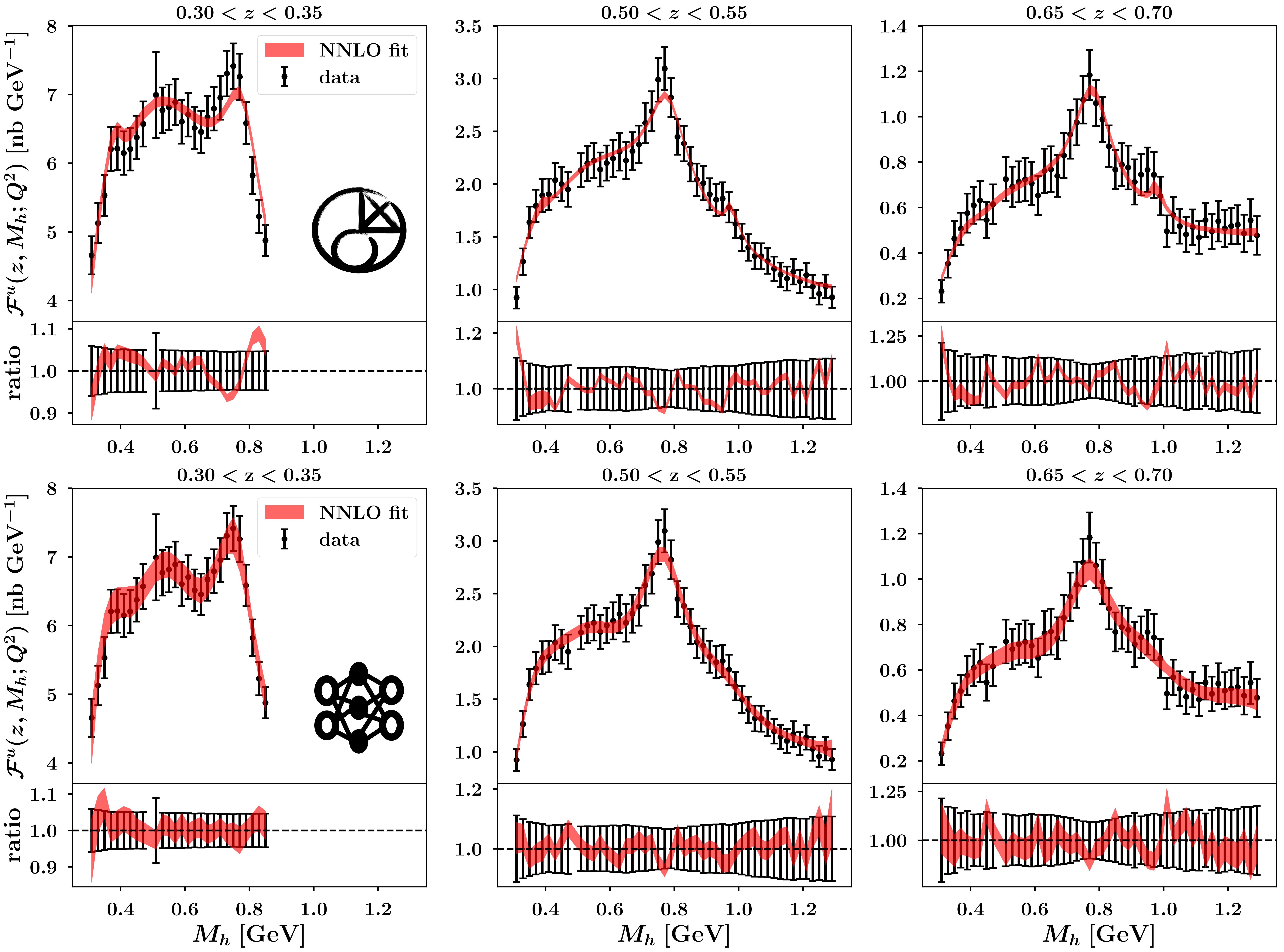
100 replicas

$\chi^2_u = 0.209$ $\chi^2_s = 0.793$ $\chi^2_{tot} = 0.535$

$\chi^2_d = 0.530$ $\chi^2_c = 0.606$

Comparison

vs

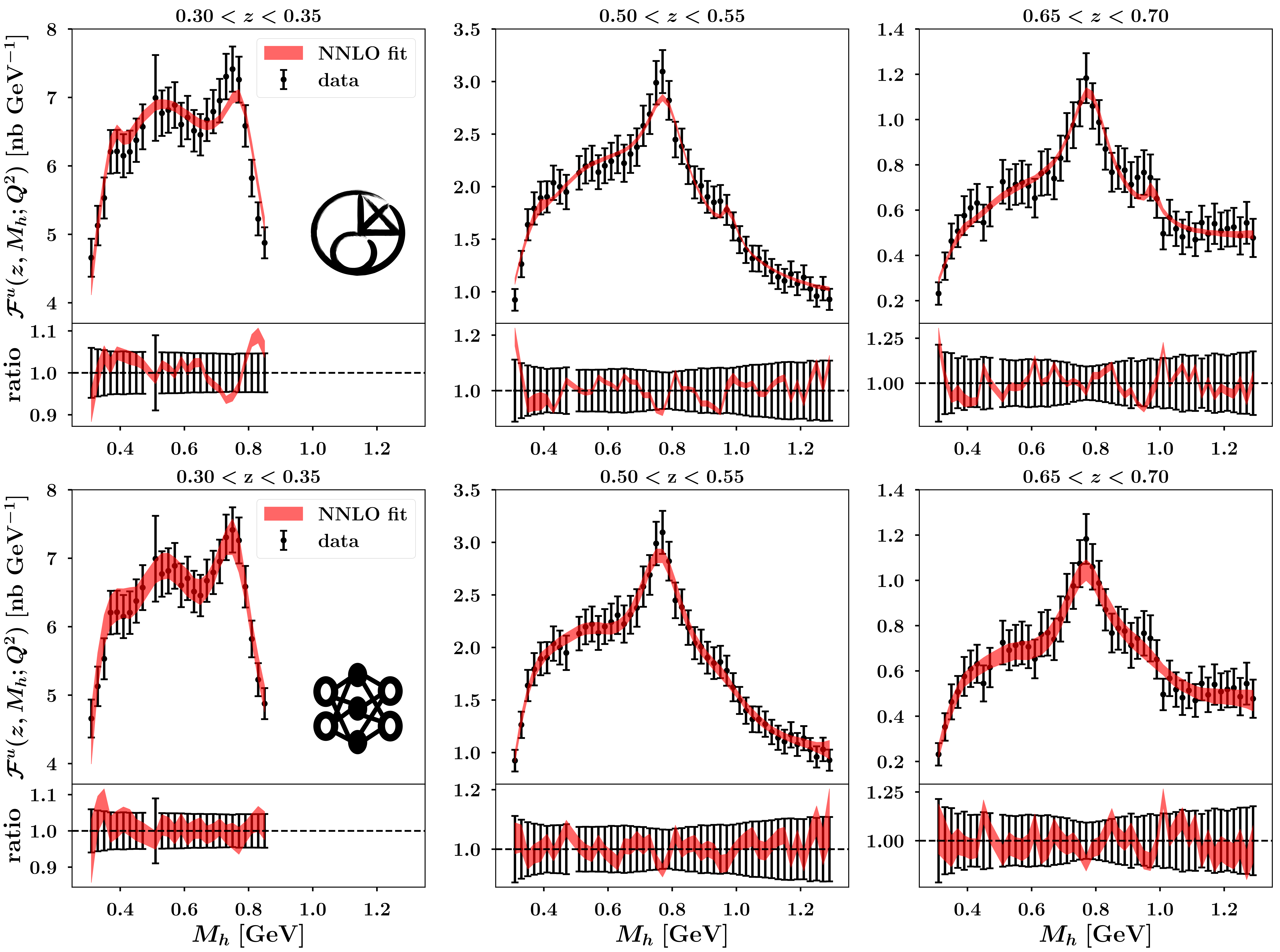


Comparison

vs

NN seems to describe better data

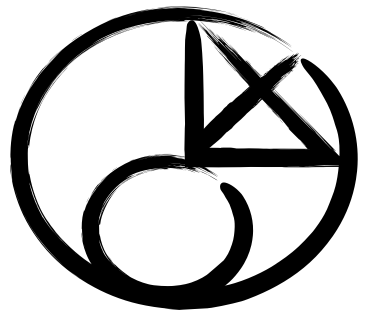
P.I. better captures the resonant structure



DI-HADRON FF

Physics informed

$\mathbf{u = d}$



gluon → assumptions

$$D_1^g(z, M_h; Q^2) = N\, z^{\alpha_1} (1 - z)^{1+\alpha_2} \cdot D_1^u(z, M_h; Q^2)$$

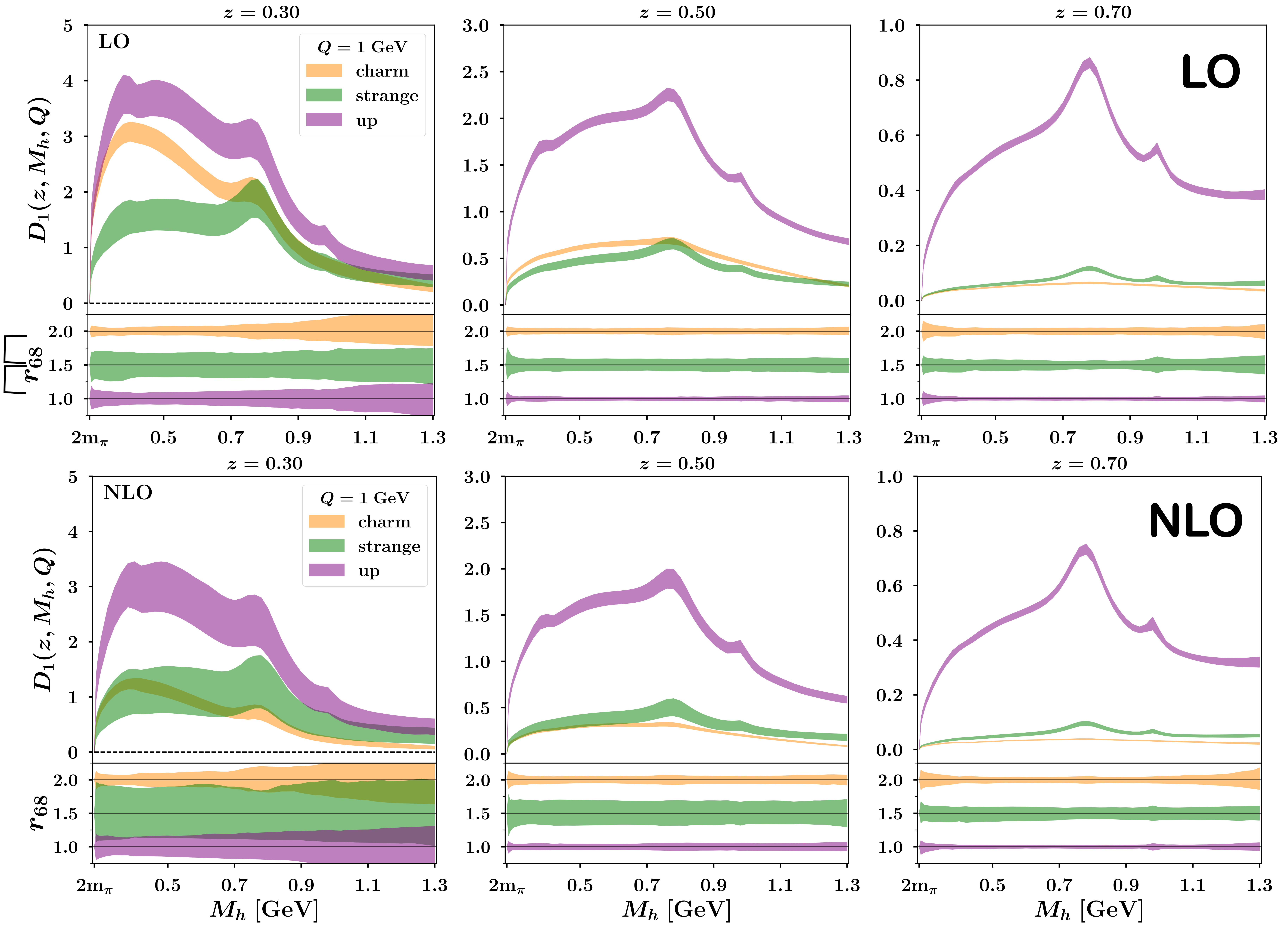
N random unif. in (0,2)

α_1, α_2 random unif. in (0,1)

DI-HADRON FF
Physics informed

u = d

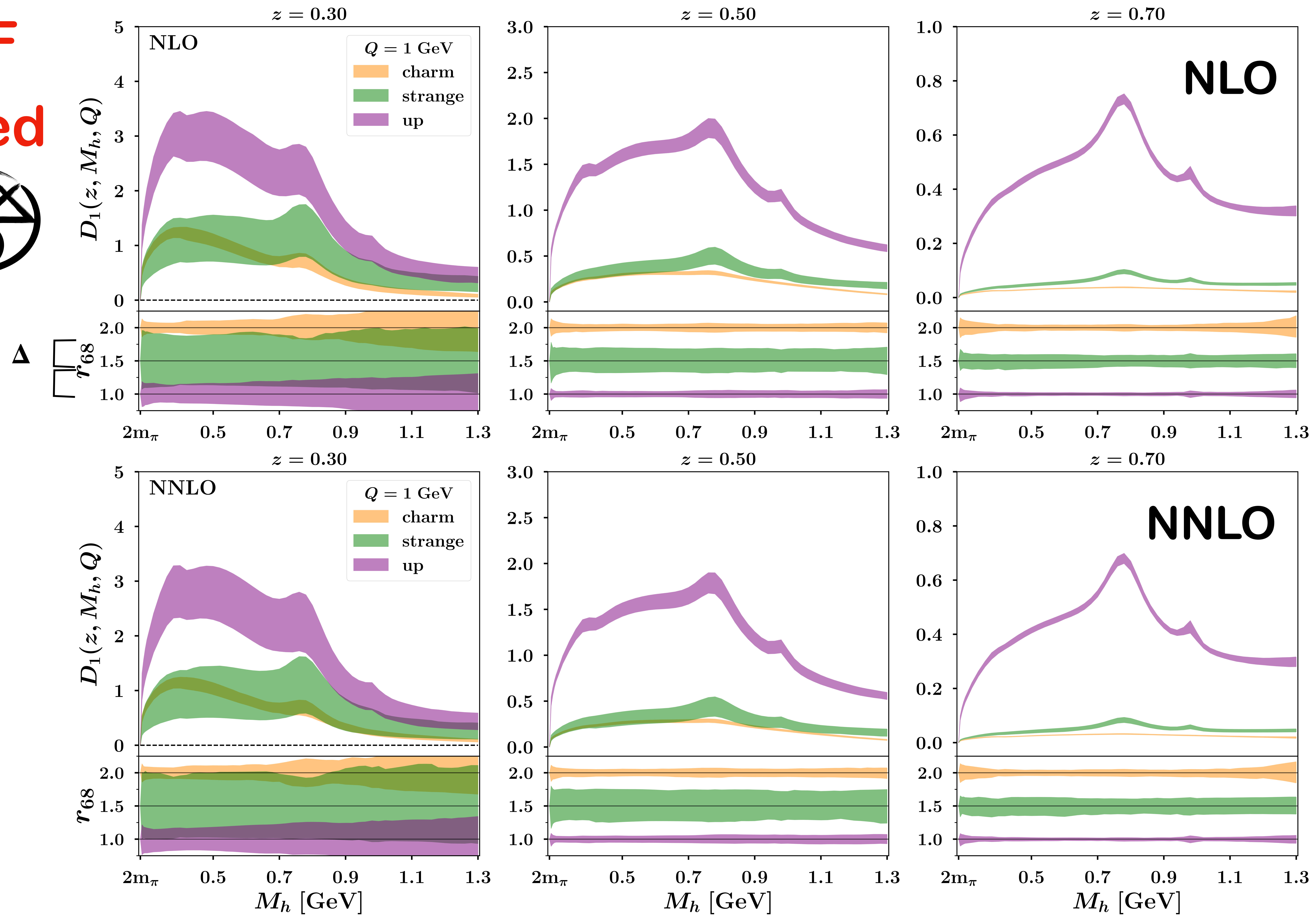
ratio offset **Δ** = 0.5



DI-HADRON FF
Physics informed

u = d

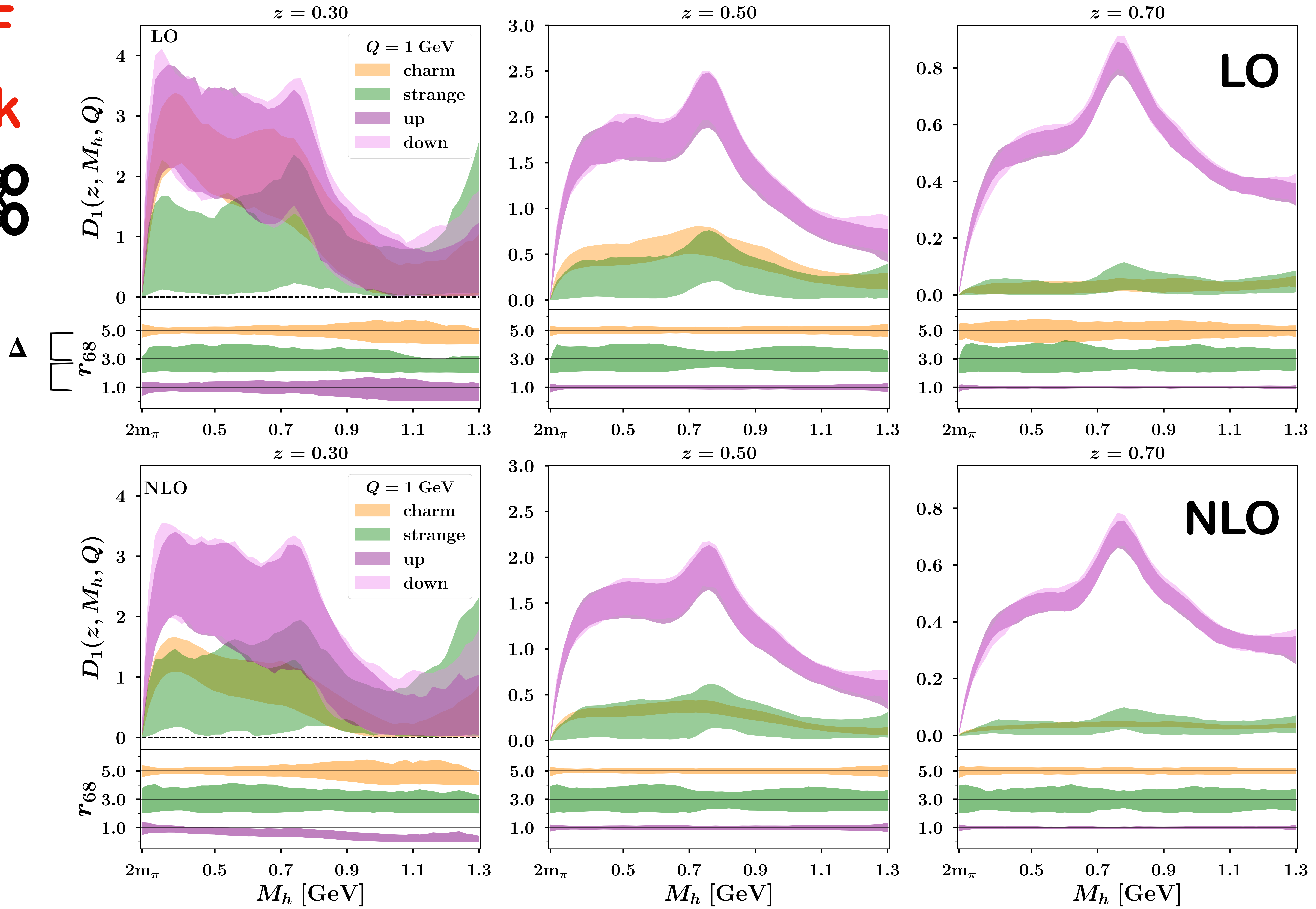
ratio offset **Δ** = 0.5



DI-HADRON FF
Neural Network

u ≠ d

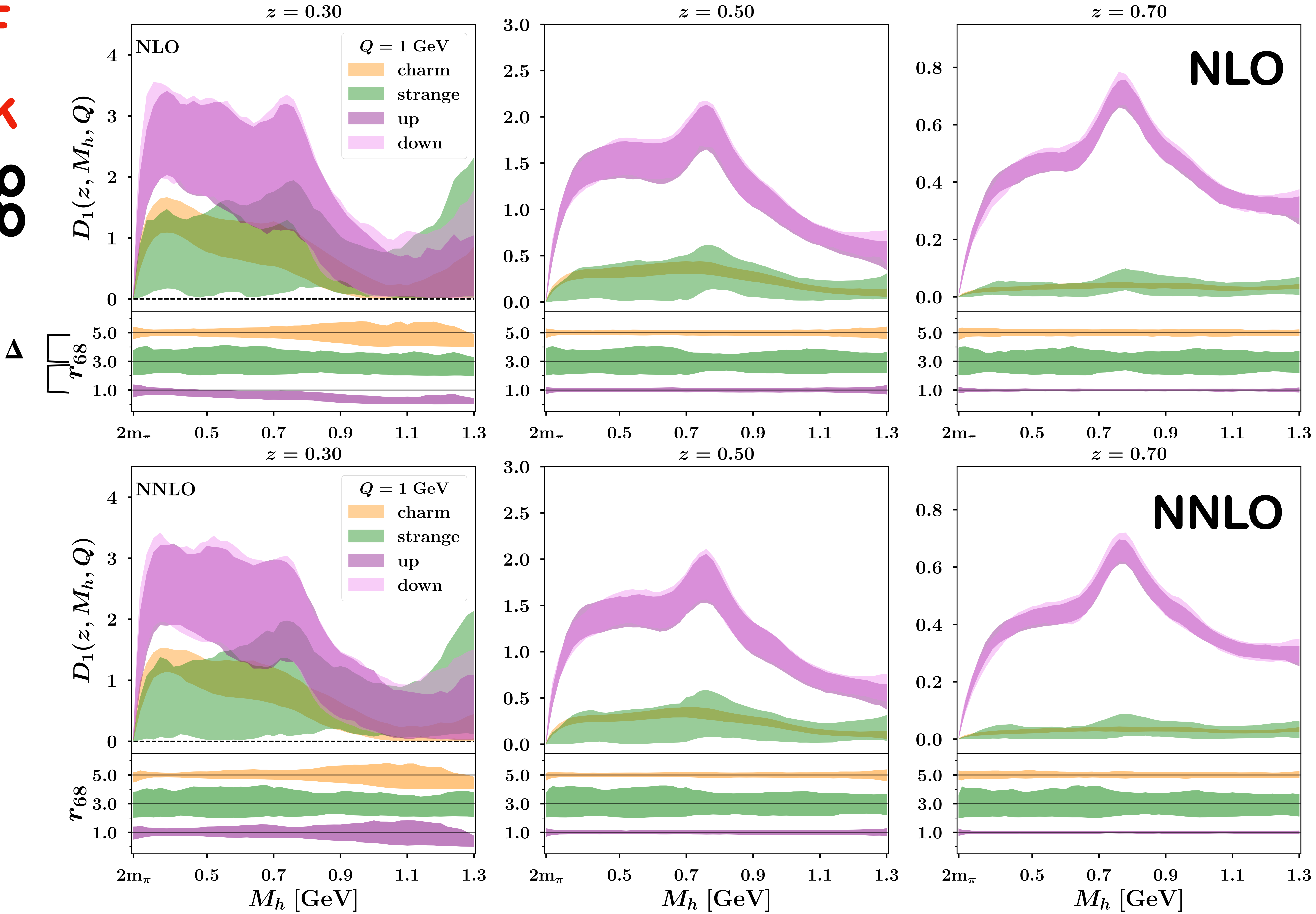
ratio offset **Δ** = 2.5



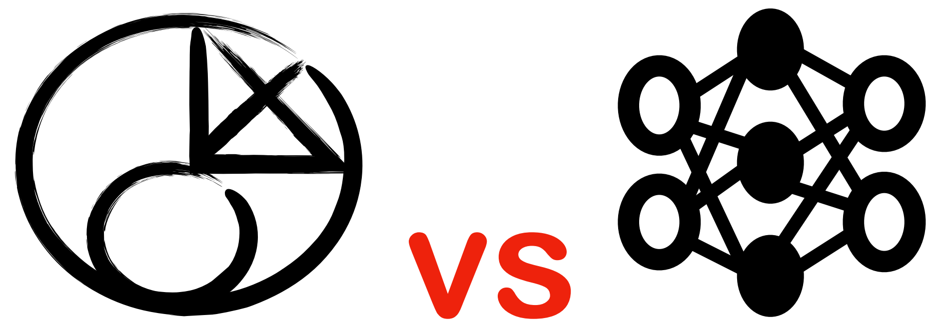
DI-HADRON FF
Neural Network

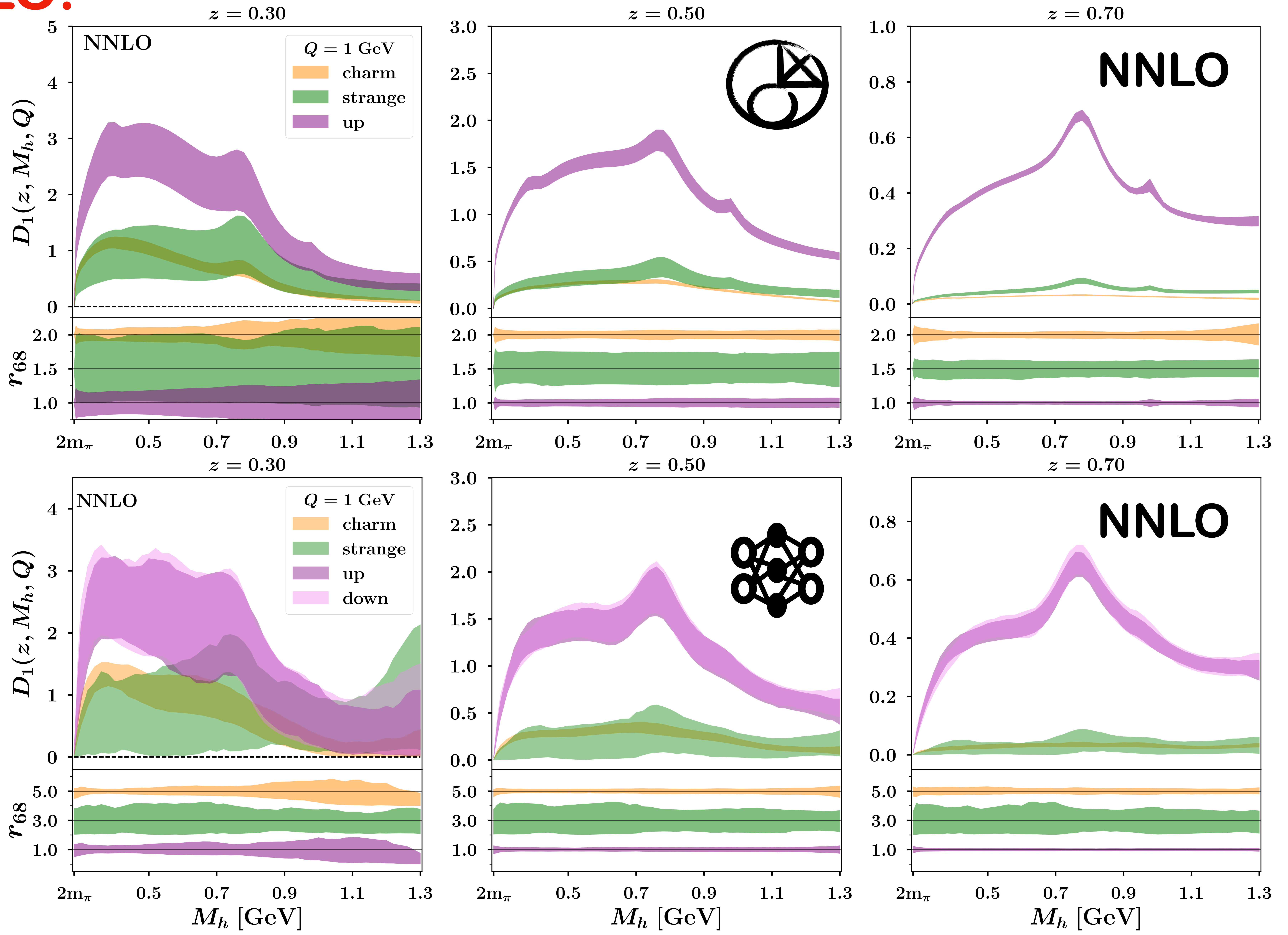
$\mathbf{u \neq d}$

ratio offset Δ = 2.5



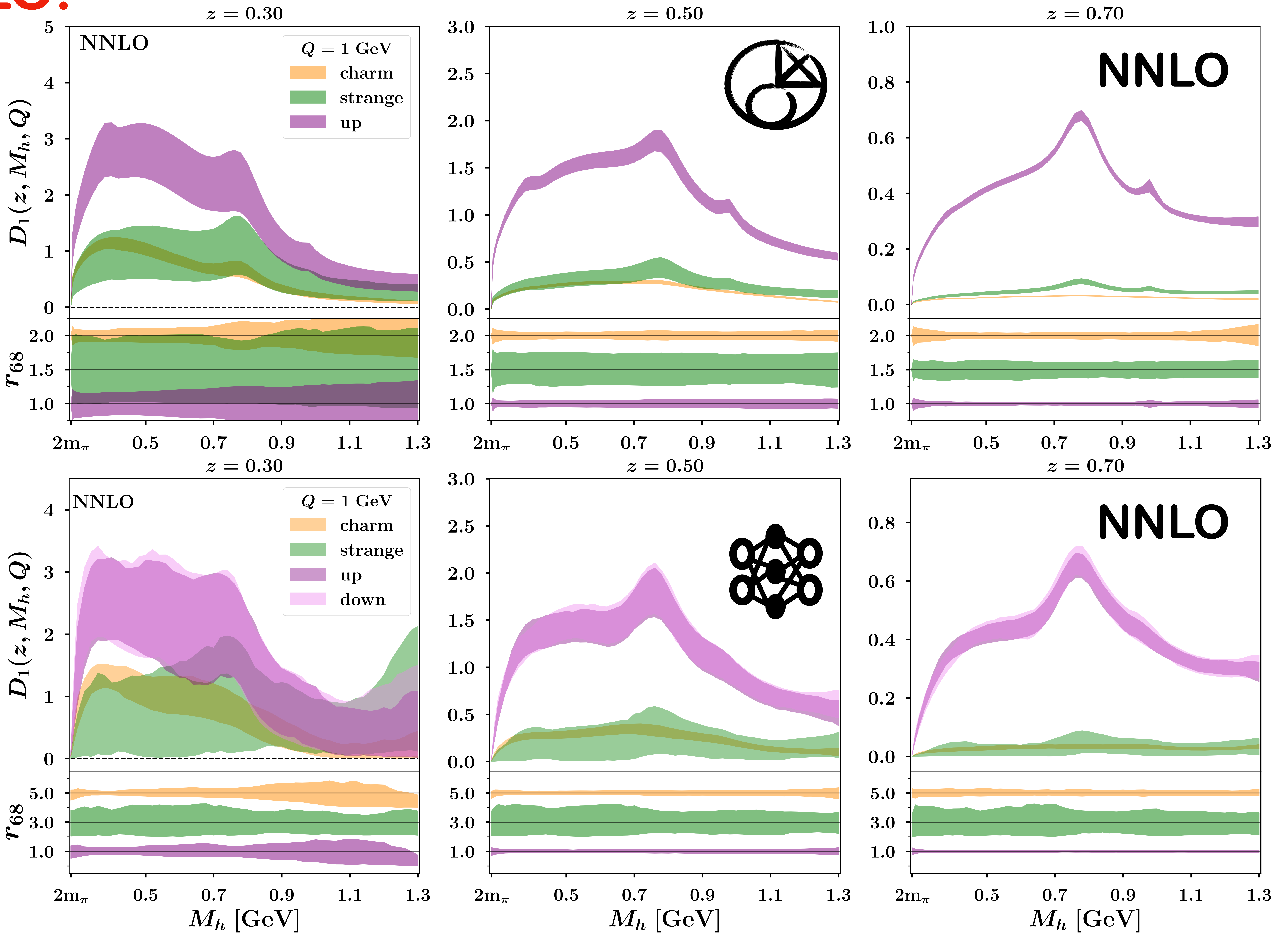
Comparison at NNLO:





Comparison at NNLO:

vs



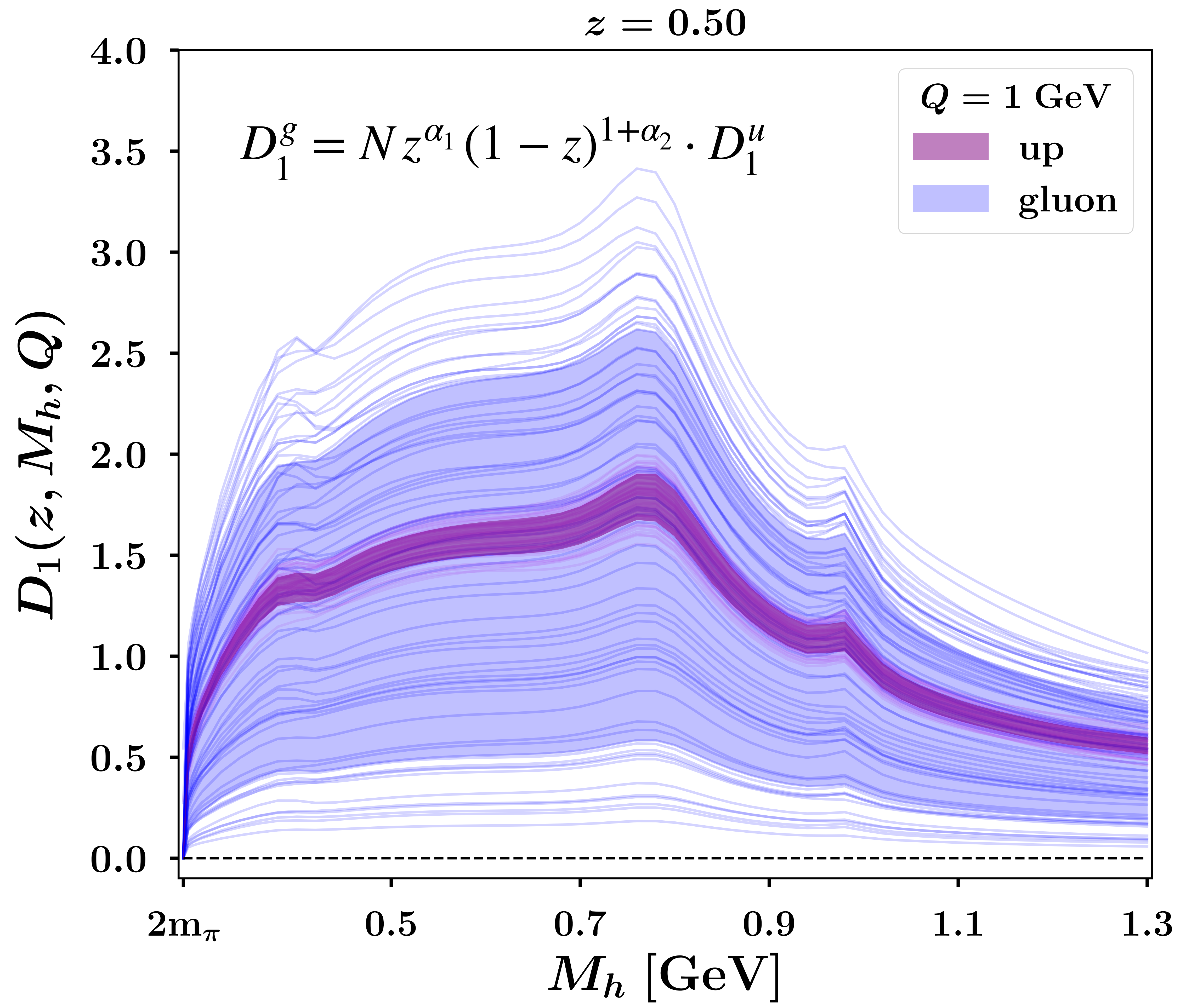
Similar features and compatible results

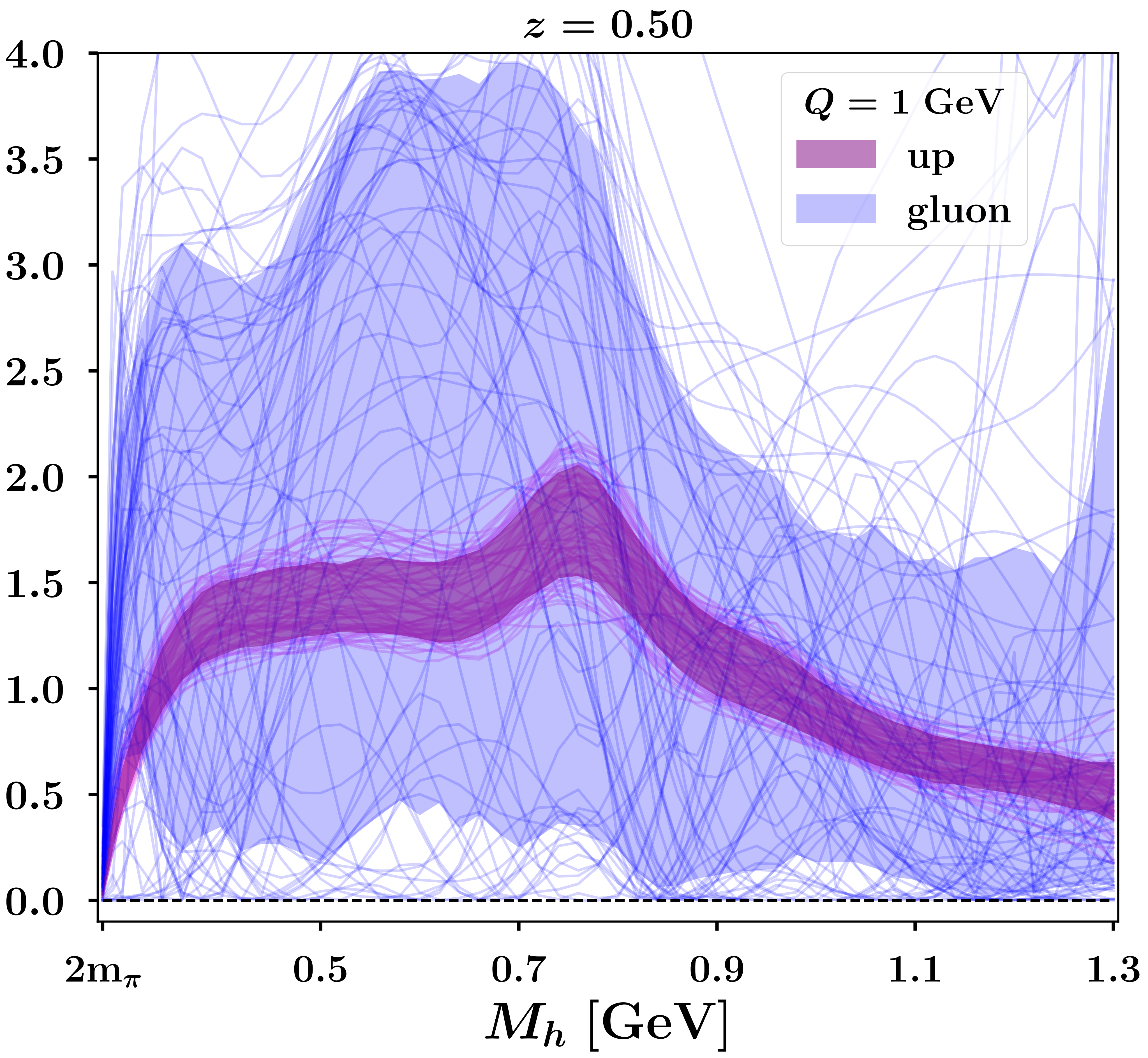
P.I. well reproduce the resonance structure, better than NN

NN has larger uncertainty bands

What about the gluon?

Gluon and up bands at NNLO





UNPOLARIZED RECAP

- Good agreement between both fits and data, small χ^2

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UNPOLARIZED RECAP

- Good agreement between both fits and data, small χ^2

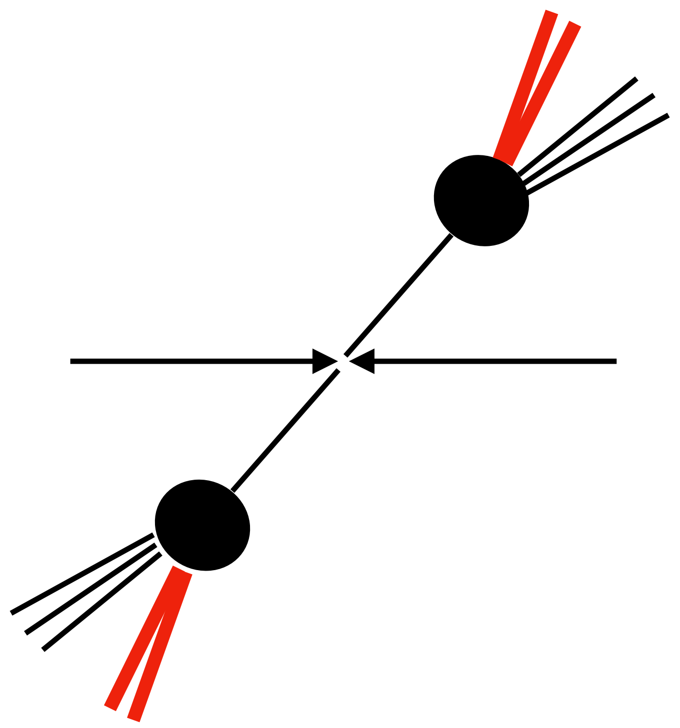
- The different parameterisations allow to grasp aspects of the Di-Hadron not accessible with the single ones

- A hierarchy among quark flavors is clear and stable
  - Gluon uncostrained with e^+e^- data
    Need for SIDIS, hadron-hadron collisions data

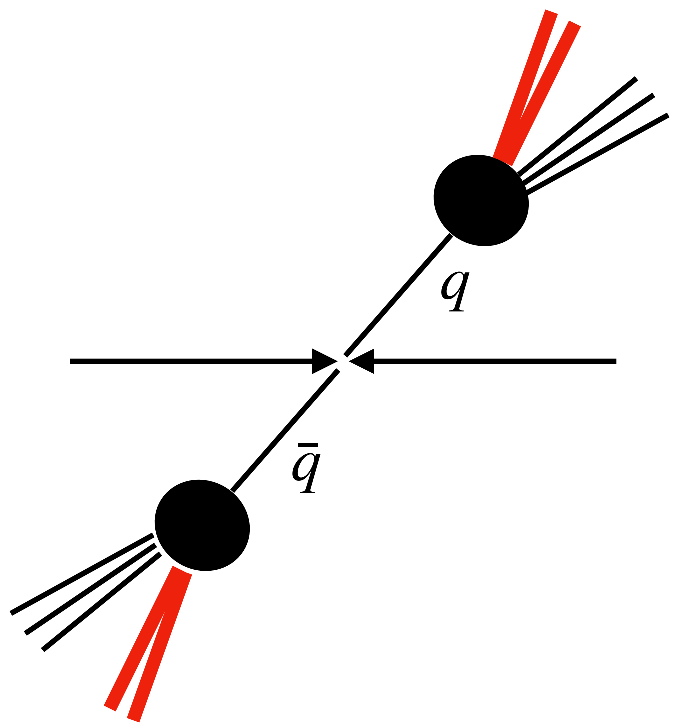
$H_1^{\measuredangle}$ updates

$H_1^{\sphericalangle}$ backgorund

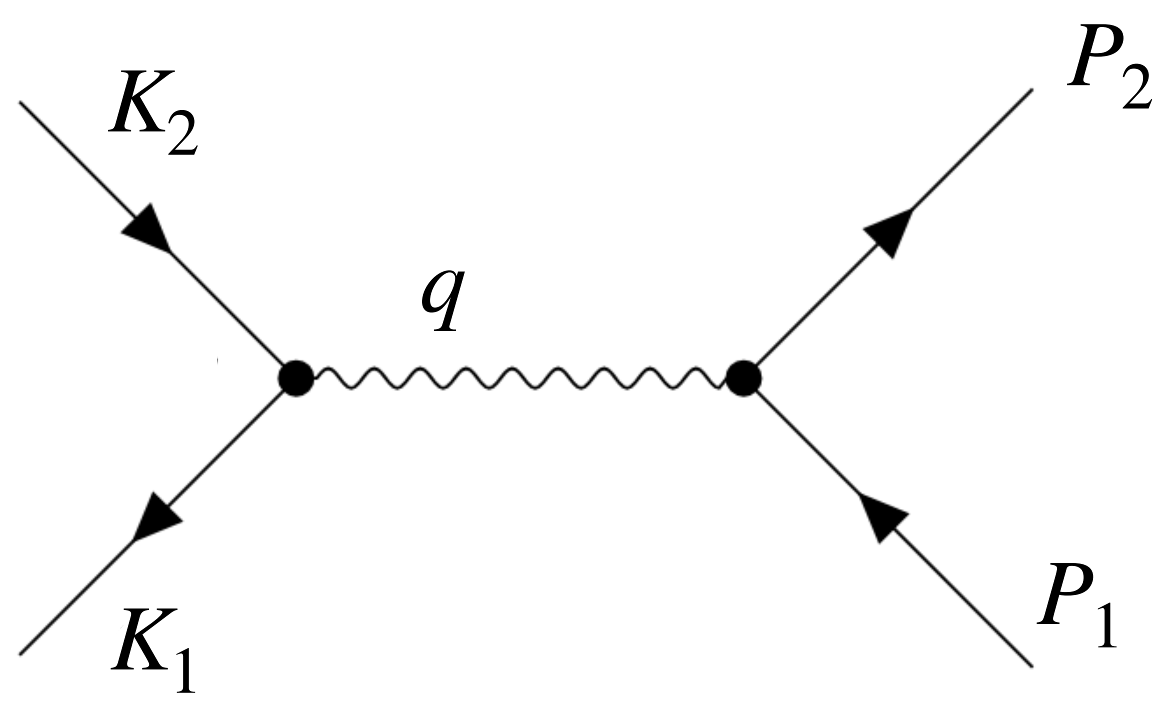
$$A_{UT}^{hh} \sim \frac{\sum_q e_q^2 \cdot H_1^{\sphericalangle,q}(z, M_h) \cdot H_1^{\sphericalangle,q}(\bar{z}, \bar{M}_h)}{\sum_q e_q^2 \cdot D_1^q(z, M_h) \cdot D_1^q(\bar{z}, \bar{M}_h)}$$



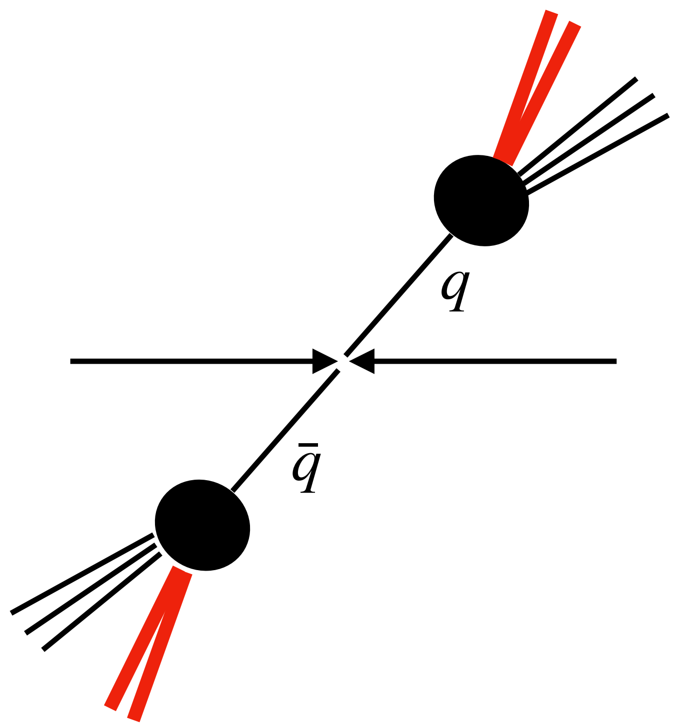
$H_1^{\sphericalangle}$ backgorund



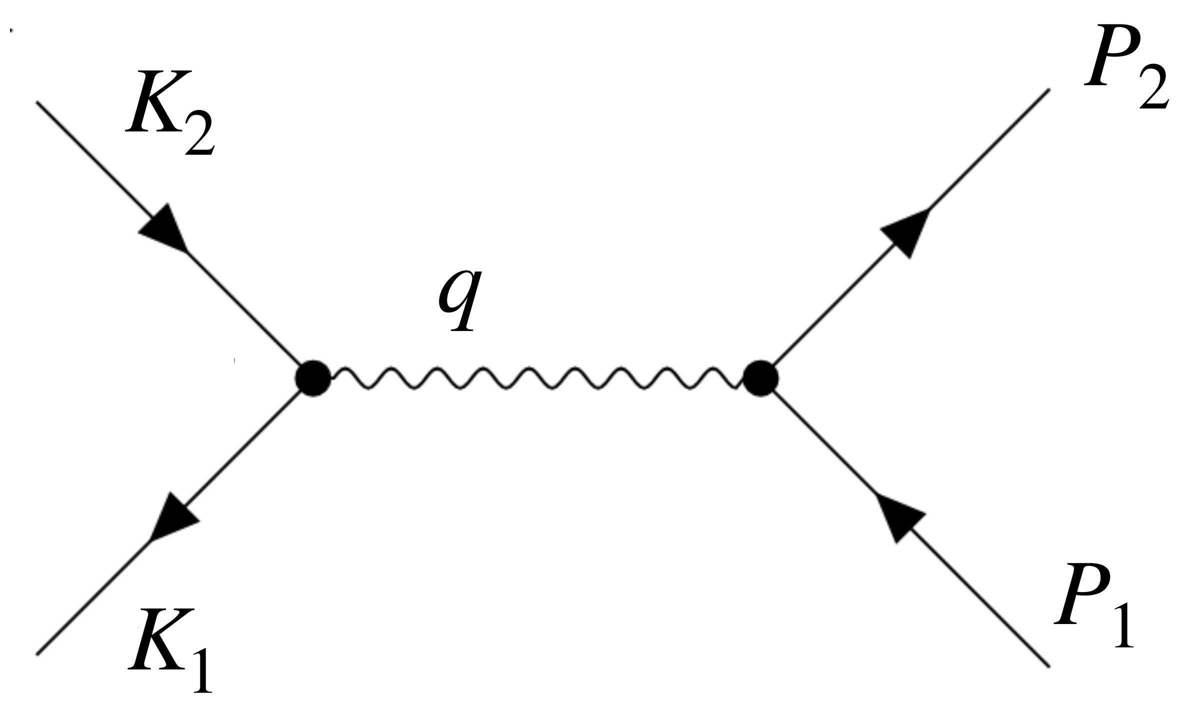
$$A_{UT}^{hh} \sim \frac{\sum_q e_q^2 \cdot H_1^{\sphericalangle,q}(z, M_h) \cdot H_1^{\sphericalangle,q}(\bar{z}, \bar{M}_h)}{\sum_q e_q^2 \cdot D_1^q(z, M_h) \cdot D_1^q(\bar{z}, \bar{M}_h)}$$

$\Big|$  $\Big|^2 \sim L_{\mu\nu} W^{\mu\nu}$

$H_1^{\sphericalangle}$ backgorund

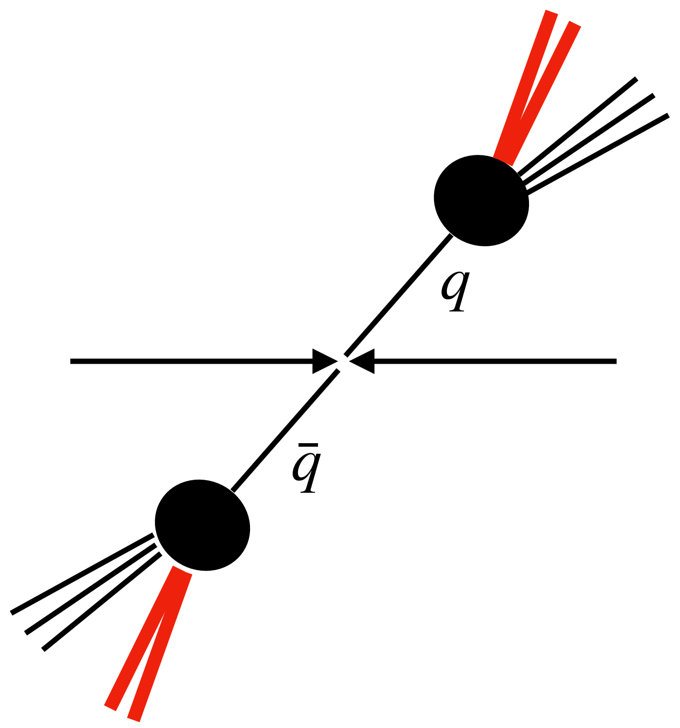


$$A_{UT}^{hh} \sim \frac{\sum_q e_q^2 \cdot H_1^{\sphericalangle,q}(z, M_h) \cdot H_1^{\sphericalangle,q}(\bar{z}, \bar{M}_h)}{\sum_q e_q^2 \cdot D_1^q(z, M_h) \cdot D_1^q(\bar{z}, \bar{M}_h)}$$



$$\left| \cdots \right|^2 \sim L_{\mu\nu}\left[g_T^{\mu\nu}(S_{T2} \cdot S_{T1}) - S_{T2}^{\mu} S_{T1}^{\nu} - S_{T2}^{\nu} S_{T1}^{\mu}\right] \sim Q^4 \frac{\sin^2(\theta)}{4} \cos(\phi_{S_{T2}} + \phi_{S_{T1}})$$

$H_1^{\sphericalangle}$ backgorund



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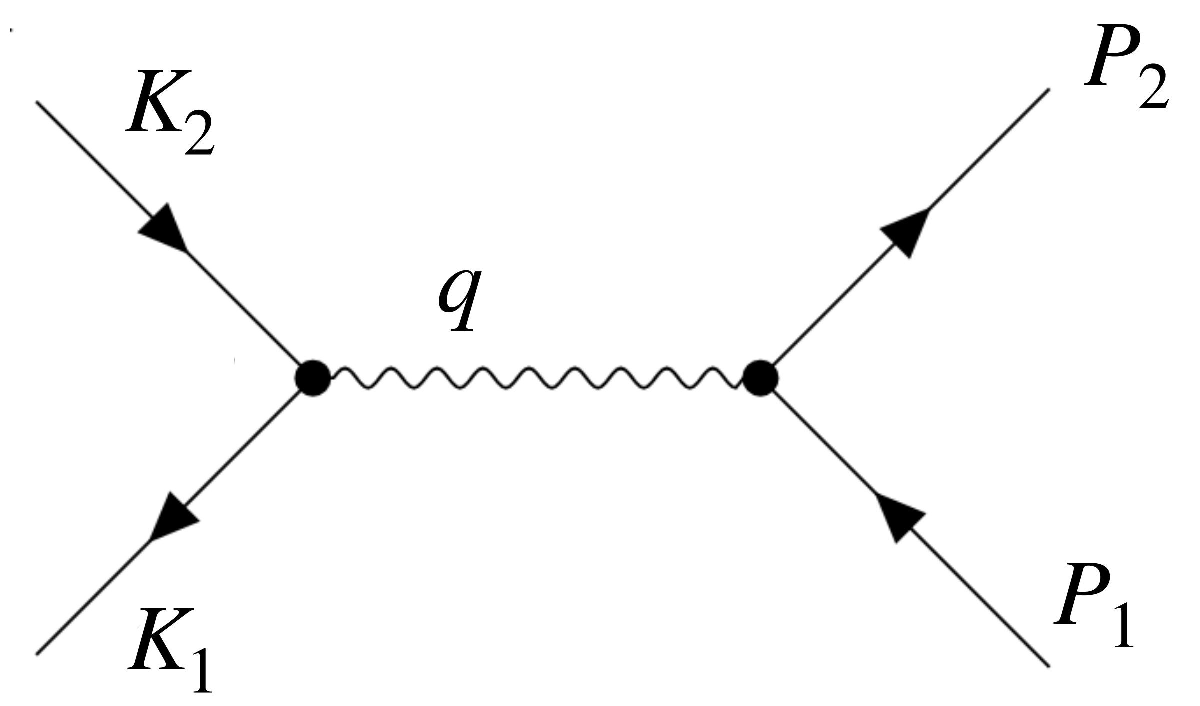

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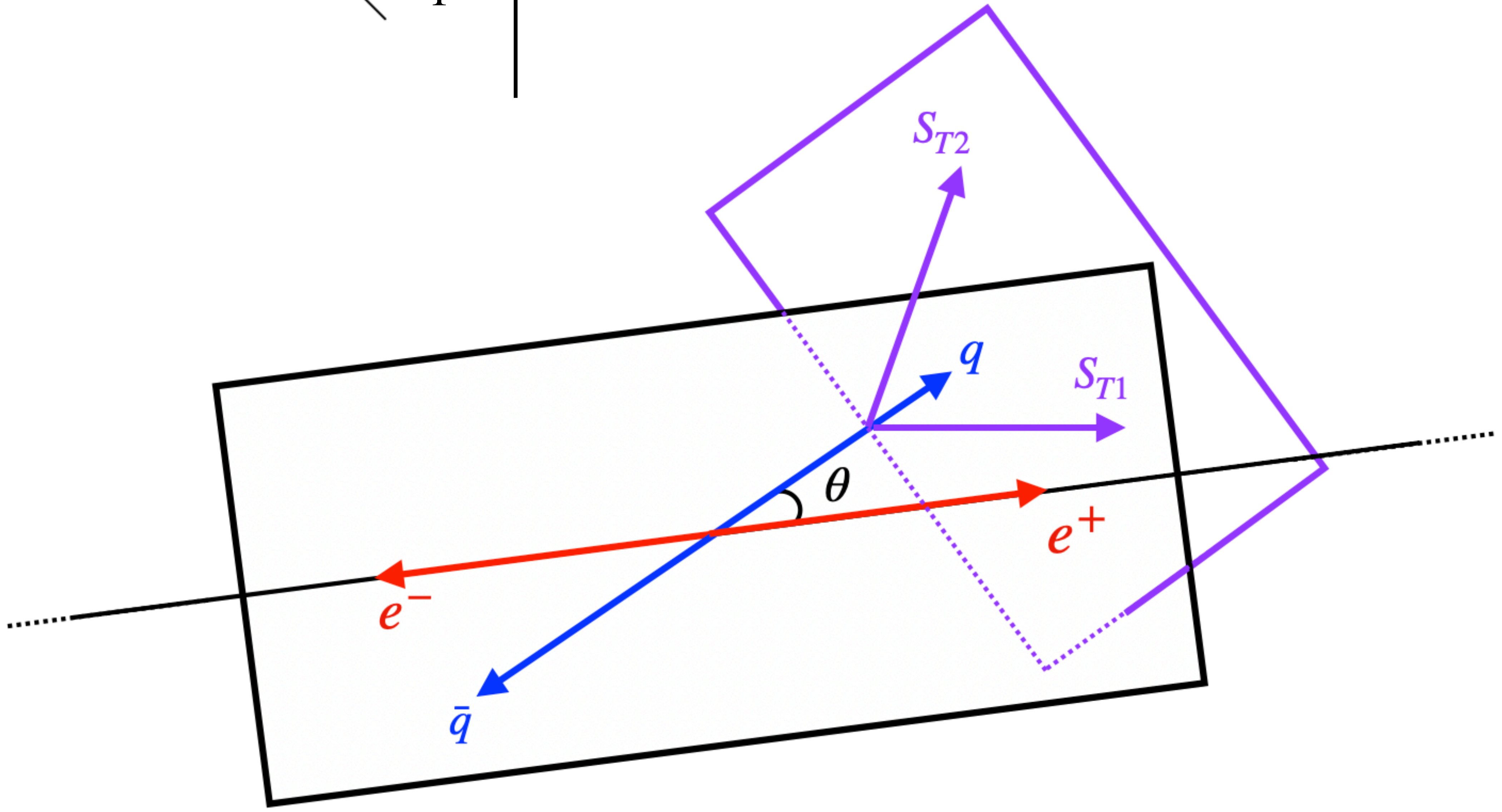
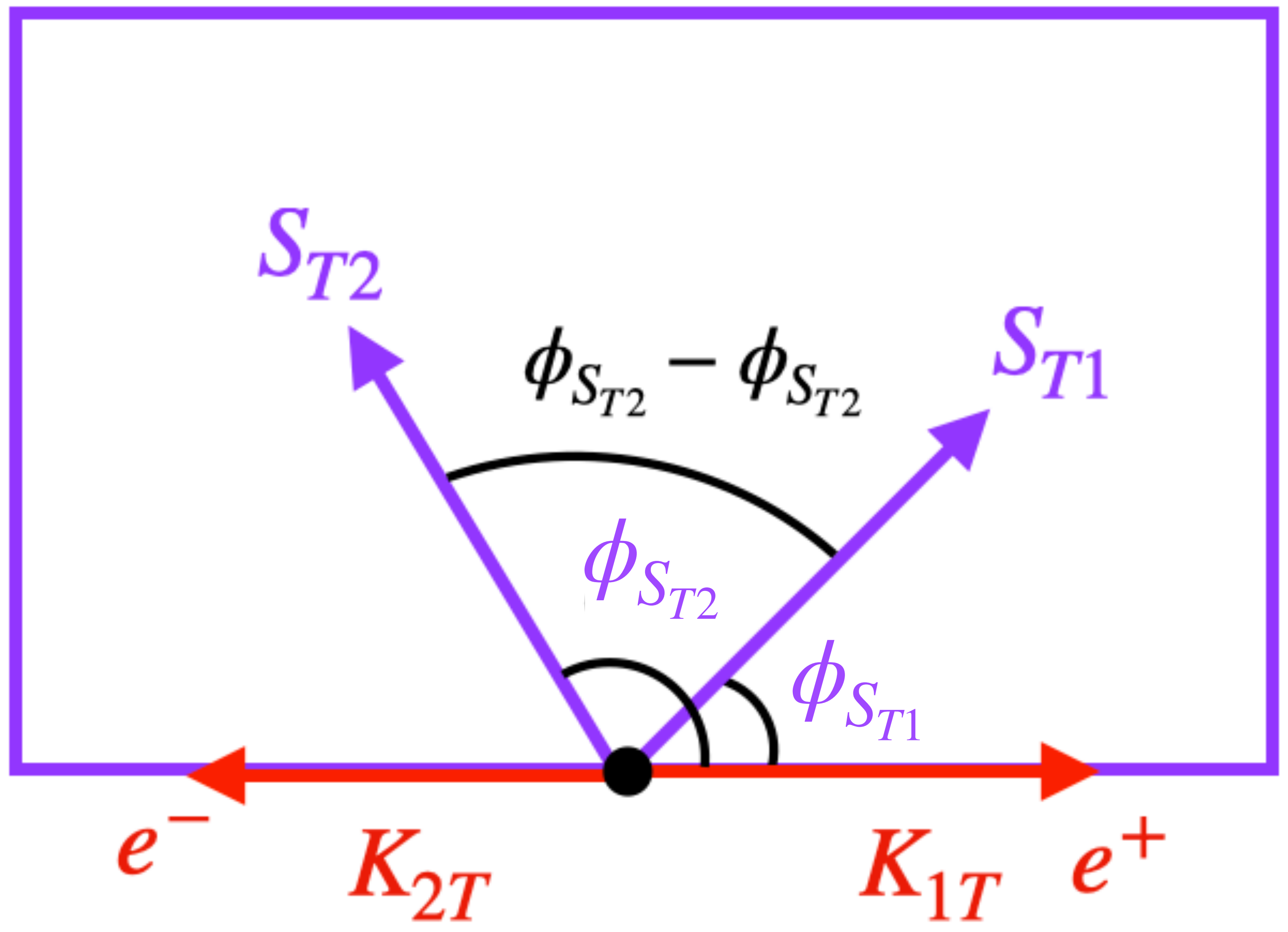
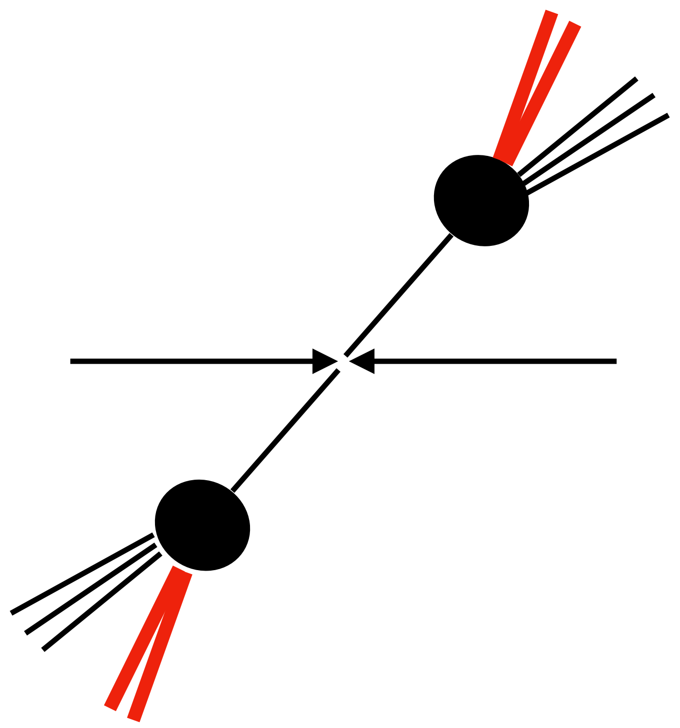




$H_1^{\sphericalangle}$ backgorund



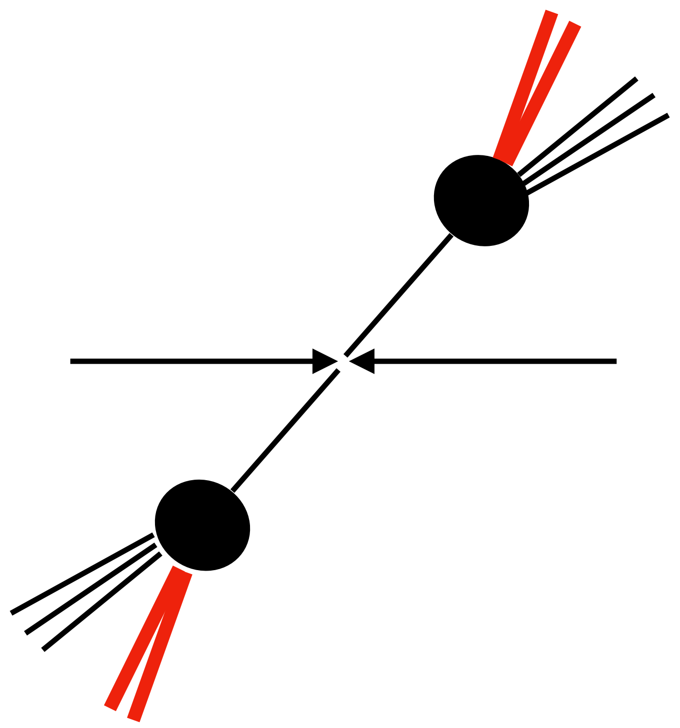
$$A_{UT}^{hh} \sim \frac{\sum_q e_q^2 \cdot H_1^{\sphericalangle,q}(z, M_h) \cdot H_1^{\sphericalangle,q}(\bar{z}, \bar{M}_h)}{\sum_q e_q^2 \cdot D_1^q(z, M_h) \cdot D_1^q(\bar{z}, \bar{M}_h)}$$

For $\pi^+\pi^-$

Symmetry relations for $H_1^{\sphericalangle}$:

- **Isopsin symmetry**
- **Charge conjugation**

$H_1^{\sphericalangle}$ backgorund



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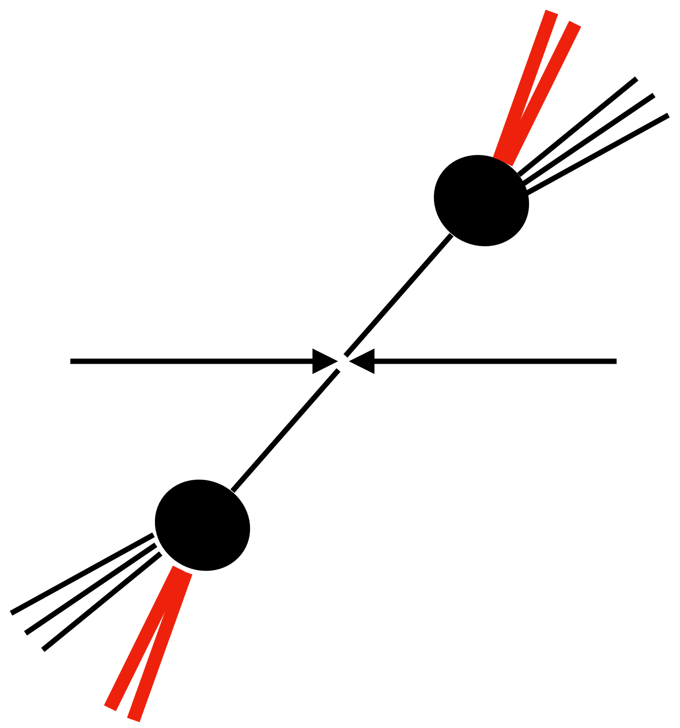
For $\pi^+\pi^-$

Symmetry relations for $H_1^{\sphericalangle}$:

- **Isopsin symmetry** $H_1^{\sphericalangle,\mathbf{u}\to\pi^+\pi^-} = H_1^{\sphericalangle,\bar{\mathbf{d}}\to\pi^+\pi^-}$, $H_1^{\sphericalangle,\bar{\mathbf{u}}\to\pi^+\pi^-} = H_1^{\sphericalangle,\mathbf{d}\to\pi^+\pi^-}$
- **Charge conjugation**

$H_1^{\sphericalangle}$ backgorund

$$A_{UT}^{hh} \sim \frac{\sum_q e_q^2 \cdot H_1^{\sphericalangle,q}(z, M_h) \cdot H_1^{\sphericalangle,q}(\bar{z}, \bar{M}_h)}{\sum_q e_q^2 \cdot D_1^q(z, M_h) \cdot D_1^q(\bar{z}, \bar{M}_h)}$$



For $\pi^+\pi^-$

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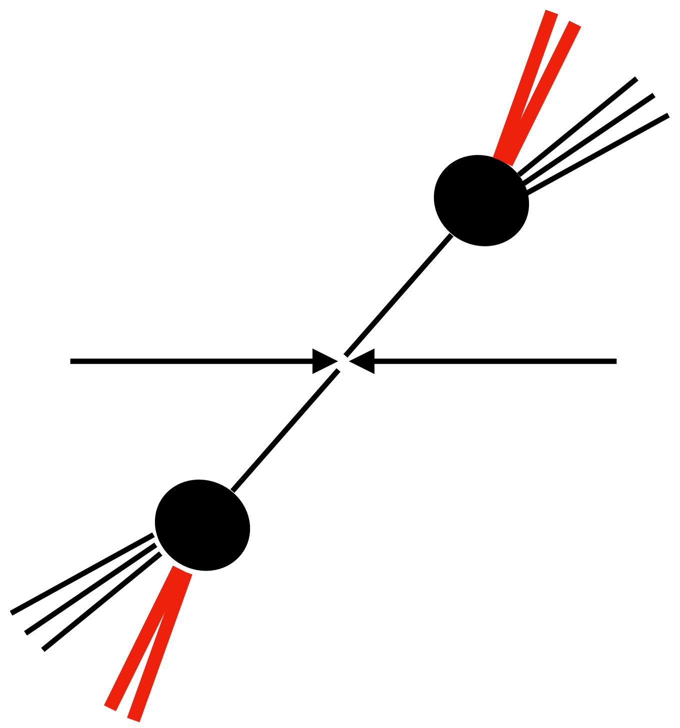
$H_1^{\sphericalangle}$ backgorund



$$A_{UT}^{hh} \sim \frac{\sum_q e_q^2 \cdot H_1^{\sphericalangle,q}(z, M_h) \cdot H_1^{\sphericalangle,q}(\bar{z}, \bar{M}_h)}{\sum_q e_q^2 \cdot D_1^q(z, M_h) \cdot D_1^q(\bar{z}, \bar{M}_h)}$$

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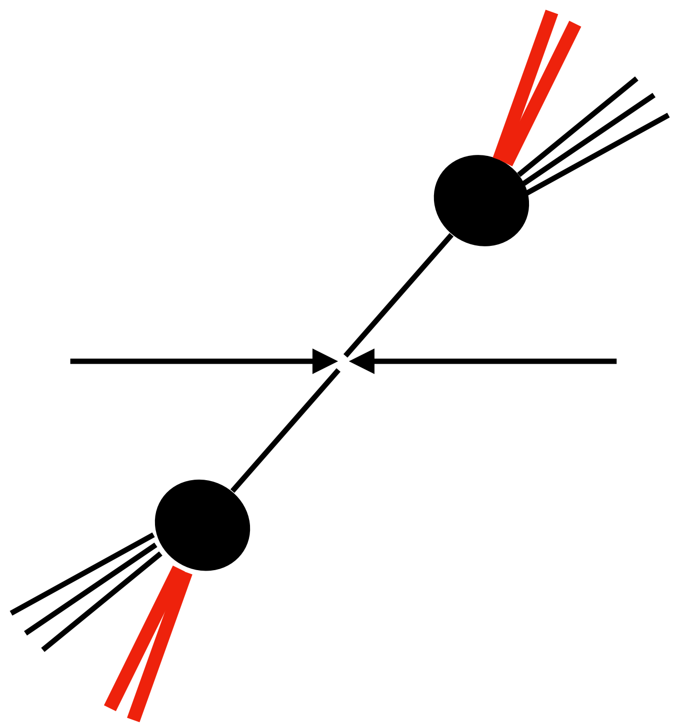
Valence $H_1^{\sphericalangle/,\mathbf{u}} = -H_1^{\sphericalangle/,\mathbf{d}} = -H_1^{\sphericalangle/,\bar{\mathbf{u}}} = H_1^{\sphericalangle/,\bar{\mathbf{d}}}$

Sea: $q = s, c, b$ $H_1^{\sphericalangle/,\mathbf{q}} = -H_1^{\sphericalangle/,\bar{\mathbf{q}}} = -H_1^{\sphericalangle/,\mathbf{q}} = 0$

$H_1^{\sphericalangle}$ backgorund



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Latest $H_1^{\sphericalangle}$ extractions

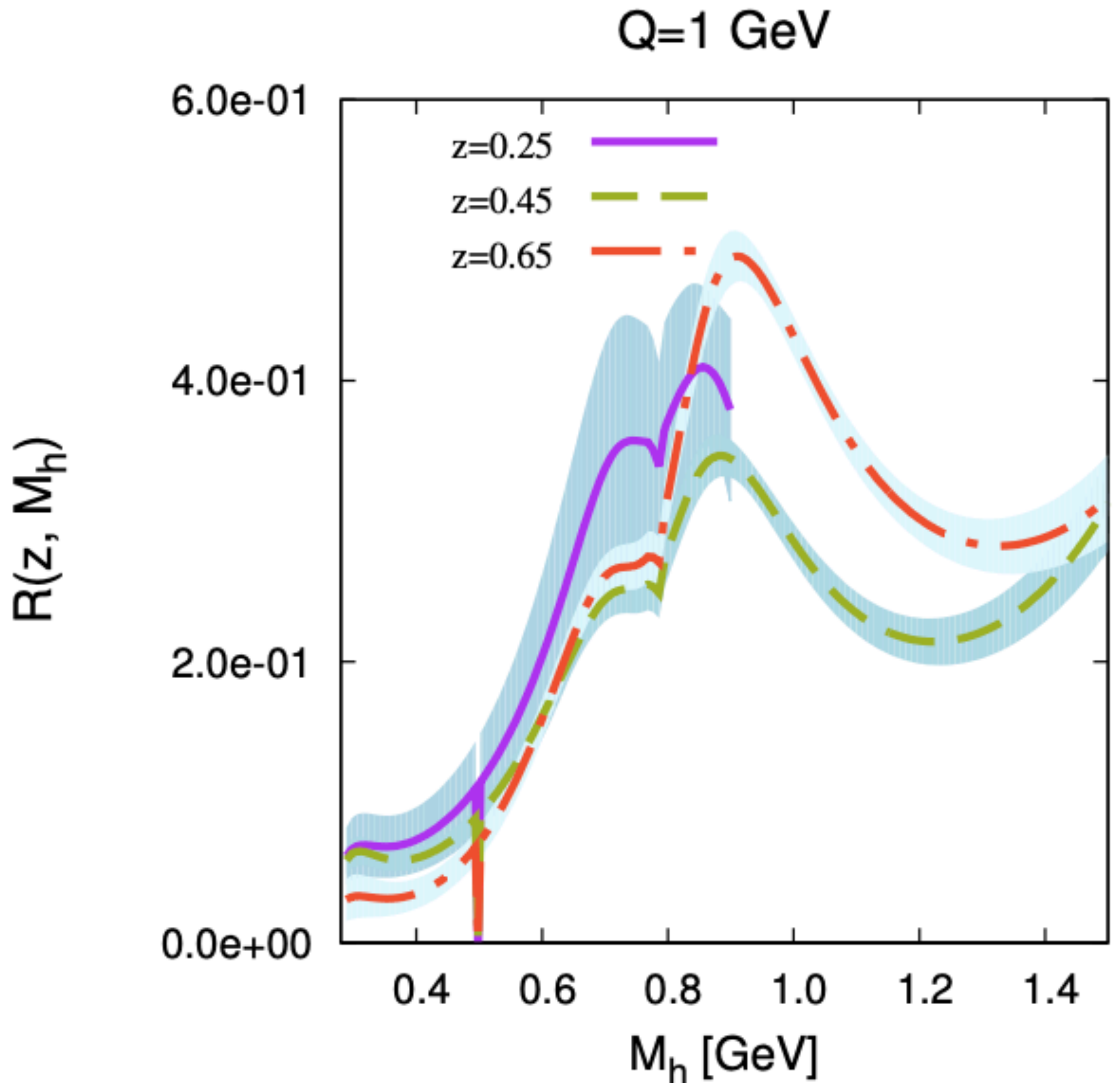
Latest $H_1^{\sphericalangle}$ extractions

Pavia 2012

2011 BELLE data at $\sqrt{S} = 10.58$ GeV of $e^+e^- \to (\pi^+\pi^-)(\bar{\pi}^+\bar{\pi}^-)X$

9 free parameters

LO



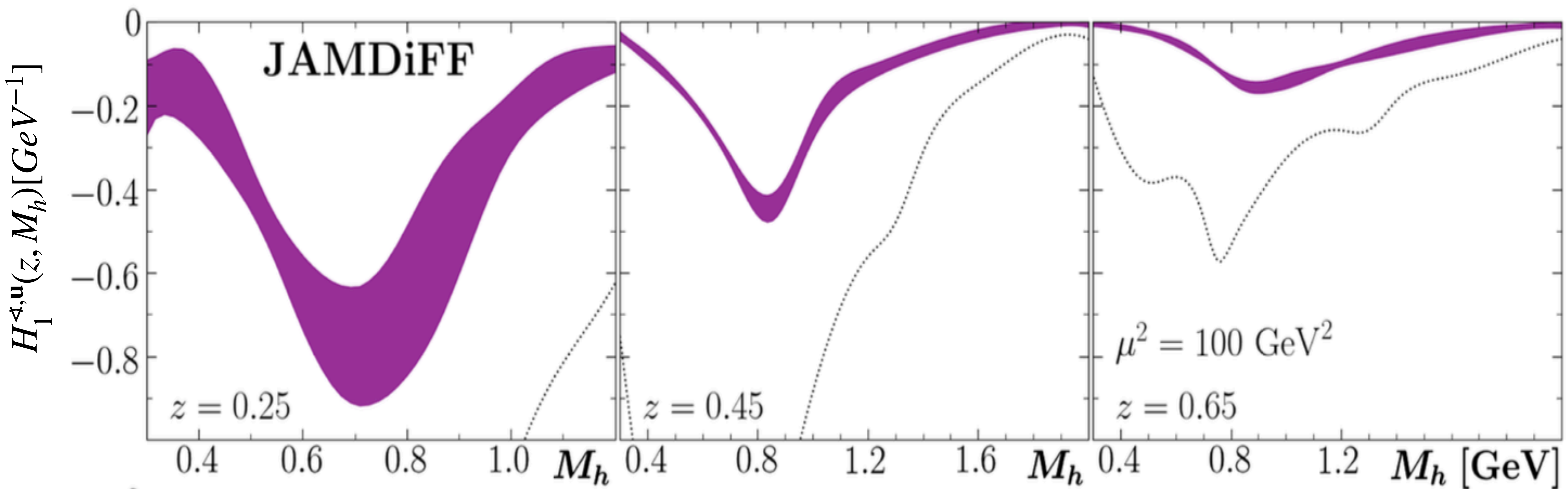
$$R(z, M_h) = \frac{|\vec{p_{h1}} - \vec{p_{h2}}|}{M_h} \frac{H_1^{\sphericalangle,u}(z, M_h; Q^2)}{D_1^u(z, M_h; Q^2)}$$

JAM 2024

2011 BELLE data at $\sqrt{S} = 10.58$ GeV of $e^+e^- \to (\pi^+\pi^-)(\bar{\pi}^+\bar{\pi}^-)X$

48 free parameters

LO



Set up for $H_1^{\sphericalangle}$ at NLO

Symmetries old also at NLO:

$$H_1^{\sphericalangle/,\mathbf{u}} = -H_1^{\sphericalangle/,\mathbf{d}} = -H_1^{\sphericalangle/,\bar{\mathbf{u}}} = H_1^{\sphericalangle/,\bar{\mathbf{d}}}$$

$$H_1^{\sphericalangle/,\mathbf{q}} = -H_1^{\sphericalangle/,\bar{\mathbf{q}}} = -H_1^{\sphericalangle/,\mathbf{q}} = 0$$

Set up for $H_1^{\sphericalangle}$ at NLO

Symmetries old also at NLO:

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Observable at NLO:

NLOcalculation in: A.P. Contogouris et al, Phys. Lett. B 334, (1995) → **Possible inconsistencies between the equations**

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Symmetries old also at NLO:

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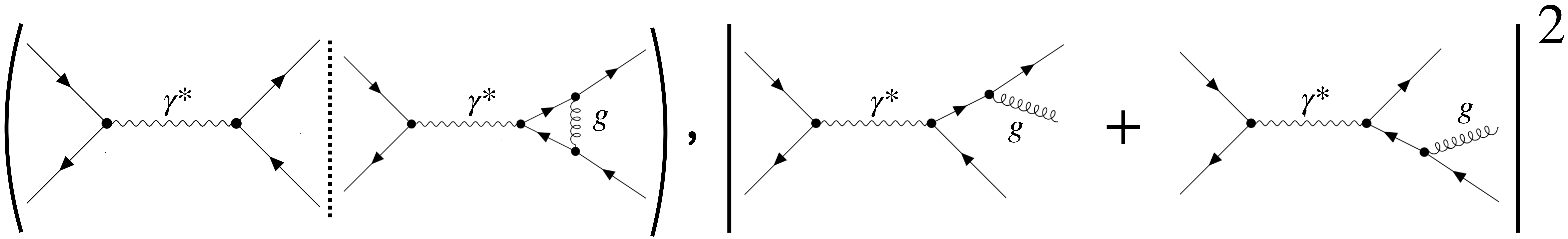
$$H_1^{\sphericalangle/,\mathbf{q}} = -\,H_1^{\sphericalangle/,\bar{\mathbf{q}}} = -\,H_1^{\sphericalangle/,\mathbf{q}} = 0$$

Observable at NLO:

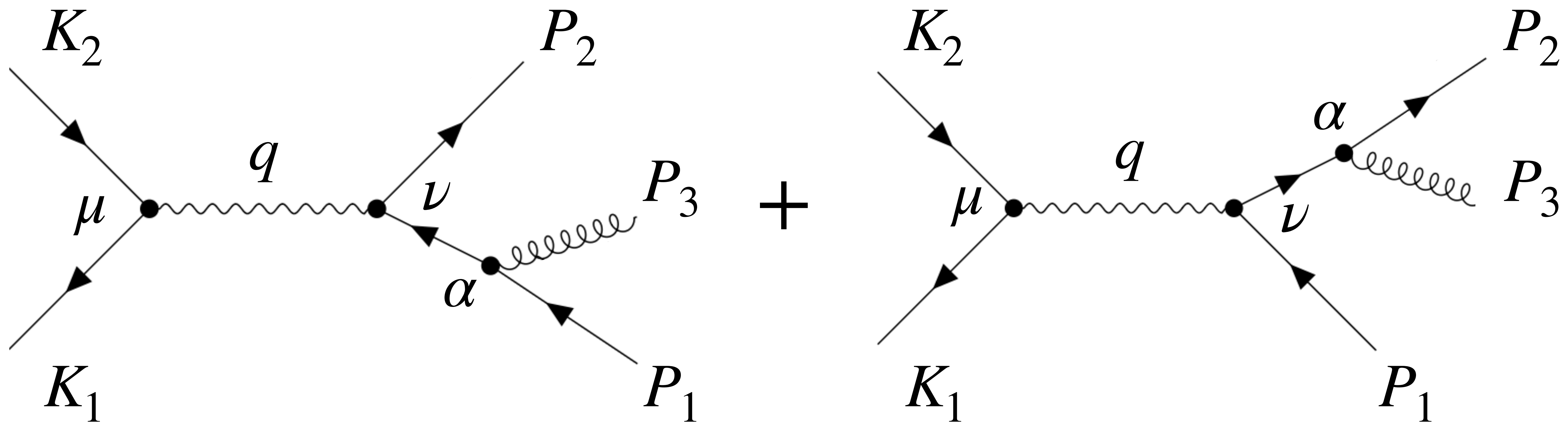
NLOcalculation in: A.P. Contogouris et al, Phys. Lett. B 334, (1995) → **Possible inconsistencies between the equations**

I checked the calculations, and I found some differences in the final result.

$\mathcal{O}(\alpha_S) \sim$



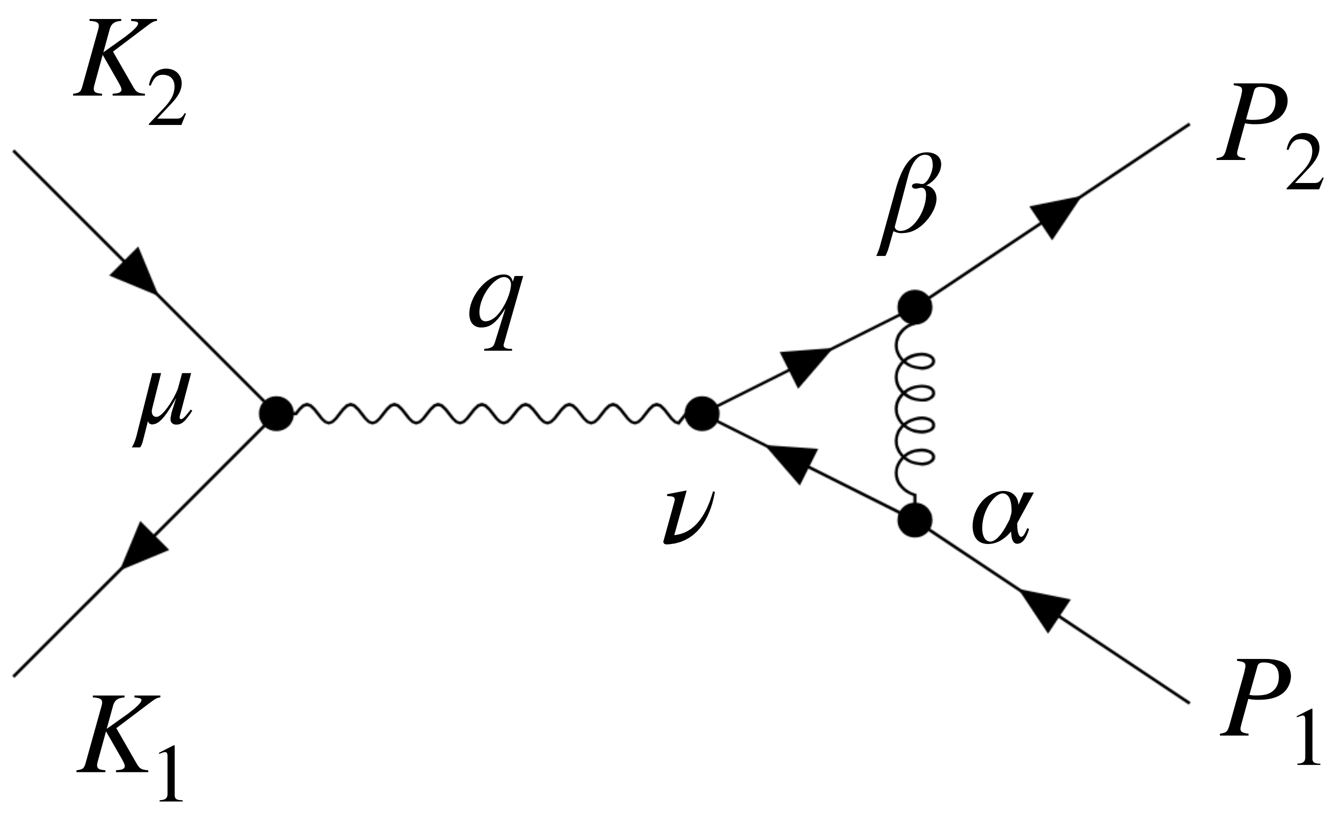
Asymmetry at NLO



$$u(P_2)\bar{u}(P_2) = \frac{1}{2}\not{P}_2\left(1 + \gamma_5 \not{S}_{T2}\right)$$

$$v(P_1)\bar{v}(P_1) = \frac{1}{2}\not{P}_1\left(1 + \gamma_5 \not{S}_{T1}\right)$$

d-**dimensions**

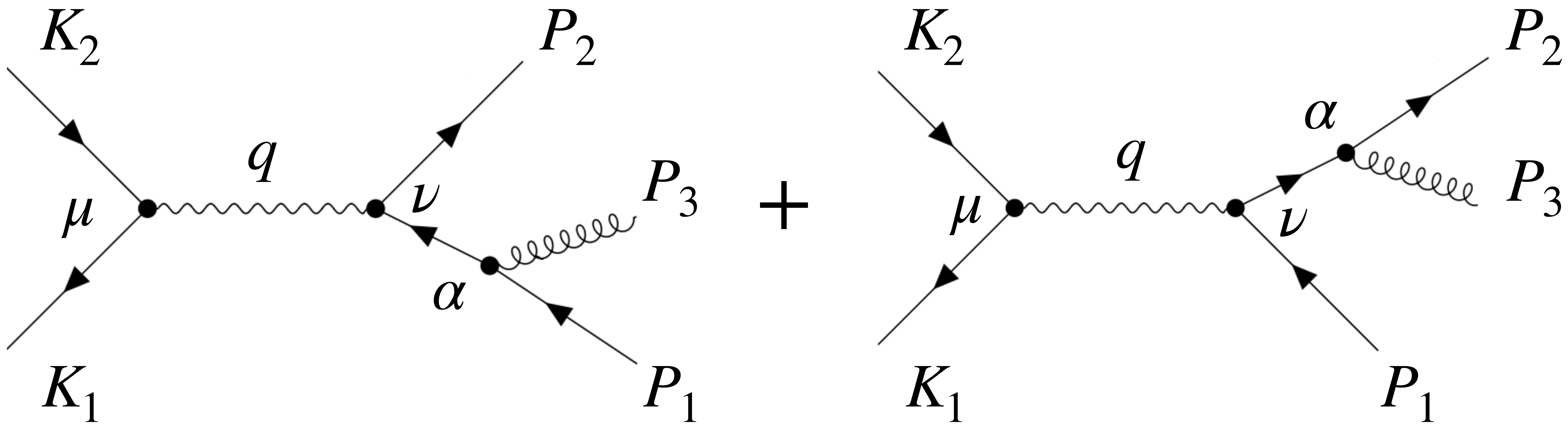


$$\sim 2Re(f(q^2))\,|M_{0,T}|^2$$

$$\frac{1}{\sigma^d_{0,T}}\frac{d\sigma^R}{du\,dz} \quad + \quad \frac{1}{\sigma^d_{0,T}}\sigma^V\delta(1-u)\delta(1-z)$$

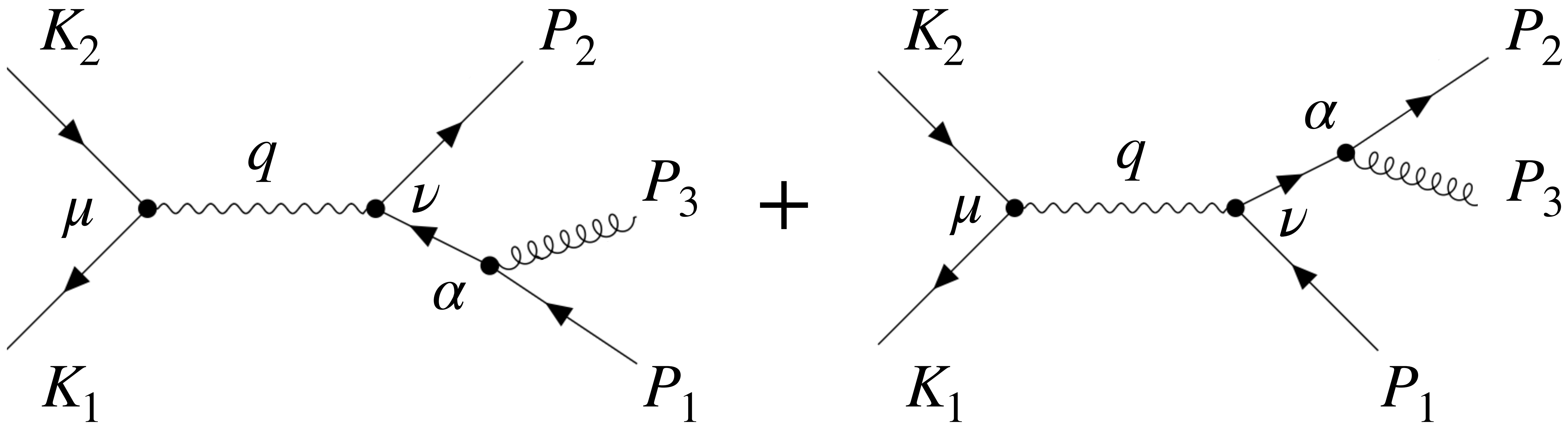
The sum isolates collinear singularities

Real g emission



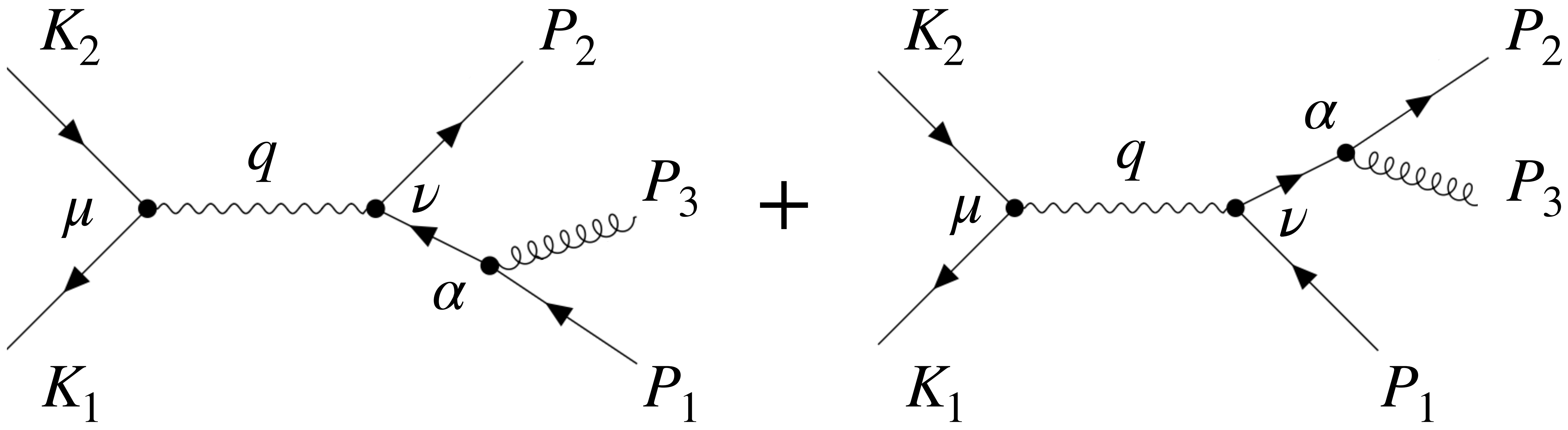
d-dimensions

Real g emission



d-dimensions

$$d\Pi_{LIPS} = \frac{d^{d-1}P_1}{(2\pi)^{d-1}}\frac{d^{d-1}P_3}{(2\pi)^{d-1}}(2\pi)^4\delta^{(d)}(q - P_1 - P_2 - P_3)$$

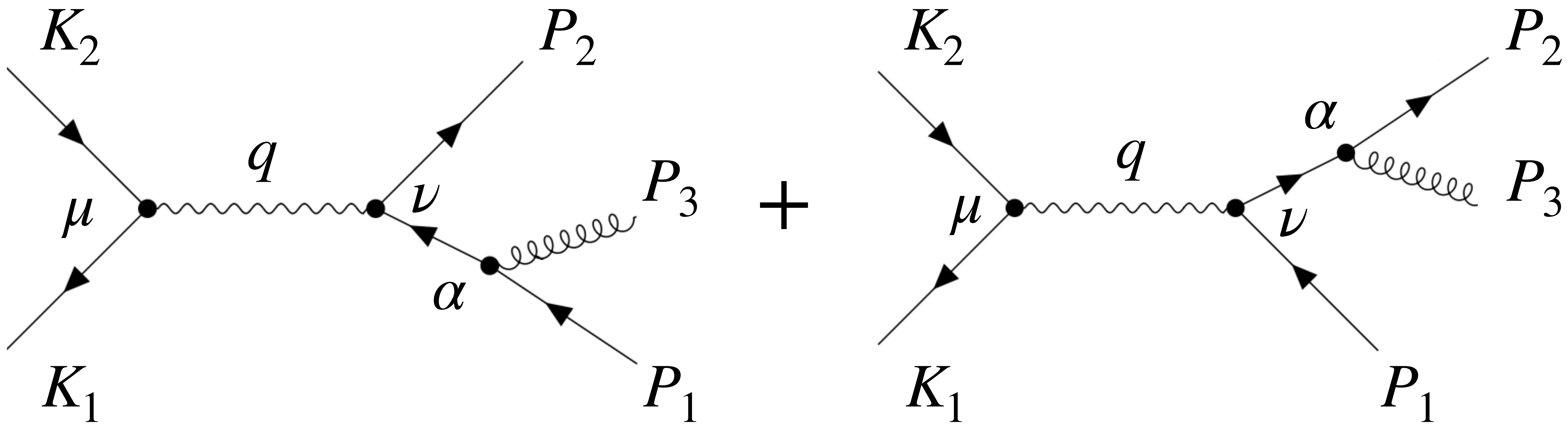


Real g emission

d-dimensions

$$\left[g_T^{\mu\nu}(S_{T2}\cdot S_{T1}) - S_{T2}^{\mu}S_{T1}^{\nu} - S_{T2}^{\nu}S_{T1}^{\mu}\right]$$

Real g emission

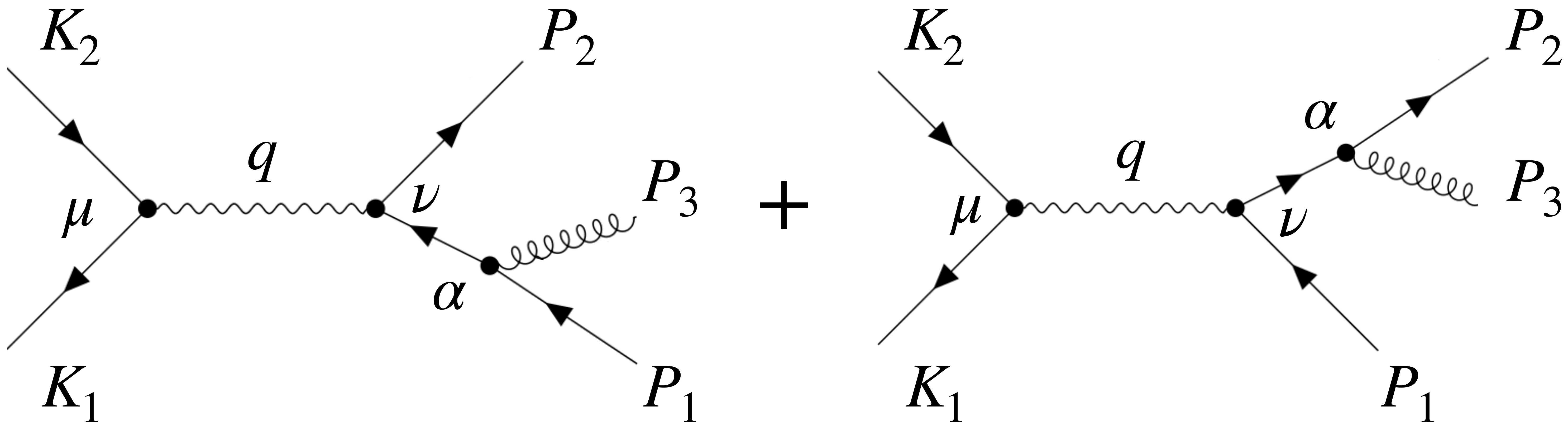


$$\begin{cases} u = \dfrac{P_1 \cdot P_2}{P_1 \cdot q} \\ z = \dfrac{2\, P_1 \cdot q}{Q^2} \end{cases}$$

d-dimensions

$$\frac{1}{\sigma_0^{d,T}} \frac{d\sigma^R}{du\, dz} = e_q^2 \frac{4}{3} \frac{\alpha_s}{2\pi} \left(\frac{Q^2}{4\pi\mu^2} \right)^{-\epsilon} \frac{1}{\Gamma[1-\epsilon]}\, z\, dz\, du \;\; (1-z)^{-1-\epsilon}(1-u)^{1-\epsilon} u^{-\epsilon} z^{-2\epsilon} \left[2uz - \epsilon(1-u\, z)^2 \right]$$

Real g emission



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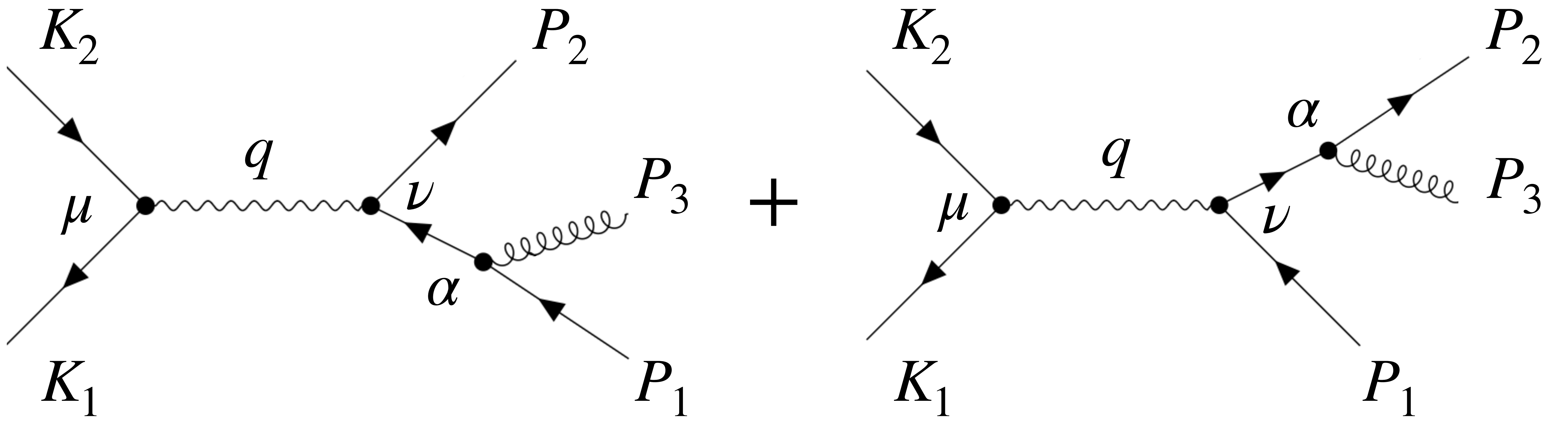

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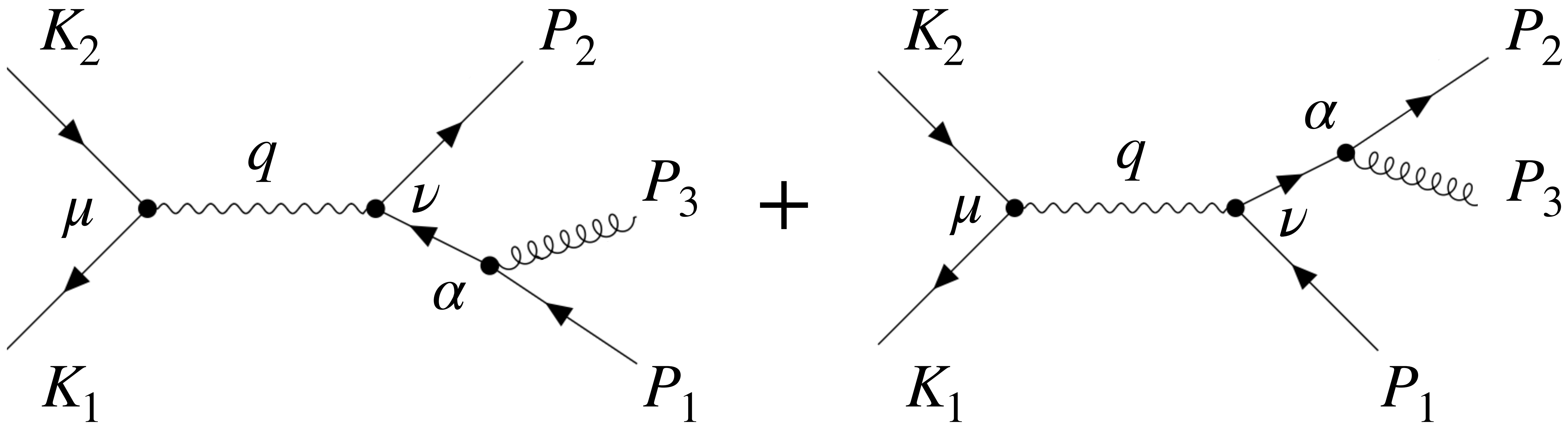
$$(1-z)^{-1-\epsilon} = -\frac{1}{\epsilon}\delta(1-z) + \frac{1}{(1-z)_+} - \epsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ + \mathcal{O}(\epsilon^2)$$

$$\int_0^1 dz\,\frac{f(z)-f(1)}{1-z} = \int_0^1 dz\,\frac{f(z)}{(1-z)_+}$$

Real g emission



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d-dimensions

$$\frac{1}{\sigma_0^{d,T}} \frac{d\sigma^R}{du\,dz} = e_q^2 \frac{4}{3} \frac{\alpha_s}{2\pi} \left(\frac{Q^2}{4\pi\mu^2 e^{-\gamma_E}} \right)^{-\epsilon} \left(1 - \frac{\pi^2}{12}\epsilon^2 \right) \times$$

$$\times \left\{ \frac{2}{\epsilon^2}\delta(1-u)\delta(1-z) + \frac{1}{\epsilon}(\ldots)\left[-\frac{\delta(1-u)}{(1-z)_+} - \frac{\delta(1-z)}{(1-u)_+} \right] + \delta(1-u)(\ldots) + \delta(1-z)(\ldots) + \frac{(\ldots)}{(1-u)_+(1-z)_+} + \mathcal{O}(\epsilon) \right\}$$

$$\frac{1}{\sigma_0^d}\sigma^V \delta(1-u)\delta(1-z) = -\,e_q^2 \frac{4}{3} \frac{\alpha_s}{2\pi} \left(\frac{Q^2}{4\pi\mu^2 e^{-\gamma_E}} \right)^{-\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{7\pi^2}{6} + 8 + \mathcal{O}(\epsilon) \right).$$

$$\frac{1}{\sigma^d_{0,T}}\frac{d\sigma^R}{du\,dz} \quad + \quad \frac{1}{\sigma^d_{0,T}}\sigma^V\delta(1-u)\delta(1-z)$$

After $\bar{MS}$ subtraction:

$$\frac{1}{\sigma_0^T}\frac{d\sigma}{du\,dz} = e_q^2\frac{\alpha_s}{2\pi}\Big[\delta(1-z)P_q^T(u)+\delta(1-u)P_q^T(z)\Big]\ln\left(\frac{Q^2}{\mu^2}\right)$$

$$\begin{cases} u = \dfrac{P_1\cdot P_2}{P_1\cdot q} \\ z = \dfrac{2\,P_1\cdot q}{Q^2} \end{cases}$$

NLO

coefficient

$$+e_q^2\frac{4}{3}\frac{\alpha_s}{2\pi}\Bigg\{(\pi^2-8)\delta(1-u)\delta(1-z)$$

$$+\delta(1-z)\left[(1-u)+2u\left(\left(\frac{\log(1-u)}{1-u}\right)_+ + \frac{\log u}{(1-u)_+}\right)\right]$$

$$+\delta(1-u)\left[(1-z)+2z\left(\left(\frac{\log(1-z)}{1-z}\right)_+ + \frac{2\log z}{(1-z)_+}\right)\right]$$

$$+\frac{2uz}{(1-u)_+(1-z)_+}+O(\epsilon)\Bigg\}.$$

CONCLUSIONS AND OUTLOOKS

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- **Fit of the Artu-Collins asymmetry data to extract** $H_1^{\sphericalangle}$ **at NLO.**

Backup

Why do we need transversity beyond LO?

- Have a better knowledge of the proton structure.
  A solid theoretical background is fundamental, especially in view of new data (EIC)

- Transversity can be used to investigate physics beyond the Standard Model

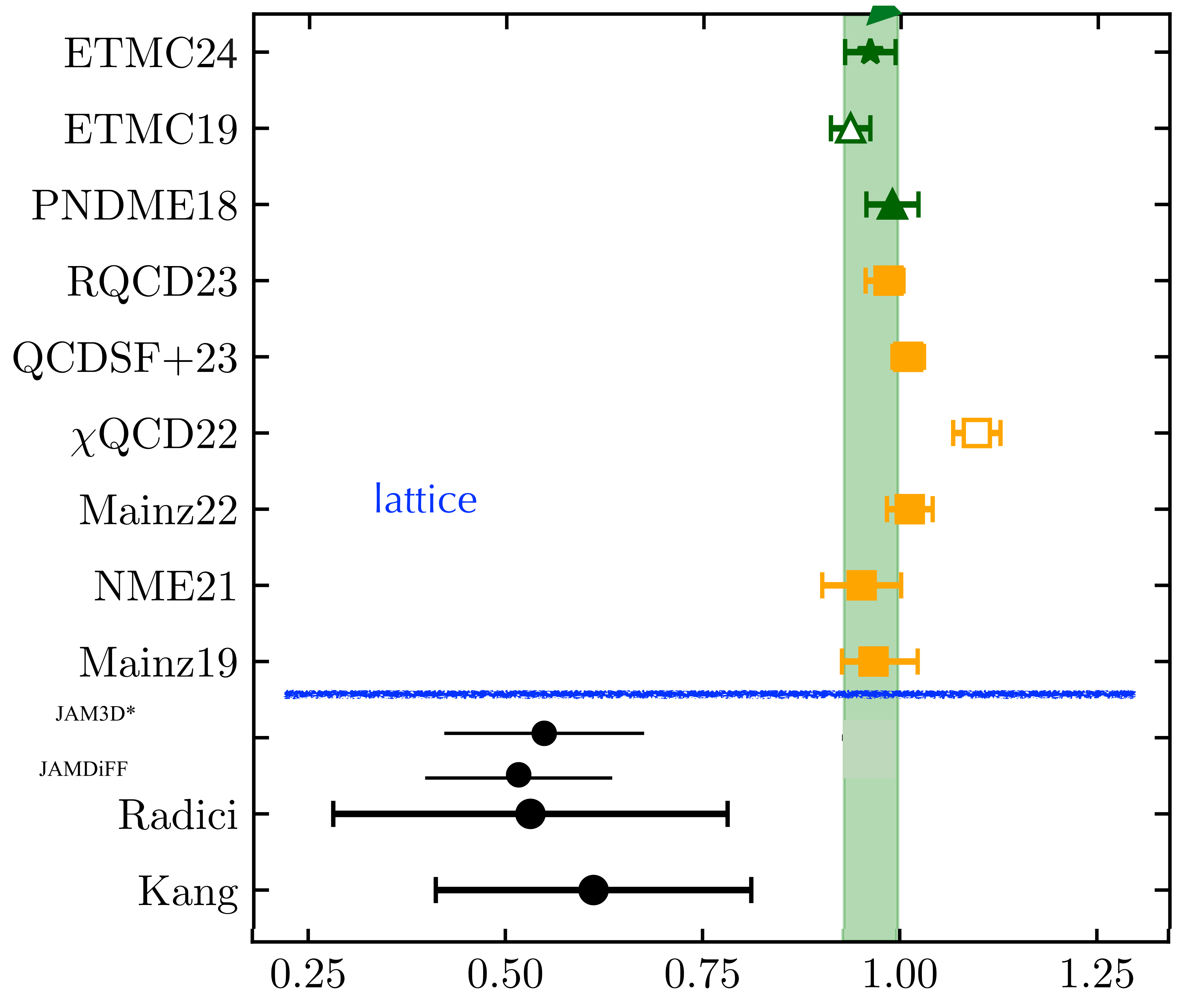
Chiral-odd structures do not appear in the SM
Tree level Lagrangian

<u>Neutrino β-decay</u> $$g_T = \delta u - \delta d$$

$$\delta q = \int_0^1 dx\, h_1^q(x)$$

<u>constraints on CP violation</u>

adapted from C. Alexandrou, QCD Evolution 24



Flavour analysis

$$\frac{d\sigma}{dzdM_hdQ^2} = \frac{4\pi\alpha^2}{Q^2}\sum_q e_q^2 D_1^q(z, M_h, Q) = \sum_q \frac{d\sigma^{\mathbf{q}}}{dzdM_hdQ^2}$$

Sum over $q, \bar{q}$

$$= 2\cdot\left(\frac{4\pi\alpha^2}{Q^2}e_{\mathbf{u}}^2 D_1^{\mathbf{u}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2}e_{\mathbf{d}}^2 D_1^{\mathbf{d}}(z, M_h, Q) + \frac{4\pi\alpha^2}{Q^2}e_{\mathbf{s}}^2 D_1^{\mathbf{s}}(z, M_h, Q) + \dots\right)$$

$$= 2\cdot\frac{d\sigma^{exp}}{dzdM_hdQ^2}\cdot\left(R_{MC}^{\mathbf{u}}(z, M_h, Q) + R_{MC}^{\mathbf{d}}(z, M_h, Q) + R_{MC}^{\mathbf{s}}(z, M_h, Q) + R_{MC}^{\mathbf{c}}(z, M_h, Q)\right)$$

$$R^{\mathbf{q}}(z, M_h, Q) = \frac{d\sigma^{\mathbf{q}}}{d\sigma} = \frac{e_{\mathbf{q}}^2 D_1^{\mathbf{q}}(z, Mh, Q)}{\sum_p e_p^2 D_1^p(z, Mh, Q)}$$

accessible from Monte Carlo simulations

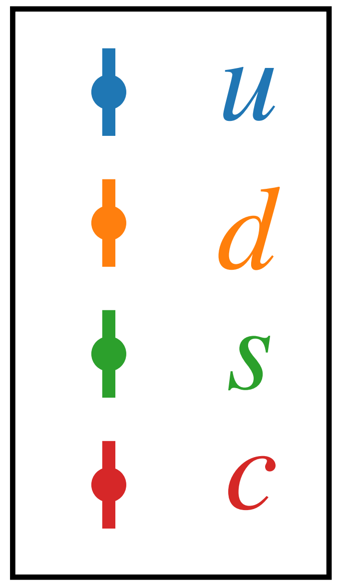
$$\mathscr{F}^{\mathbf{q}}(z, M_h; Q^2) = 2\cdot\frac{d\sigma^{exp}}{dzdM_hdQ^2}\cdot R_{MC}^{\mathbf{q}}(z, M_h, Q)$$

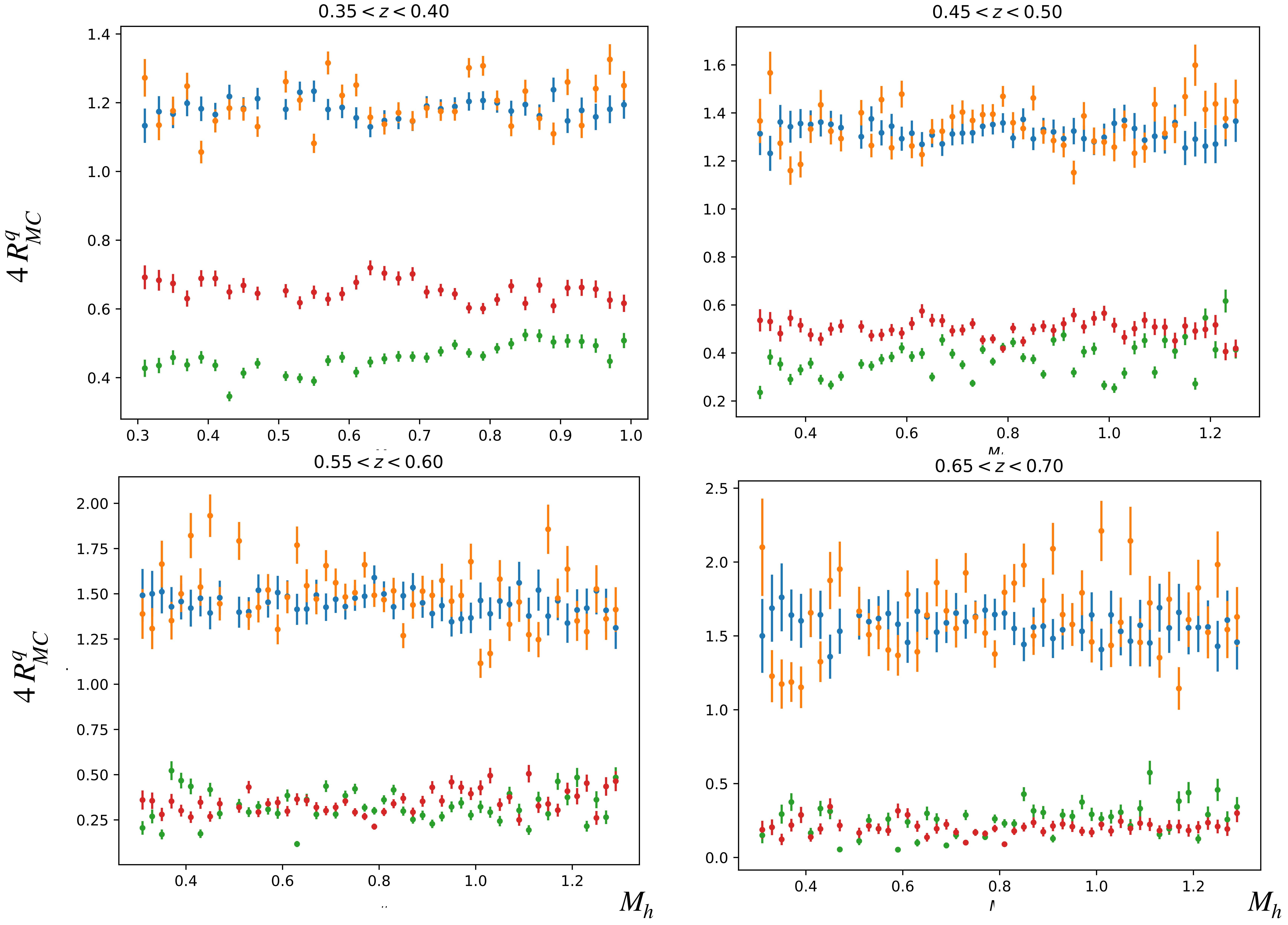
$$\mathscr{F}^q$$

$$q = u, d, s, c$$

$$R^{q}_{MC}(z, M_h, Q)$$

$$\frac{d\sigma}{dz dM_h dQ^2} = 2 \cdot \frac{d\sigma^{exp}}{dz dM_h dQ^2} \cdot \left(R^{\mathbf{u}}_{MC}(z, M_h, Q) + R^{\mathbf{d}}_{MC}(z, M_h, Q) + R^{\mathbf{s}}_{MC}(z, M_h, Q) + R^{\mathbf{c}}_{MC}(z, M_h, Q) \right)$$





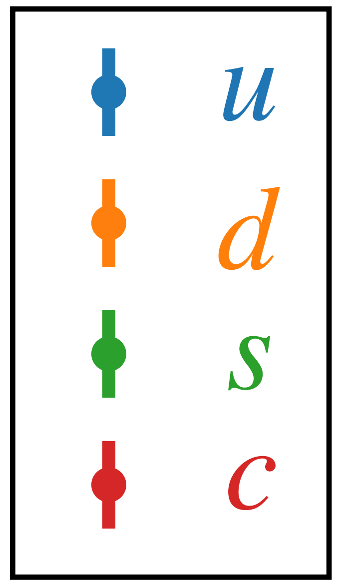
Normalized by the charge and multiplied by 4

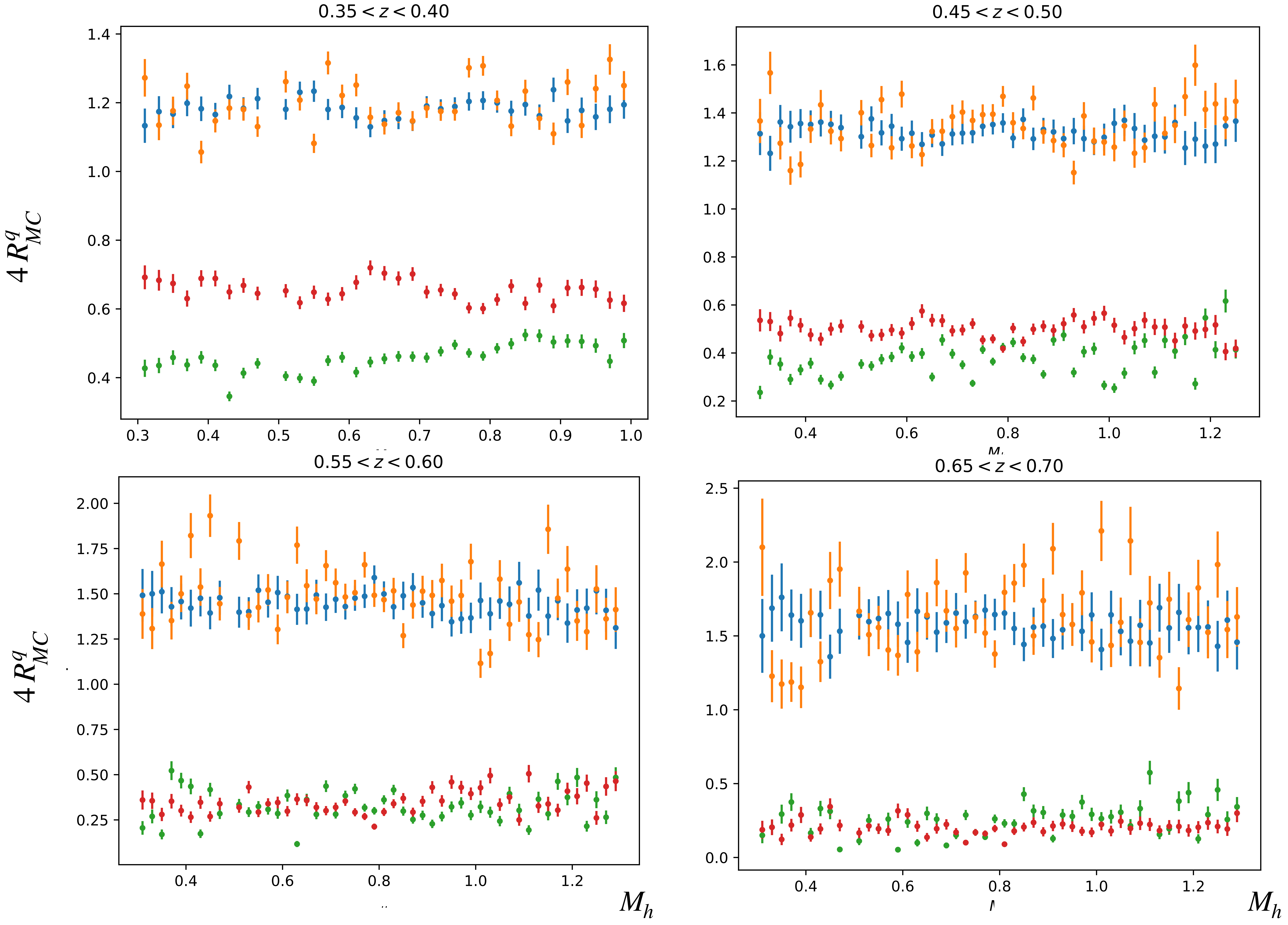
$$4 \frac{D_1^{\mathbf{q}}(z, M_h, Q)}{\sum_p e_p^2 D_1^p(z, Mh, Q)}$$

Luca Polano

$$R^q_{MC}(z, M_h, Q)$$

$$\frac{d\sigma}{dz dM_h dQ^2} = 2 \cdot \frac{d\sigma^{exp}}{dz dM_h dQ^2} \cdot \left(R^{\mathbf{u}}_{MC}(z, M_h, Q) + R^{\mathbf{d}}_{MC}(z, M_h, Q) + R^{\mathbf{s}}_{MC}(z, M_h, Q) + R^{\mathbf{c}}_{MC}(z, M_h, Q) \right)$$





Normalized by the charge and multiplied by 4

$$4\,\frac{D_1^{\mathbf{q}}(z, M_h, Q)}{\sum_p e_p^2\, D_1^p(z, Mh, Q)}$$

$$\mathbf{u} \sim \mathbf{d}$$

Luca Polano

$$W^{\mu\nu} \rightsquigarrow P_i^\mu P_j^\nu,\ P_i^\mu S_{T1}^\nu,\ P_i^\mu S_{T2}^\nu,\ S_{T1}^\mu S_{T2}^\nu,\ g^{\mu\nu}$$

$$i,j = 1,2,3$$

We are integrating over

$$d\Pi_{LIPS} = \frac{d^{d-1}P_1}{(2\pi)^{d-1}}\frac{d^{d-1}P_3}{(2\pi)^{d-1}}(2\pi)^4\delta^{(d)}(q - P_1 - P_2 - P_3)$$

That allow use to substitute $P_3^\mu = q^\mu - P_1^\mu - P_2^\mu$

Great amount of terms, leading do different azimuthal modulations.

$$\left[g_T^{\mu\nu}(S_{T2}\cdot S_{T1}) - S_{T2}^\mu S_{T1}^\nu - S_{T2}^\nu S_{T1}^\mu\right]$$

$$g_T^{\mu\nu} = g^{\mu\nu} - \frac{P_1^\mu P_2^\nu + P_1^\nu P_2^\mu}{P_1\cdot P_2}$$

Any contribution of the kind $P_i^\mu S_{Tj}^\nu$ **should be removed.**

Terms proportional to q^μ **vanish after the contraction with** $L_{\mu\nu}$**.**

$$\sim \int d\Pi_{LIPS}\,\frac{8\,u\,z + 2(D-4)\,(1-u\,z)^2}{z(1-z)(1-u)}\left[g_T^{\mu\nu}(S_{T2}\cdot S_{T1}) - S_{T2}^\mu S_{T1}^\nu - S_{T2}^\nu S_{T1}^\mu\right]$$

Unpolarized

$$\frac{1}{\sigma_0^d}\frac{d\sigma}{du\,dz} = e_q^2\frac{\alpha_s}{2\pi}\left[\delta(1-z)P_{qq}(u)+\delta(1-u)P_{qq}(z)\right]\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$+\,e_q^2\frac{4}{3}\frac{\alpha_s}{2\pi}\left\{\left(\pi^2-8\right)\delta(1-u)\delta(1-z)\right.$$
$$+\,\delta(1-z)\left[(1-u)+(1+u^2)\left(\left(\frac{\ln(1-u)}{1-u}\right)_+ + \frac{\ln u}{(1-u)_+}\right)\right]$$
$$+\,\delta(1-u)\left[(1-z)+(1+z^2)\left(\left(\frac{\ln(1-z)}{1-z}\right)_+ + \frac{2\ln z}{(1-z)_+}\right)\right]$$
$$\left.+\left(z^2+(1+z(u-1))^2\right)\frac{1}{(1-u)_+(1-z)_+}+\mathcal{O}(\epsilon)\right\}.$$

T polarized

$$\frac{1}{\sigma_0^T}\frac{d\sigma}{du\,dz} = e_q^2\frac{\alpha_s}{2\pi}\left[\delta(1-z)P_q^T(u)+\delta(1-u)P_q^T(z)\right]\ln\left(\frac{Q^2}{\mu^2}\right)$$
$$+e_q^2\frac{4}{3}\frac{\alpha_s}{2\pi}\left\{(\pi^2-8)\delta(1-u)\delta(1-z)\right.$$
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