Momentum-Space BK Evolution with RG-Improved Resummation and Kinematical Constraints

Wanchen Li
wanchen1995@outlook.com

Department of Physics, Fudan University

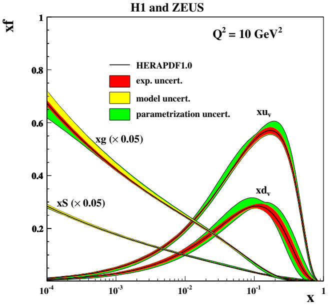
Base on: Krzysztof Kutak, Wanchen Li, Anna Stasto, Robert Straka, Phys.Rev.D 111 (2025) 11, 11602.

QCD Evolution — *May 14th, 2026*

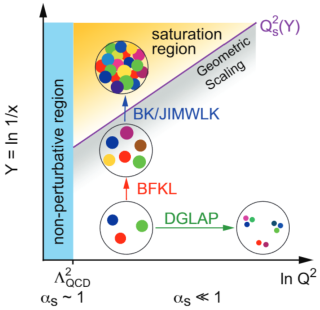
Outline

Gluon saturation in the small x region

- Gluon saturation: one of the major goals of the EIC to address.
- Ensuring unitarity for QCD or any non-Abelian theory.
- Initial condition for heavy-ion collisions, etc.



H1 and ZEUS, 2010.



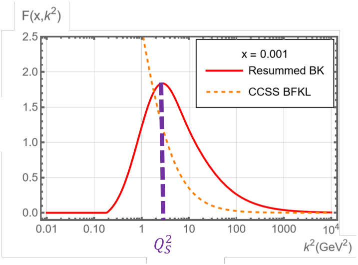
Ciesielski, Kovchegov, Scapparone, DIS2015.

Gluon saturation in the small x region

- The strong rise (orange dashed line) in small k_T^2 region, predicted by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation violates the unitarity.
- When the transverse momentum k_T smaller than the saturation scale Q_s, that is

$$k^2 \equiv k_T^2 < Q_s^2(x).$$

  The gluons start to recombine, regulating the infra-red divergence in the linear BFKL .
- The existence of energy dependent saturation scale is prediction of CGC.



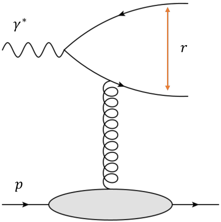
Ciafaloni, Colferai, Salam, Stasto (CCSS)

WL, Stasto '22

BK Equation in the Dipole Model

- The BK equation in the coordinate space:

$$\frac{\partial N(\boldsymbol{r},\boldsymbol{b},x)}{\partial \ln(1/x)} = \bar{\alpha}_s \int \frac{d^2\boldsymbol{r}'\, r^2}{(\boldsymbol{r}'+\boldsymbol{r})^2(\boldsymbol{r}')^2}\Bigg[N\left(\boldsymbol{r}',\boldsymbol{b}+\frac{\boldsymbol{r}'+\boldsymbol{r}}{2},x\right) + N\left(\boldsymbol{r}'+\boldsymbol{r},\boldsymbol{b}+\frac{\boldsymbol{r}'}{2},x\right) - N(\boldsymbol{r},\boldsymbol{b},x) - N\left(\boldsymbol{r}',\boldsymbol{b}+\frac{\boldsymbol{r}'+\boldsymbol{r}}{2},x\right)N\left(\boldsymbol{r}'+\boldsymbol{r},\boldsymbol{b}+\frac{\boldsymbol{r}'}{2},x\right)\Bigg]$$



Dipole scattering in DIS.

  - $N(\boldsymbol{r},\boldsymbol{b},x)$ is the dipole amplitude,
  - $\boldsymbol{r}$ is the size of the dipole,
  - $\boldsymbol{b}$ is the impact parameter.
- The linear terms in the first two lines are BFKL terms
- The $(N\cdot N)$ term in the last line corresponds to the non-linear evolution.

BK Equation in the Momentum Space

- The BK equation in the momentum space:
  Kutak, Kwiecinski '03; Nikolaev, Schafer '06; Bartels, Kutak '08.

$$\mathcal{F}(x,k^2) = \mathcal{F}^{(0)}(x,k^2) + \overline{\alpha}_s \int_x^1 \frac{dz}{z} \int dk'^2 \left[\frac{\mathcal{F}\left(\frac{x}{z},k'^2\right)}{|k^2-k'^2|} - \frac{k'^2}{k^2} \frac{2\min(k^2,k'^2)\,\mathcal{F}(\frac{x}{z},k)}{(k^2+k'^2)|k^2-k'^2|} \right]$$
$$- \frac{2\overline{\alpha}_s^2\pi^2}{N_c^2R^2} \int_x^1 \frac{dz}{z} \left\{ \left[\int_{k^2}^{\infty} \frac{dl^2}{l^2} \mathcal{F}\left(\frac{x}{z},l^2\right) \right]^2 + \mathcal{F}\left(\frac{x}{z},k^2\right) \int_{k^2}^{\infty} \frac{dl^2}{l^2} \ln\left(\frac{l^2}{k^2}\right) \mathcal{F}\left(\frac{x}{z},l^2\right) \right\}.$$

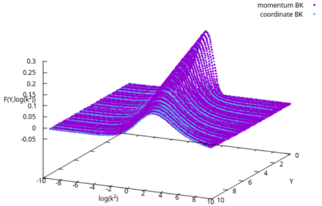
- The $\mathcal{F}(x,k^2)$ is the unintegrated gluon density, related to the usual gluon PDF $g(x,k^2)$ in double leading log approximation (DLLA) by

$$\mathcal{F}(x,k^2) = \frac{\mathrm{d}}{\mathrm{d}k^2} xg(x,k^2).$$

- The arguments of the running coupling $\overline{\alpha}_s$ are associated with unintegrated gluon density respectively.
- R is the size of the hadron's radius, N_c is the number of colors.

BK Equation in the Momentum Space

- Derivation:
  - Fourier transforming from dipole amplitudes, see review in K. Kutak, arXiv:2602.00864.
  - Using the three Pomeron vertex approach, K. Kutak, arXiv:hep-ph/0703068.



- Motivations:
  - Clear pattern on how nonlinear term prevents diffusion into the infrared. K. J. Golec-Biernat, L. Motyka and A. M. Stasto, arXiv:hep-ph/0110325.
  - Unambiguous implementation of kinematical constraints. J. Kwiecinski, A.D. Martin, P.J. Sutton, arXiv:hep-ph/9602320, B. Andersson, G. Gustafson, J. Samuelsson, NPB, 1996, M. Deak, K. Kutak, WL, and A. M. Stasto, arXiv:1906.09062.
  - Convenient implementation of the DGLAP and running coupling.
  - Potential compatibility with high energy factorization ← work in progress.

Resummations

- To summarize the structure of our resummed equation, we give

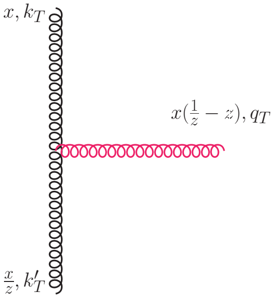
$$\mathcal{F}(x,k^2) = \mathcal{F}^{(0)}(x,k^2) + \mathcal{K}_{\text{res}} \otimes \mathcal{F}(x,k^2) - \mathcal{V} \otimes \mathcal{F}^2(x,k^2).$$

  - $\mathcal{F}^{(0)}(x,k^2)$ is the initial condition,
  - $\mathcal{V} \otimes \mathcal{F}^2(x,k^2)$ is the nonlinear term.
- $\mathcal{K}_{\text{res}} \otimes \mathcal{F}(x,k^2)$ is the resummed linear term

$$\mathcal{K}_{res} \otimes \mathcal{F} = \mathcal{K}_0^{\text{kc}}(z;\boldsymbol{k},\boldsymbol{k}') \overset{z,\boldsymbol{q}}{\otimes} \mathcal{F}(\frac{x}{z},k') + \mathcal{K}_0^{\text{coll}}(z;k,k') \overset{z,k'}{\otimes} \mathcal{F}(\frac{x}{z},k')\ ,$$

  - ‘kc’ stands for kinematical constraints,
  - ‘coll’ stands for the collinear DGLAP evolution.

Kinematical Constraints



Kwiecinski, Martin, Sutton '96

Andersson, Gustafson, Samuelsson '96

- The kinematical constraints can take different form due to the what approximation implemented, we use the first one below:

$$k'^2 < \frac{k^2}{z},$$
$$q^2 < \frac{k^2}{z},$$
$$q^2 < \frac{1-z}{z} k^2.$$

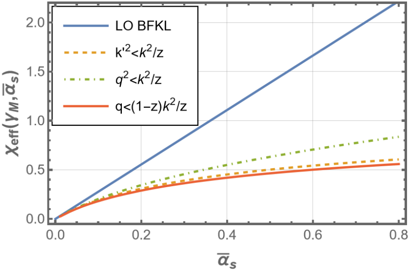
- The Kinematical constraints are due to that dominance of the transverse momentum in the small x region.

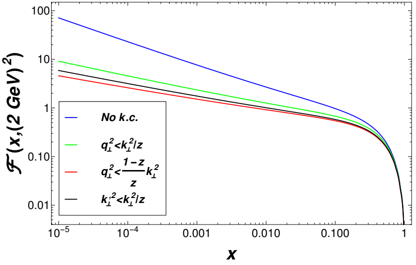
Kinematical Constraints in the BFKL equation

- The kinematical constraint implement in the BFKL equation as

$$\mathcal{K}_0^{\rm kc}(z;\boldsymbol{k},\boldsymbol{k}') \overset{z,\boldsymbol{q}}{\otimes} \mathcal{F}\left(\frac{x}{z},k'\right)$$
$$= \int_x^1 \frac{dz}{z}\int\frac{d^2\boldsymbol{q}}{\pi\boldsymbol{q}^2}\,\bar{\alpha}_s(\boldsymbol{q}^2)\left[\mathcal{F}\left(\frac{x}{z},|\boldsymbol{k}+\boldsymbol{q}|\right)\Theta\left(\frac{k^2}{z}-k'^2\right)-\Theta(k-q)\mathcal{F}(\frac{x}{z},k)\right].$$

- $\chi_{\rm eff}$ is the Pomeron intercept from the BFKL kernel in the Mellin space, indicating the power growth of the Reggeized cross section $\sigma\sim s^{\chi_{\rm eff}}$ with $\chi_{\rm eff}^{\rm LO\ BFKL}=4\ln 2\,\overline{\alpha}_s$.





Deak, Kutak, WL, Stasto '19

Implementing DGLAP Splitting functions

- The resummation techniques that resums collinear and small x logarithms has been developed by several groups: Altarelli-Ball-Forte (ABF); Ciafaloni-Colferai-Salam-Stasto (CCSS), Thorne-White (TW).
- The DGLAP evolution is implemented as

$$\mathcal{K}_0^{\mathrm{coll}}(z;k,k') \overset{z,k'}{\otimes} \mathcal{F}\left(\frac{x}{z},k'\right) = \int_x^1 \frac{dz}{z}\int_0^{k^2}\frac{dk'^2}{k^2}\,\bar{\alpha}_s(k^2) z\tilde{P}_{gg}(z)\mathcal{F}\left(\frac{x}{z},k'\right) + \int_x^1 \frac{dz}{z}\int_{k^2}^{k^2/z}\frac{dk'^2}{k'^2}\,\bar{\alpha}_s(k'^2) z\frac{k'^2}{k^2}\tilde{P}_{gg}(z\frac{k'^2}{k^2})\mathcal{F}\left(\frac{x}{z},k'\right).$$

- The two terms corresponds to the collinear and anti-collinear contribution, where the non-singular part of the splitting function is

$$\tilde{P}_{gg}^{(0)} = P_{gg}^{(0)} - \frac{1}{z}\ ,$$

Structure Function from the k_T Factorization

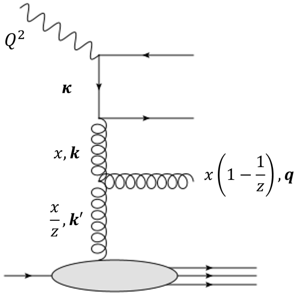
- The structure function F_2 from the k_T factorization is given by

$$F_2(x,Q^2) = \sum_q e_q^2 \, S_q(x,Q^2) \; ,$$

where the sum is over the quark flavors and general expression for $S_q(x,Q^2)$ is

$$S_q(x,Q^2) \; = \; \int_x^1 \frac{dz}{z} \int dk^2 S^q_{\rm box}(z,m_q^2,k^2,Q^2)\mathcal{F}\left(\frac{x}{z},k^2\right) .$$

- k is the gluon transverse momentum, κ is the quark transverse momentum.



DIS in the k_T factorization.

Perturbative Contributions

- The off-shell photon-gluon partonic calculation gives

$$\begin{aligned} S_q(x,Q^2) = & \frac{Q^2}{4\pi^2}\int\frac{dk^2}{k^2}\int_0^1 d\beta\int d\kappa'\alpha_s\left\{\left[\beta^2+(1-\beta^2)\right]\right. \\ & \left(\frac{\boldsymbol{\kappa}}{D_{1q}}-\frac{\boldsymbol{\kappa}-\boldsymbol{k}}{D_{2q}}\right)^2+\left[m_q^2+4Q^2\beta^2(1-\beta)^2\right] \\ & \left.\left(\frac{1}{D_{1q}}-\frac{1}{D_{2q}}\right)^2\right\}\mathcal{F}\left(\frac{x}{z},k^2\right)\Theta\left(1-\frac{x}{z}\right). \end{aligned}$$

- The shifted quark transverse momentum is $\boldsymbol{\kappa}'=\boldsymbol{\kappa}-(1-\beta)\boldsymbol{k}$.
- The energy denominators are

$$\begin{aligned} D_{1q} &= \kappa^2+\beta(1-\beta)Q^2+m_q^2\,, \\ D_{2q} &= (\boldsymbol{\kappa}-\boldsymbol{k})^2+\beta(1-\beta)Q^2+m_q^2\,. \end{aligned}$$

Non-perturbative Contributions

- Approach in (Kwiecinski, Martin, Stasto '97),
  - Depending on the values of transverse momenta k, κ and a typical perturbative cut $k_0 \sim 1\,\text{GeV}$, , the perturbative and non-perturbative contributions are categorized as:

$$S_q = S_q^{(a)} + S_q^{(b)} + S_q^{(c)}.$$

  - $S_q^{(a)}$: $k^2 < k_0^2, \quad \kappa^2 < k_0^2$; Modeled soft Pomeron contribution.
  - $S_q^{(b)}$: $k^2 < k_0^2 < \kappa^2$; Previously modeled with collinear approximation.
  - $S_q^{(c)}$: $k^2 > k_0^2$; Perturbative contribution from the k_T factorization.
- Now we extend the lower bound $k_{\text{min}}^2 \ll k_0^2$ in the BK evolution, thus $S_q^{(b)}$ is also captured by the perturbative calculation.

$$S_q = S_q^{(a)} + S_q^{\text{pert}} \tag{1}$$

The Fit to the Reduced Cross Sections

- We employ Golec-Biernat-Wuesthoff (GBW) inspired initial condition

$$\mathcal{F}^{(0)}(x,k) \ = A\,\alpha_s(k^2)\,(1-x)^{\alpha}\,x^{\beta}\,(k^2)^{\gamma}\,e^{-B^2k^2}.$$

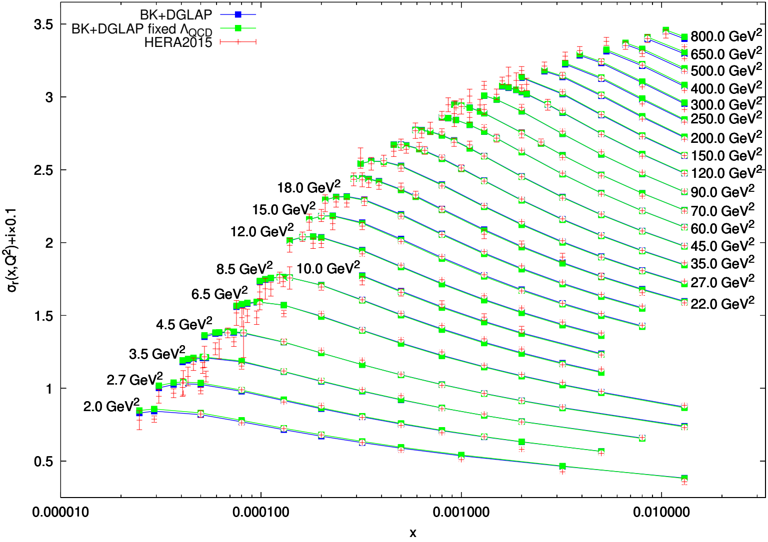
- The soft Pomeron contribution is parametrized as

$$S_q^{(a)} = C_{I\!P}\,x^{-\lambda}\,(1-x)^8.$$

- We fit to the HERA reduced cross section σ_r data (2015), with
  - $\chi^2/\text{dof} = 1.6$ for fitted Λ_{QCD},
  - $\chi^2/\text{dof} = 2.0$ for fixed $\Lambda_{\text{QCD}} = 0.289$ GeV.
- Together with proton's radius R, the parameters are given as follows

	α	β	A	B^2	γ	$C_{I\!P}$	λ	R^2	Λ_{QCD}
$\chi^2/\text{dof} = 1.6$	0.80662	-0.42651	0.74038	0.64239	1.09987	0.6163	-0.02361	2.50003	0.499
$\chi^2/\text{dof} = 2.0$	0.67356	-0.41625	0.65917	0.47992	1.05338	0.572	-0.00223	2.54913	fixed

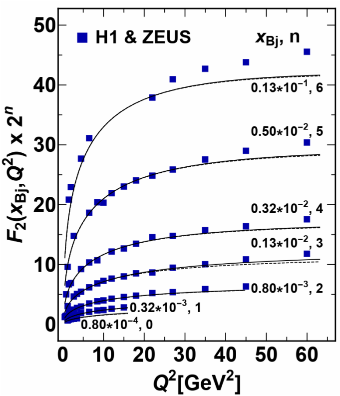
Fit to the Reduced Cross Sections

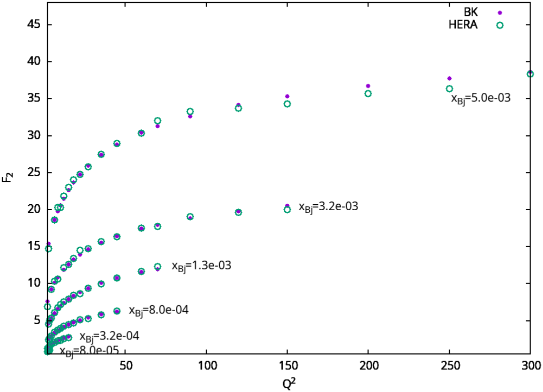


Fit to the Structure Function

BK without DGLAP

BK + DGLAP (our fit)

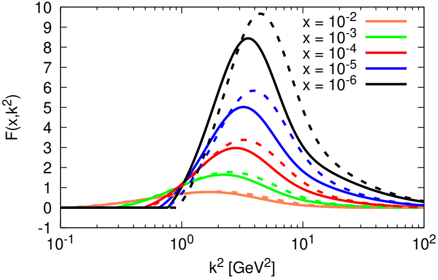


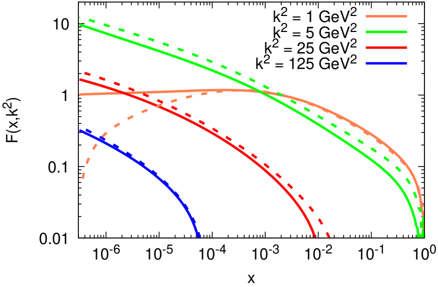


Dipole BK fits from Sanhueza, Carrido, Guevara, 2024.

- Including DGLAP on top of small x evolution enables a better fit for a wider range of Q^2.

The Unintegrated Gluon Density





- Our approach with an extensive range of k^2 allows the extracting of the saturation scale Q_S^2.
- The saturation scale corresponds the k^2 position for max unintegrated gluon density.
- One can see the Q_S^2 increases when x decreases, as expected from the map of high energy QCD.

Conclusion

- We implement the kinematical constraint and DGLAP resummation into the BK evolution.
- We achieve good fits to the structure functions and reduced cross sections, while the unintegrated gluon density is extended to low values of transverse momenta.
- Information of saturation scales can be extracted from our fit.
- Work in Progress
  - Incorporating of the CCSS resummation with NLL BFKL into the framework.
  - Related phenomenological studies in high energy factorization.
  - We plan to make our code more public available.

Back-ups: LO BK in two spaces

- For the comparison between coordinate and momentum space BK solutions, we use the GBW input:

$$N(r,x) = 1 - \exp\left[-\frac{1}{4}r^2 Q_{s0}^2\right]$$

$$\mathcal{F}^{(0)}(x,k^2) = \frac{N_c}{2\pi^2\alpha_s}\frac{k^2}{Q_{s0}^2}\exp\left(-\frac{k^2}{Q_{s0}^2}\right).$$

with $Q_{s0} = 1$ GeV and $R = 1$ GeV^{-1}.

Back-ups: the b dependency

- We make ansatz that the dipole amplitude can be factorized

$$N(r, b, x) = N(r, x)\, S(b)$$

where the $S(b)$ is an profile function of the proton with normalizations

$$\int d^2\boldsymbol{b}\, S(b) = 1,$$
$$\int d^2\boldsymbol{b}\, S^2(b) = \frac{1}{\pi R^2},$$

and R is the proton size.