TMD factorization in diffractive heavy quark production in $\gamma^* A$

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Outline

Production of heavy quarks in diffraction

1) Gluon diffractive TMD
    a. Introduction
    b. Color Glass Condensate theory
    c. Light-cone perturbation theory (LCPT)
    d. TMD factorization
2) Quark diffractive TMD
3) Summary

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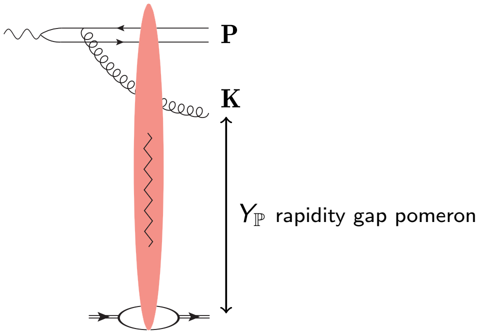
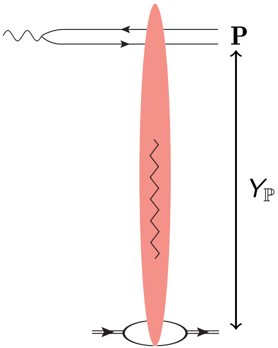
Introduction

Motivation



Diffractive jets

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}^2\mathbf{P}\mathrm{d}^2\mathbf{K}\mathrm{d}Y_{\mathbb{P}}} \sim \frac{1}{P_\perp^4}}$$



Exclusive jets

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}^2\mathbf{P}} \sim \frac{1}{P_\perp^6}} \rightarrow \text{More suppressed!}$$

- New experimental data on diffractive $J/\psi, D, \Upsilon$ production

Introduction

General picture

- Correlation limit (back-to-back)

$$\text{hard jets:} \quad m,\ k_{1\perp} \simeq k_{2\perp} \sim Q \gg Q_s$$
$$\text{semi-hard jet:} \quad k_{3\perp} \sim Q_s$$

quark mass m
saturation scale $Q_s \sim$ GeV

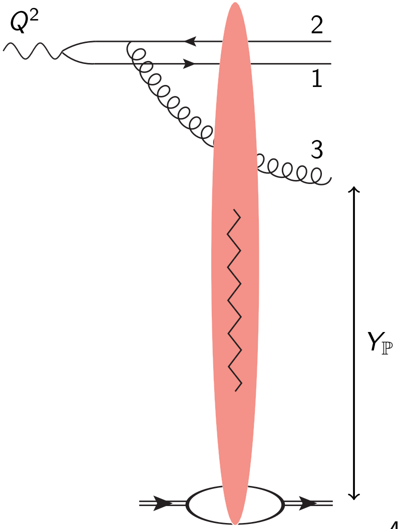
- Goal $\rightarrow$ TMD factorization from dipole picture

"hard" factor

$$\frac{d\sigma}{dz_1 dz_2 d^2\mathbf{K} d^2\mathbf{P} dY_{\mathbb{P}}} = H(z_1, z_2, P_\perp, m) \, \frac{dx\mathcal{G}_{\mathbb{P}}(x, x_{\mathbb{P}}, K_\perp)}{d^2\mathbf{K}}$$

gluon diffractive TMD

relative momentum $\mathbf{P} \simeq \mathbf{k}_1 \simeq -\mathbf{k}_2$
imbalance $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = -\mathbf{k}_3$



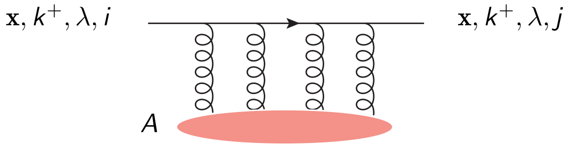
Outline

Production of heavy quarks in diffraction

Color Glass Condensate

Eikonal interaction

- At small x many gluons in the target $\rightarrow$ described as a classical gluon field A
- Eikonal interaction probe with color field in mixed space $(\mathbf{x}, k^+)$:



λ: helicity, i, j: color

Wilson line

Path ordered exponential $\rightarrow$ resums multiple scatterings (not perturbative)

$$V_{F,A}(\mathbf{x}) = \mathcal{P} \exp\left(-ig \int dx^+ A_a^-(x^+, \mathbf{x})\, t^a\right)$$

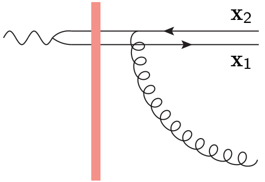
t^a : $SU(3)$ color matrix

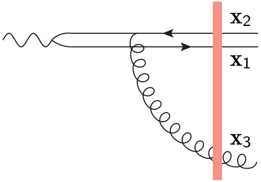
Color Glass Condensate

Eikonal interaction

- scattering operator $\hat{S}$ on states $\Rightarrow$ color rotation $(i \to j)$ in mixed space $\Rightarrow$ Wilson line

$$\hat{S}\,\big|q(k^+, \mathbf{x}, i)\big\rangle = \sum_j \underbrace{V_F(\mathbf{x})_{ji}}_{\text{Wilson line}}\,\big|q(k^+, \mathbf{x}, j)\big\rangle$$





$$S_{q\bar{q}}(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{N_c}\big\langle \operatorname{Tr}\big[V_F(\mathbf{x}_1)V_F^\dagger(\mathbf{x}_2)\big]\big\rangle$$

$$S_{q\bar{q}g}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{1}{N_c C_F}\big\langle V_A^{ba}(\mathbf{x}_3)\operatorname{Tr}\big[t^b V_F(\mathbf{x}_1) t^a V_F^\dagger(\mathbf{x}_2)\big]\big\rangle$$

Hard quarks

- Change of variables

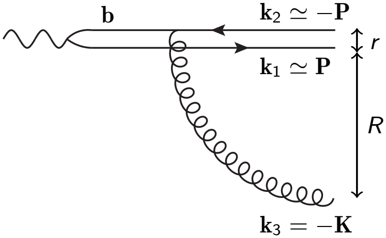
relative momentum: $\mathbf{P} \equiv z_2\mathbf{k}_1 - z_1\mathbf{k}_2$

total momentum: $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$

center of mass: $\mathbf{b} \equiv z_1\mathbf{x}_1 + z_2\mathbf{x}_2$

$q\bar{q}$ size: $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$

separation $g - q\bar{q}$: $\mathbf{R} = \mathbf{x}_3 - \mathbf{b}$



- Correlation limit: $\mathbf{P}, m \sim Q \gg K_\perp = k_{3\perp} \sim Q_s$
  - Exponent of the Fourier transform has $\mathbf{P} \cdot \mathbf{r}$
  - $\Psi^{\gamma^* \to q\bar{q}g}$ from LCPT has Bessel function $K_n(r\sqrt{Q^2 z_1 z_2 + m^2})$

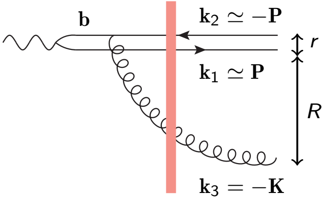
$\implies$ $r \ll R$

Color Glass Condensate

Effective gg dipole

- hard quarks $\to r \ll R$ in coordinate space

$$\mathcal{S}_{q\bar{q}g}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \simeq \mathcal{S}_{q\bar{q}g}(\mathbf{b},\mathbf{b},\mathbf{R}+\mathbf{b}) \sim \left\langle U^{ba}(\mathbf{R}+\mathbf{b})\,\mathrm{Tr}\left[t^b U^{\dagger ab}(\mathbf{b}) t^b\right]\right\rangle \sim \left\langle \mathrm{Tr}\left[U(\mathbf{R}+\mathbf{b})U^\dagger(\mathbf{b})\right]\right\rangle \sim \mathcal{S}_g(\mathbf{R})\,.$$



$S_g(\mathbf{R})$: Effective gluon-gluon dipole of transverse separation $\mathbf{R}$

$$S_{q\bar{q}g}(\mathbf{b},\mathbf{b},\mathbf{R}+\mathbf{b}) - S_{q\bar{q}}(\mathbf{b},\mathbf{b}) \simeq S_g(\mathbf{R}) - 1 = -\mathcal{N}_g(\mathbf{R})$$ independent of $\mathbf{r}$

- small dipole $S_{q\bar{q}}(\mathbf{b},\mathbf{b}) \to$ color transparency

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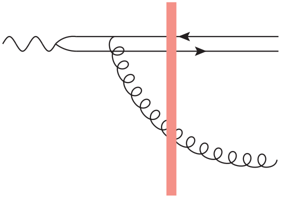
Light Cone Perturbation Theory

Invariant amplitude

- Fock state expansion projectile (dipole frame)



$$\left|\gamma^*\right\rangle^{\text{in}} \sim \left|\gamma^*\right\rangle_0 + \Psi^{\gamma^*\to q\bar{q}}\left|q\bar{q}\right\rangle_0 + \Psi^{\gamma^*\to q\bar{q}g}\left|q\bar{q}g\right\rangle_0 + \ldots$$

$$\left|q\bar{q}g\right\rangle_{\text{s}}^{\text{out}} \sim \left|q\bar{q}g\right\rangle_0 + \Psi^{q\bar{q}g\to q\bar{q}}\left|q\bar{q}\right\rangle_0 + \cdots$$

color singlet
($\mathbb{P}$ exchange)

$\Psi(\mathbf{k}, z)$: perturbative light cone wavefunctions

- Invariant amplitude from scattering matrix element:

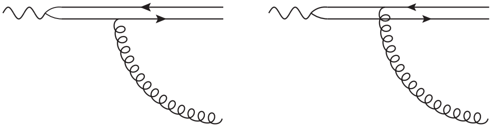
$$\mathcal{M}^{\gamma^*\to q\bar{q}g} \sim {}^{\text{out}}_{\text{s}}\left\langle q\bar{q}g\right| 1 - \hat{S} \left|\gamma^*\right\rangle^{\text{in}}$$

$\hat{S}$: scattering operator
z: fraction of q^+ carried by final partons

Light Cone Perturbation Theory

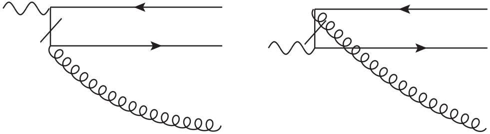
Light cone wavefunction

- $\Psi^{\gamma^* \to q\bar{q}g} = \Psi^{\gamma^* \to q\bar{q}g}_{\text{regular}} + \Psi^{\gamma^* \to q\bar{q}g}_{\text{instantaneous}}$



Regular contribution

$+$



Instantaneous contribution

Light Cone Perturbation Theory

Light cone wavefunction

$$\Psi^{\gamma^*\to q\bar{q}g} = \Psi^{\gamma^*\to q\bar{q}g}_{\text{regular}} + \Psi^{\gamma^*\to q\bar{q}g}_{\text{instantaneous}} = \varepsilon^i_\lambda \varepsilon^{*m}_\sigma H^{in}(z_1, z_2, \mathbf{P}, m, Q)\frac{k_3^n k_3^m + \delta^{nm}\Omega^2/2}{k_{3\perp}^2 + \Omega^2}$$

$$= \varepsilon^i_\lambda \varepsilon^{*m}_\sigma \left(H^{in}(z_1, z_2, \mathbf{P}, m, Q)\frac{k_3^n k_3^m - \delta^{nm} k_{3\perp}^2/2}{k_{3\perp}^2 + \Omega^2} + \frac{H^{in}(z_1, z_2, \mathbf{P}, m, Q)}{2} \right)$$

traceless 2D tensor ($\mathbb{P}$) diagonal term

- H^{in}: hard factor

$$H^{in}(z_1, z_2, \mathbf{P}, m, Q) = \frac{\varphi^{ij}(z_1, h_1)\delta^{jn}\delta_{h_1 - h_2}}{P_\perp^2 + \tilde{Q}^2} - \frac{\varphi^{ij}(z_1, h_1) 2 P^j P^n \delta_{h_1 - h_2} - 2\sqrt{2} m \varepsilon^{i*}_{h_1} P^n \delta_{h_1 h_2}}{(P_\perp^2 + \tilde{Q}^2)^2}$$

$\tilde{Q}^2 = Q^2 z_1 z_2 + m^2$

$\Omega = \frac{z_3}{z_1 z_2}(P_\perp^2 + \tilde{Q}^2)$

Light Cone Perturbation Theory

Light cone wavefunction

- Fourier Transform to mixed space $(z, \mathbf{x})$

  (!) small dipole $\mathbf{r}$ (color transparency)$\rightarrow$ neglect the change due to the scattering of $\mathbf{P}$.

$$\Psi^{\gamma^* \to q\bar{q}g}(z_1, z_2, z_3, \mathbf{P}, \mathbf{R}) = \int \frac{\mathrm{d}^2\mathbf{k}_3}{(2\pi)^2} e^{i\mathbf{k}_3 \cdot \mathbf{R}} \left[\Psi^{im}_{\mathbb{P}}(z_1, z_2, z_3, \mathbf{P}, \mathbf{k}_3) + \cancel{\Psi^{im}_{\text{diag}}(z_1, z_2, \mathbf{P})}\right]$$

$$= H^{in}(z_1, z_2, \mathbf{P}, m, Q) \frac{1}{2\pi} \left[\frac{\delta^{nm}}{2} - \left(\frac{R^n R^m}{R^2}\right)\right] \Omega^2 K_2(\Omega R) \,.$$

Light Cone Perturbation theory

Invariant amplitude

$$\mathcal{M}^{\gamma^* \to q\bar{q}g} \sim \varepsilon^i_\lambda \varepsilon^{*m}_\sigma H^{in}(z_1, z_2, \mathbf{P}, m, Q) G^{nm}(\mathbf{K}, \Omega, Y_{\mathbb{P}})$$

where

$$G^{nm}(\mathbf{K}, \Omega, Y_{\mathbb{P}}) = \frac{1}{2\pi} \int \mathrm{d}^2\mathbf{R} e^{i\mathbf{KR}} \left(\delta^{nm} - \frac{2R^n R^m}{R^2} \right) \Omega^2 K_2(\Omega R) \mathcal{N}_g(R) = \left(\frac{K^n K^m}{K_\perp^2} - \frac{\delta^{nm}}{2} \right) \mathcal{G}(K_\perp, \Omega, Y_{\mathbb{P}})$$

and

$$\mathcal{G}(K_\perp, \Omega, Y_{\mathbb{P}}) = 2(2\pi)\Omega^2 \int_0^\infty \mathrm{d}R R J_2(K_\perp R) K_2(\Omega R) \mathcal{N}_g(R, Y_{\mathbb{P}})$$

- In collinear factorization: semi-hard gluon TMD is a target distribution. But here (!)

$$\Omega^2 \simeq \frac{z_3}{z_1 z_2}(P_\perp^2 + \tilde{Q}^2)$$

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TMD factorization

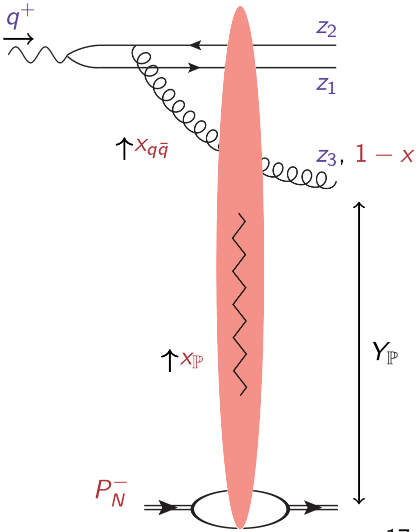
Strategy

	dipole picture (LCPT)
q^+	longitudinal momentum photon
z	fraction of q^+ carried by final partons

$\downarrow$

	target variables
P_N^-	longitudinal momentum target
$x_{\mathbb{P}}$	fraction of P_N^- carried by the $\mathbb{P}$
$x_{q\bar{q}}$	fraction of P_N^- transferred to $q\bar{q}$
$x = x_{q\bar{q}}/x_{\mathbb{P}}$	fraction of $\mathbb{P}P_N^-$ carried by the gluon

gluon DTMD $\rightarrow$ target distribution $\dfrac{\mathrm{d}x\mathcal{G}_{\mathbb{P}}(x, x_{\mathbb{P}}, K_\perp)}{\mathrm{d}^2\mathbf{K}}$



TMD factorization

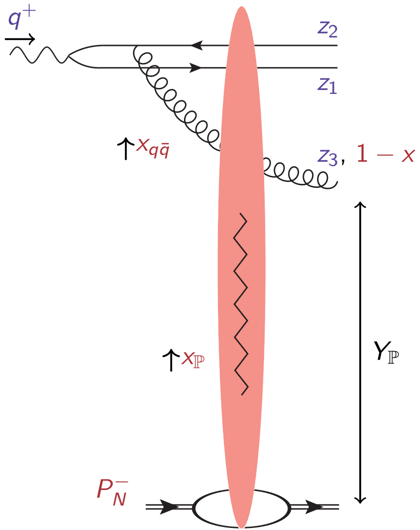
Change of variables

$$\Omega^2 \simeq \frac{z_3}{z_1 z_2}(P_\perp^2 + \tilde{Q}^2) \quad \rightarrow \quad \Omega^2 = \frac{x}{1-x}K_\perp^2$$

and

$$\frac{\mathrm{d}z_3}{z_3} = \frac{\mathrm{d}x_{\mathbb{P}}}{x_{\mathbb{P}} - x_{q\bar{q}}} = \frac{\mathrm{d}x_{\mathbb{P}}}{x_{\mathbb{P}}}\frac{1}{1-x}$$

→ Gluon is seen as emitted from the $\mathbb{P}$.



TMD factorization

hard factor — gluon DTMD

$$\frac{\mathrm{d}\sigma_{\mathrm{D}}^{\gamma_T^* A\to q\bar{q}gA}}{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}^2\mathbf{K}\mathrm{d}^2\mathbf{P}\mathrm{d}Y_{\mathbb{P}}} = \delta(1-z_1-z_2)\alpha_{\mathrm{e.m.}}\alpha_{\mathrm{s}}e_f^2\left[(z_1^2+z_2^2)\frac{P_\perp^4+\tilde{Q}^4}{\left[P_\perp^2+\tilde{Q}^2\right]^4}+2m^2\frac{P_\perp^2}{\left[P_\perp^2+\tilde{Q}^2\right]^4}\right]\frac{\mathrm{d}xG_{\mathbb{P}}(x,K_\perp,Y_{\mathbb{P}})}{\mathrm{d}^2\mathbf{K}}$$

$$\tilde{Q}^2 = z_1 z_2 Q^2 + m^2$$

Gluon DTMD

$$\frac{\mathrm{d}xG_{\mathbb{P}}(x,K_\perp,Y_{\mathbb{P}})}{\mathrm{d}^2\mathbf{K}} \equiv \frac{S_\perp(N_c^2-1)}{8\pi^4(1-x)}\left[\Omega^2\int_0^\infty \mathrm{d}R\, R\, J_2(K_\perp R)\, K_2(\Omega R)\,\mathcal{N}_g(R,Y_{\mathbb{P}})\right]^2$$

density of gluons with $K_\perp$ and x in the $\mathbb{P}$ wavefunction.

$$\Omega^2 = K_\perp^2 x/(1-x)$$

- Same hard factor as in inclusive DIS (Dominguez, Marquet, Xiao, Yuan (2011))🔗
- Generalizes result of Iancu, Mueller, Triantafyllopoulos, Wei (2022)🔗 to heavy quarks

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Introduction

General picture

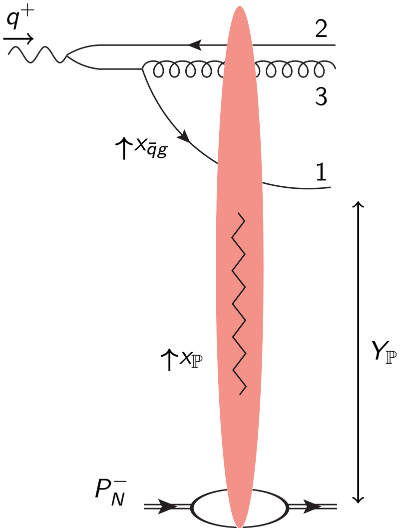
- Correlation limit (back-to-back):

$$k_{2\perp},\ k_{3\perp} \sim Q \gg k_{1\perp}, m \sim Q_s$$

- Goal $\rightarrow$ TMD factorization from dipole frame

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}^2\mathbf{K}\mathrm{d}^2\mathbf{P}\mathrm{d}Y_{\mathbb{P}}} = \overbrace{H(z_2, z_3, P_\perp, Q)}^{\text{"hard" factor}} \underbrace{\frac{\mathrm{d}x\mathcal{G}_{\mathbb{P}}(x, x_{\mathbb{P}}, K_\perp, m)}{\mathrm{d}^2\mathbf{K}}}_{\text{quark diffractive TMD}}$$

relative momentum $\mathbf{P} \simeq \mathbf{k}_2 \simeq -\mathbf{k}_3$
imbalance $\mathbf{K} = \mathbf{k}_2 + \mathbf{k}_3 = -\mathbf{k}_1$

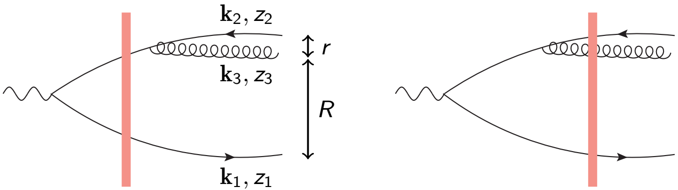


Light Cone Perturbation theory

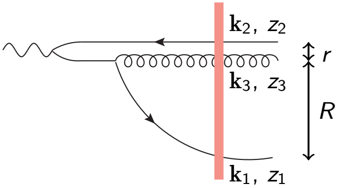
Invariant amplitude

- Diagrams that contribute to the amplitude

  - Gluon emission by antiquark before and after shockwave



  - Gluon emission by quark



- Calculation in LCPT + change of variables:

  longitudinal momentum fraction z_1 of the projectile $\rightarrow$ target longitudinal momentum x.

TMD factorization

Cross section

$$\frac{\mathrm{d}\sigma^{\gamma^* \to q\bar{q}g}}{\mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}^2\mathbf{K} \mathrm{d}^2\mathbf{P} \mathrm{d}Y_{\mathbb{P}}} = H(z_2, z_3, P_\perp) \frac{\mathrm{d}xq_{\mathbb{P}}(x, x_{\mathbb{P}}, K_\perp, m)}{\mathrm{d}^2\mathbf{K}}$$

- Hard factor:

$$H(z_2, z_3, P_\perp, Q) = \delta(1 - z_2 - z_3) \frac{\alpha_s C_F}{2\pi^2} \frac{1}{z_3} \frac{\hat{Q}^2 \left[(P_\perp^2 + \hat{Q}^2)^2 + P_\perp^4 z_3^2 + \hat{Q}^4 z_2^2\right]}{P_\perp^2 (P_\perp^2 + \hat{Q}^2)^3}$$

$$\hat{Q}^2 = z_2 z_3 Q^2$$

- Quark diffractive TMD depends on target variables and the quark mass:

$$\frac{\mathrm{d}xq_{\mathbb{P}}(x, x_{\mathbb{P}}, K_\perp, m)}{\mathrm{d}^2\mathbf{K}} = \frac{S_\perp N_c}{4\pi^3} \frac{1}{2\pi} \frac{x}{1-x} \frac{K_\perp^2 + m^2}{(xK_\perp^2 + m^2)} \left(\mathcal{Q}_1^2(x, x_{\mathbb{P}}, K_\perp) + \frac{(1-x)m^2}{xK_\perp^2 + m^2} \mathcal{Q}_0^2(x, x_{\mathbb{P}}, K_\perp) \right)$$

\# density of quarks with $K_\perp$, x and m in the $\mathbb{P}$ wavefunction.

$$\mathcal{Q}_1(x, x_{\mathbb{P}}, K_\perp) \equiv \omega^2 \int_0^\infty \mathrm{d}R\, R\, J_1(KR)\, K_1(\omega R)\, \mathcal{N}(R)$$

$$\mathcal{Q}_0(x, x_{\mathbb{P}}, K_\perp) = \omega^2 \int_0^\infty \mathrm{d}R\, R\, J_0(KR)\, K_0(\omega R)\, \mathcal{N}(R)$$

$$\omega^2 = (m^2 + xK_\perp^2)/(1-x)$$

Summary

- TMD factorization for diffractive production of a pair of heavy quarks in the CGC formalism
  - Hard $\bar{q}q$, semi-hard g

    $\rightarrow$ known massive hard factor inclusive DIS (Dominguez, Marquet, Xiao, Yuan (2011))

    $\rightarrow$ generalized result to heavy quarks of Iancu, Mueller, Triantafyllopoulos, Wei (2022)

    $\rightarrow$ mass independent gluon DTMD
  - Hard $\bar{q}g$, semi-hard q

    $\rightarrow$ new mass dependent quark DTMD
- Outlook

  $\rightarrow$ quarkonium, open charm predictions for UPCs at LHC, diffraction at EIC

Backup

- Invariant mass

$$M_{q\bar{q}}^2 = (k_1 + k_2)^2 = \left(\frac{k_{1\perp}^2}{z_1} + \frac{k_{2\perp}^2}{z_2} + \frac{m^2}{z_1 z_2} - K_\perp^2 \right)$$

$$M_{q\bar{q}g}^2 = (k_1 + k_2 + k_3)^2 = \left(\frac{k_{1\perp}^2}{z_1} + \frac{k_{2\perp}^2}{z_2} + \frac{k_{3\perp}^2}{z_3} + \frac{m^2}{z_1 z_2} \right) = M_{q\bar{q}}^2 + K_\perp^2 + \frac{k_3^2}{z_3}$$

- β variable

$$\beta \equiv \frac{Q^2}{Q^2 + M_{q\bar{q}g}^2}$$

- Target longitudinal momentum fraction

$$x_{q\bar{q}} = \frac{Q^2 + M_{q\bar{q}}^2 + K_\perp^2}{2P_N \cdot q} \qquad x_{\mathbb{P}} = \frac{Q^2 + M_{q\bar{q}g}^2}{2q \cdot P_N}$$

$$x \equiv \frac{x_{q\bar{q}}}{x_{\mathbb{P}}} = \frac{Q^2 + M_{q\bar{q}}^2 + K_\perp^2}{Q^2 + M_{q\bar{q}}^2 + K_\perp^2 + k_{3\perp}^2/z_3},$$

Backup

- Mixed space wavefunction and scattering
  - Fourier transform Pomeron term (traceless 2D tensor):

    small dipole $\mathbf{r}$ (color transparency)$\rightarrow$ neglect the change due to the scattering of $\mathbf{P}$.

$$\Psi_{\mathbb{P}}^{im}(z_1, z_2, z_3, \mathbf{P}, \mathbf{R}) = \int \frac{\mathrm{d}^2\mathbf{k}_3}{(2\pi)^2} e^{i\mathbf{k}_3\cdot\mathbf{R}} \Psi_{\mathbb{P}}^{im}(z_1, z_2, z_3, \mathbf{P}, \mathbf{k}_3)$$

$$= H^{in}(z_1, z_2, \mathbf{P}, m) \int \frac{\mathrm{d}^2\mathbf{k}_3}{(2\pi)^2} e^{i\mathbf{k}_3\cdot\mathbf{R}} \frac{k_3^n k_3^m - \delta^{nm}\frac{k_{3\perp}^2}{2}}{k_{3\perp}^2 + \Omega^2}$$

$$= H^{in}(z_1, z_2, \mathbf{P}, m) \frac{1}{2\pi} \left[\frac{\delta^{nm}}{2} - \left(\frac{R^n R^m}{R^2}\right)\right] \Omega^2 K_2(\Omega R)\,.$$

  - The Fourier transform of $\Psi_{\text{diag}}(\mathbf{P})$ gives a $\delta^2(\boldsymbol{R})$.

$$\delta^2(\boldsymbol{R})\mathcal{N}(\mathbf{R}) \rightarrow 0 \ \text{ due to color transparency}$$

$\rightarrow$ We keep only the $\Psi_{\mathbb{P}}$ term.

Backup

- Change of variables: $z_3 \to x$ or $x_{\mathbb{P}}$ using

$$\Omega^2 \simeq \frac{z_3}{z_1 z_2}(P_\perp^2 + \tilde{Q}^2) \qquad x \simeq \frac{P_\perp^2 + \tilde{Q}^2}{P_\perp^2 + \tilde{Q}^2 + z_1 z_2 \frac{K_\perp^2}{z_3}}$$

we get

$$\boxed{\Omega^2 = \frac{x}{1-x} K_\perp^2}$$

Using also $x_{\mathbb{P}}$:

$$x_{\mathbb{P}} = \frac{Q^2 + M_{q\bar{q}g}^2}{2q \cdot P_N} \qquad x_{q\bar{q}} = \frac{Q^2 + M_{q\bar{q}}^2 + K_\perp^2}{2P_N \cdot q} \qquad x \equiv \frac{x_{q\bar{q}}}{x_{\mathbb{P}}} \simeq \frac{Q^2 + M_{q\bar{q}}^2 + K_\perp^2}{Q^2 + M_{q\bar{q}}^2 + K_\perp^2 + k_{3\perp}^2/z_3}$$

we have the change of variables:

$$\frac{x_{\mathbb{P}} - x_{q\bar{q}}}{x_{q\bar{q}}} = \frac{x_{\mathbb{P}}}{x_{q\bar{q}}} - 1 = \frac{Q^2 + M_{q\bar{q}}^2 + K_\perp^2 + k_{3\perp}^2/z_3}{Q^2 + M_{q\bar{q}}^2 + K_\perp^2} - 1 = \frac{k_{3\perp}^2/z_3}{Q^2 + M_{q\bar{q}}^2 + K_\perp^2}$$

Taking the logarithmic differential:

$$\boxed{\frac{\mathrm{d}z_3}{z_3} = \frac{\mathrm{d}x_{\mathbb{P}}}{x_{\mathbb{P}} - x_{q\bar{q}}} = \frac{\mathrm{d}x_{\mathbb{P}}}{x_{\mathbb{P}}} \frac{1}{1-x}}$$