Gluon production at small-x

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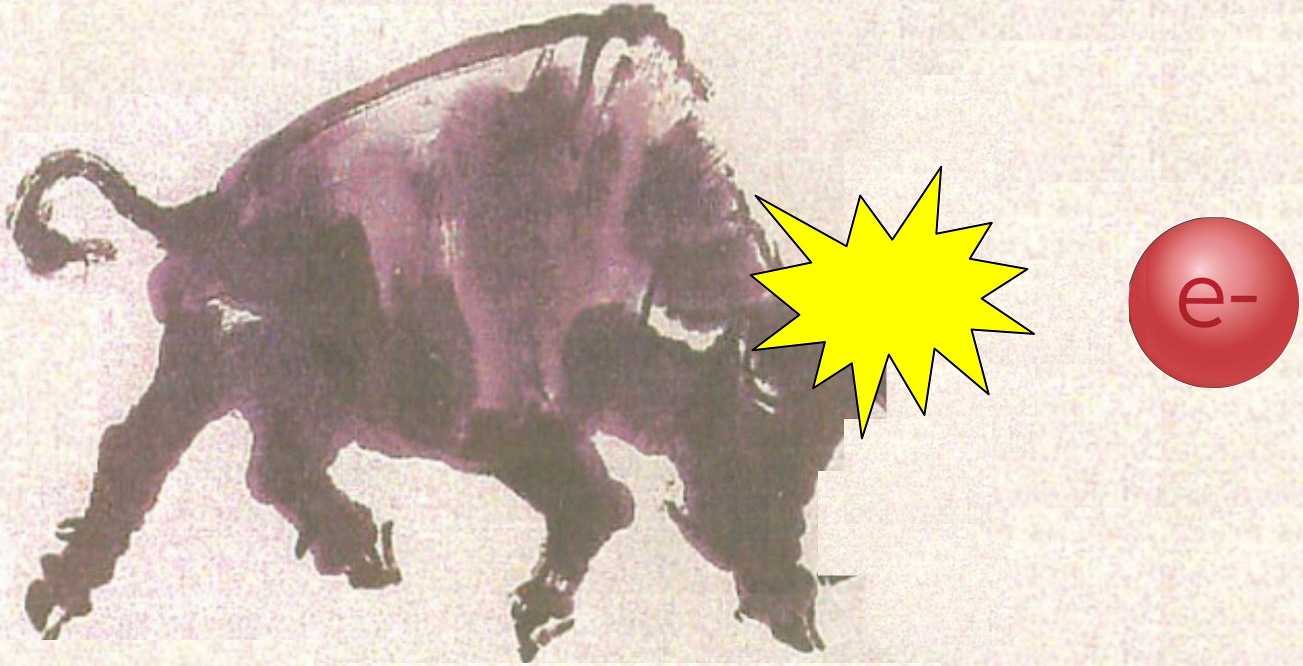

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Outline

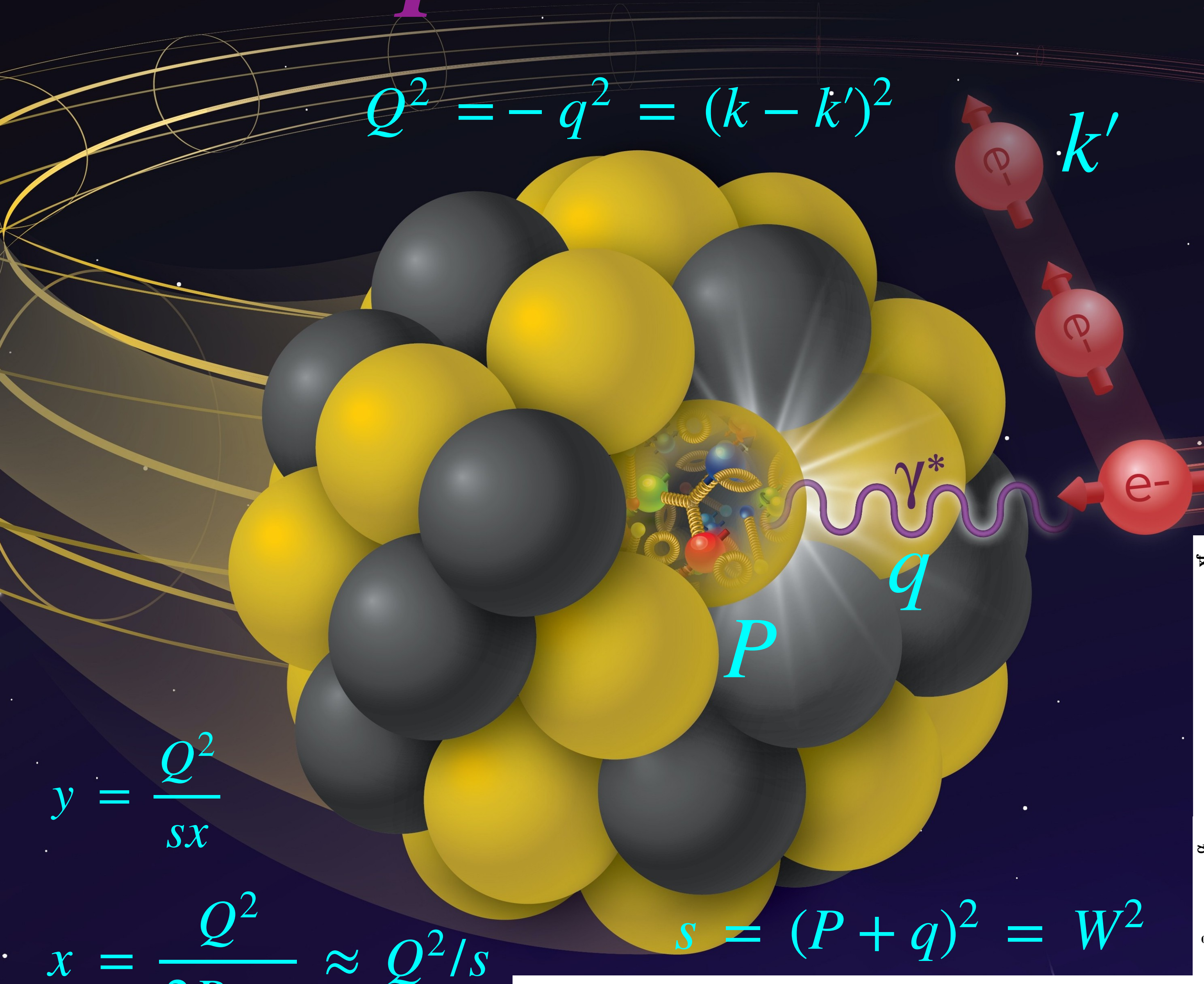
- Brief summary of our theoretical achievements in high energy DIS.
- Multiplicity distribution of the produced gluons in high energy DIS:
  - The linear but with complicated kernel and non-homogeneous evolution equations for the cross sections of n-cut BFKL Pomeron production in the final states.
  - Approximate analytical solution to the equation (homotopy and large n solution)
  - Calculation of the multiplicity distribution and related quantities
  - Future work

Based on arXiv:2603.21775 [hep-ph]

In collaboration with G. Levin and C. Contreras

Deep inelastic scattering

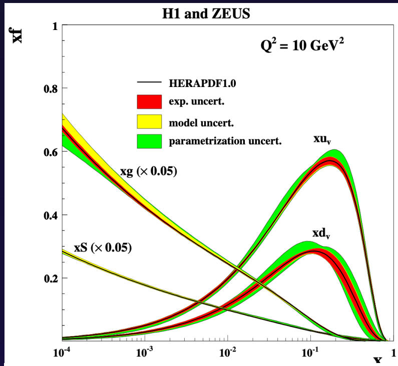
$$Q^2 = -q^2 = (k - k')^2$$

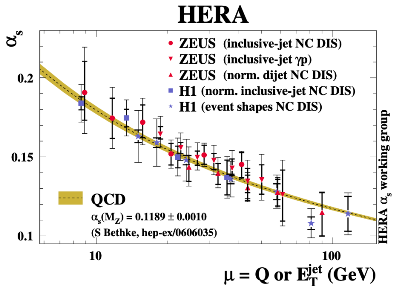


$$y = \frac{Q^2}{sx}$$

$$x = \frac{Q^2}{2P \cdot q} \approx Q^2/s$$

$$s = (P+q)^2 = W^2$$

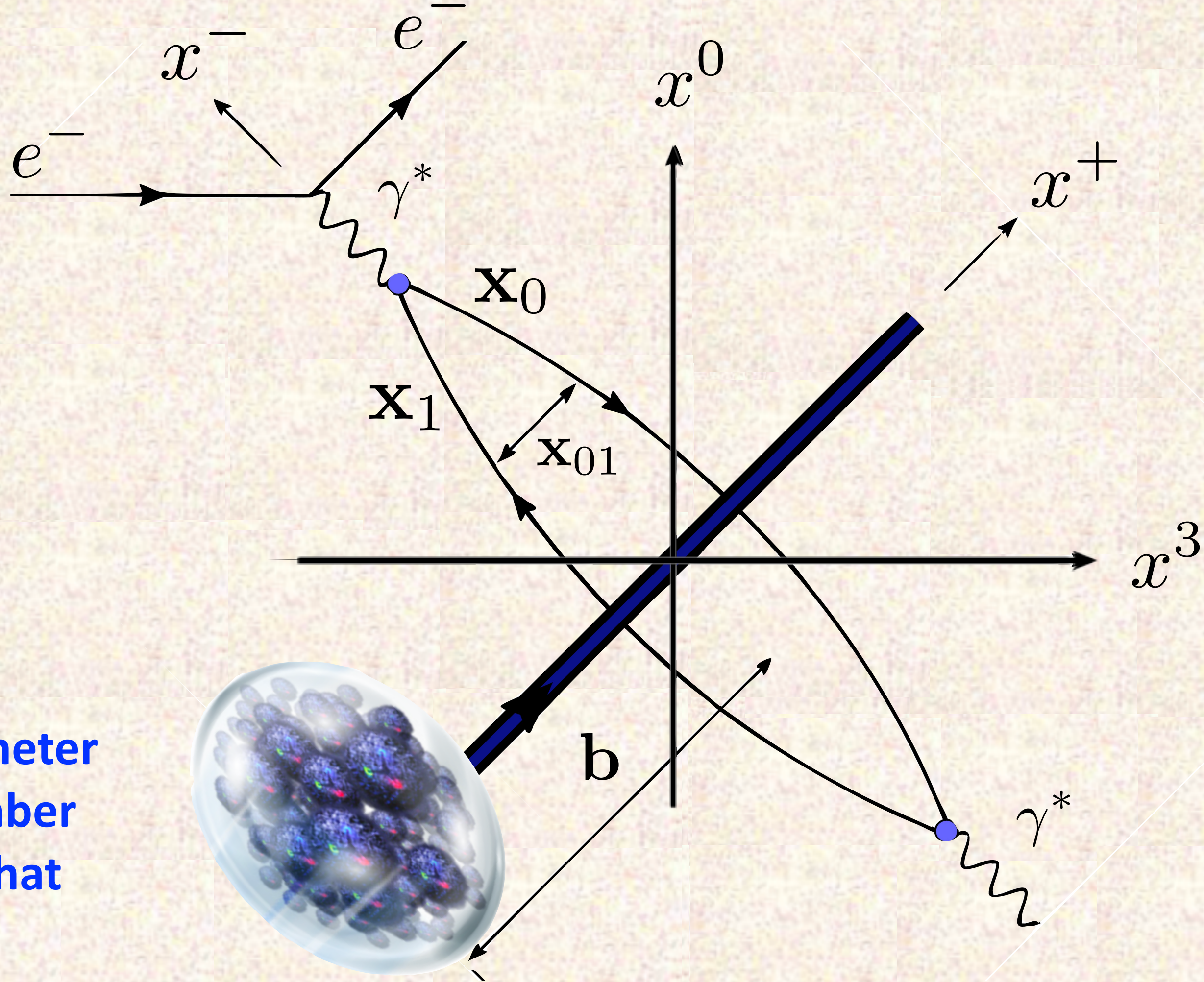




Small$-x$ $\Leftrightarrow$ High energy s

Dipole picture of DIS

- We consider DIS for $Q^2 \geq 1$ GeV^2 and in the region of small x.
- For x<0.01, the quark loop diagram dominates over the handbag diagram and we have a dipole-nucleus scattering.



- For DIS with nucleus the impact parameter fixes the number of nucleons that interact.

The scattering amplitude

- **The basics of the scattering theory: the final and initial states are related by the S-matrix operator**

$$|\psi_f\rangle = \hat{S}\,|\psi_i\rangle$$

- **The high energy scattering amplitudes must satisfy Lorentz-invariance, unitarity of S-matrix and analyticity.**
- **For S=I+iT, the unitarity of the S-matrix, $SS^\dagger=I$, leads to:**

$$i(T^\dagger - T) = T^\dagger T$$

where T is the forward scattering amplitude.

- **Define the dipole amplitude N as the imaginary part of the dipole T-matrix, to rewrite the unitarity conditions as:**

$$2\,N(Y;\mathbf{x}_{01},\mathbf{b}) = |N(Y;\mathbf{x}_{01},\mathbf{b})|^2 + G_{in}(Y;\mathbf{x}_{01},\mathbf{b})$$

where G_{in} stands for the contribution of all inelastic processes

The scattering amplitude

- **The previous constraint has a general solution if we assume that the scattering amplitude is pure imaginary at high energy, namely:**

$$N(Y;\mathbf{x}_{01},\mathbf{b}) \;=\; 1 \,-\, e^{-\Omega(Y;\mathbf{x}_{01},\mathbf{b})}$$
$$= \sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{1}{k!}\,\Omega^k$$

Here Ω is the Pomeron exchange.

- **We can rewrite the s-channel unitarity constrains as**

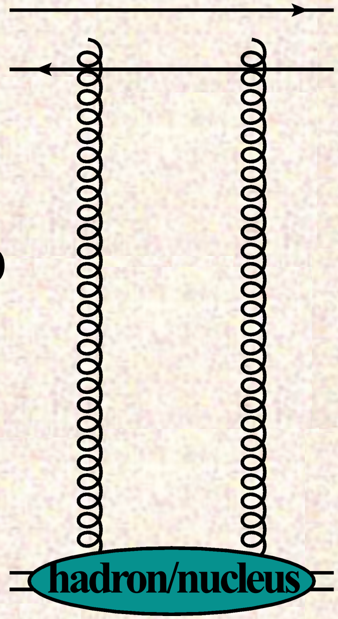
$$2\,N(Y;\mathbf{x}_{01},\mathbf{b}) \;=\; \sigma_{SD}(Y;\mathbf{x}_{01},\mathbf{b}) \,+\, \sigma_{in}(Y;\mathbf{x}_{01},\mathbf{b})$$

where σ_{sd} and σ_{in} are the single diffraction and inelastic cross sections at fixed dipole size and impact parameter b.

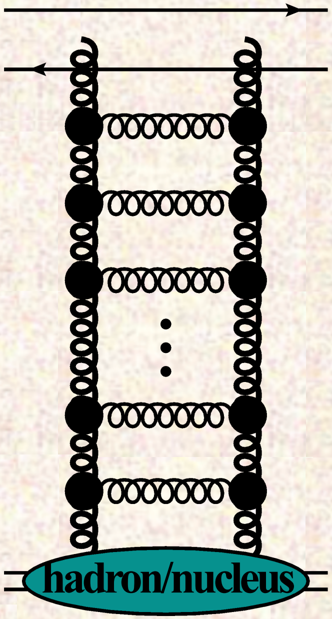
The scattering amplitude

- In QCD the interaction with the target occurs due two gluon exchange.
- The t-channel unitarity is the same as s-channel but for the t-channel reaction. In framework of QCD the reggeization of the gluon provides the t-channel unitarity for the effective theory based on BFKL Pomeron calculus.

two gluons (no energy)



Bound state of two reggeized gluons (due to evolution equation)

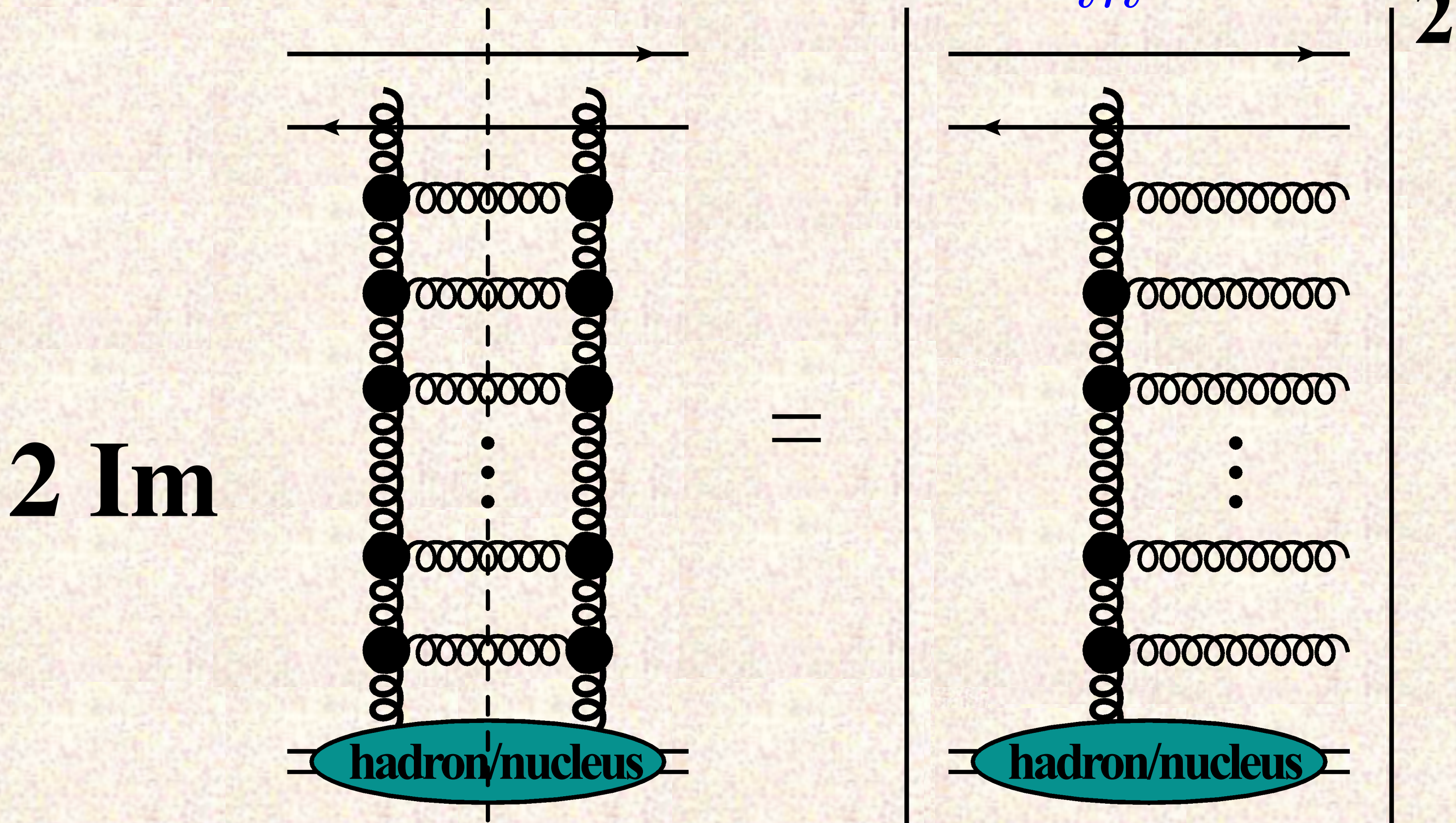


- The reggeized gluon has spin different from one.

The Cut Pomeron

- **In the single-Pomeron approximation, the scattering amplitude is small, N<<1. If N <<1, then the elastic contribution N^2 << N can be neglected:**

$$2\,\mathrm{Im}\,G_{\mathrm{BFKL}} = \sigma_{in}^{\mathrm{BFKL}}$$



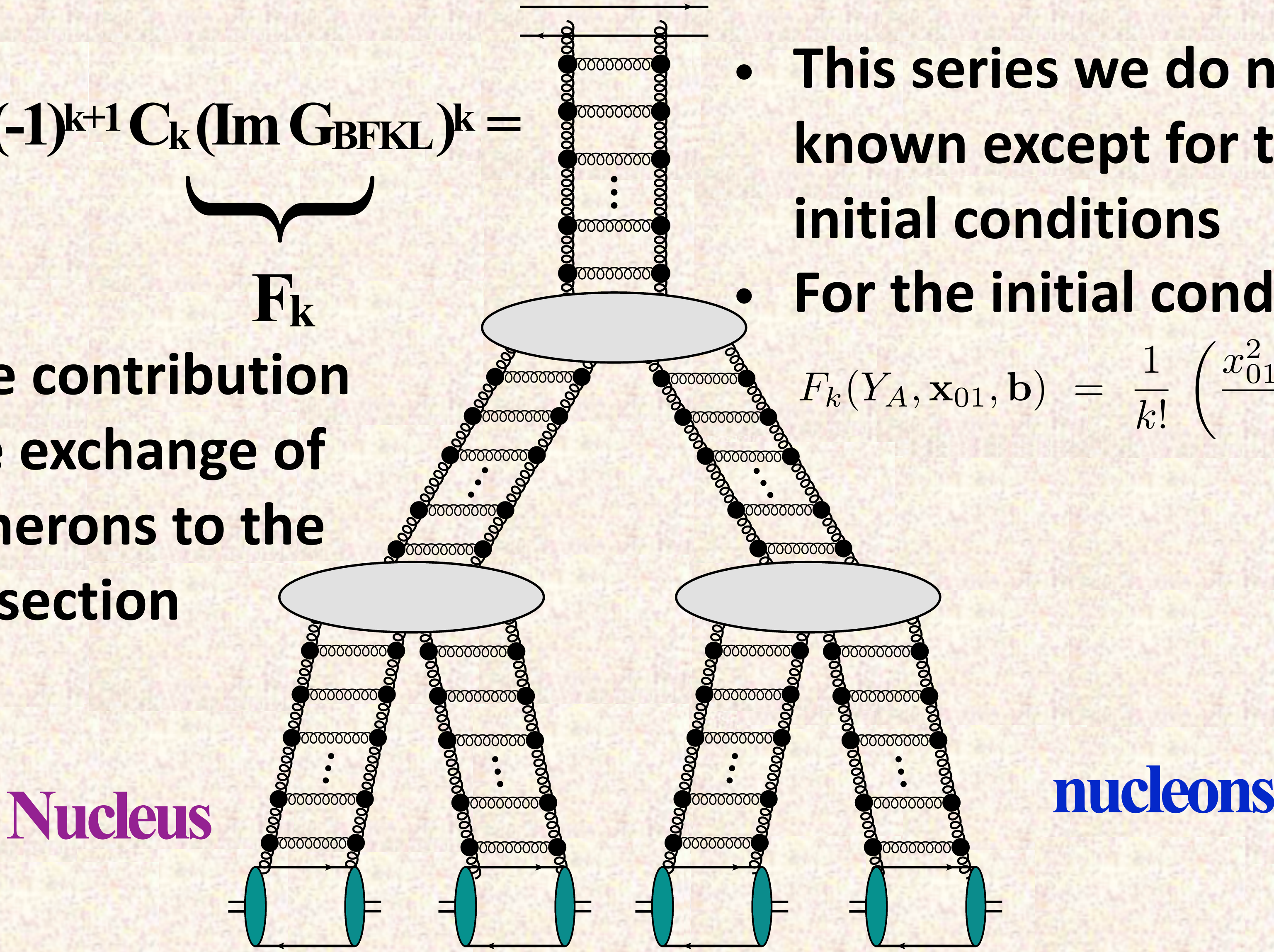
- **it gives the "cut pomeron", and, hence, particle production with average multiplicity** $n_{\mathrm{BFKL}} = \Delta_{\mathrm{BFKL}}\, Y$.

The scattering amplitude

- The scattering amplitude, can also be calculated as the sum of the "fan" diagrams of the BFKL Pomeron calculus.

$$N = \sum_{k=1}^{\infty} (-1)^{k+1} C_k \underbrace{(\mathrm{Im}\, G_{BFKL})^k}_{F_k} =$$

F_k: the contribution of the exchange of k-Pomerons to the cross section



- This series we do not known except for the initial conditions
- For the initial condition

$$F_k(Y_A, \mathbf{x}_{01}, \mathbf{b}) \;=\; \frac{1}{k!}\left(\frac{x_{01}^2\, Q_{s0}^2}{4}\right)^k$$

The scattering amplitude

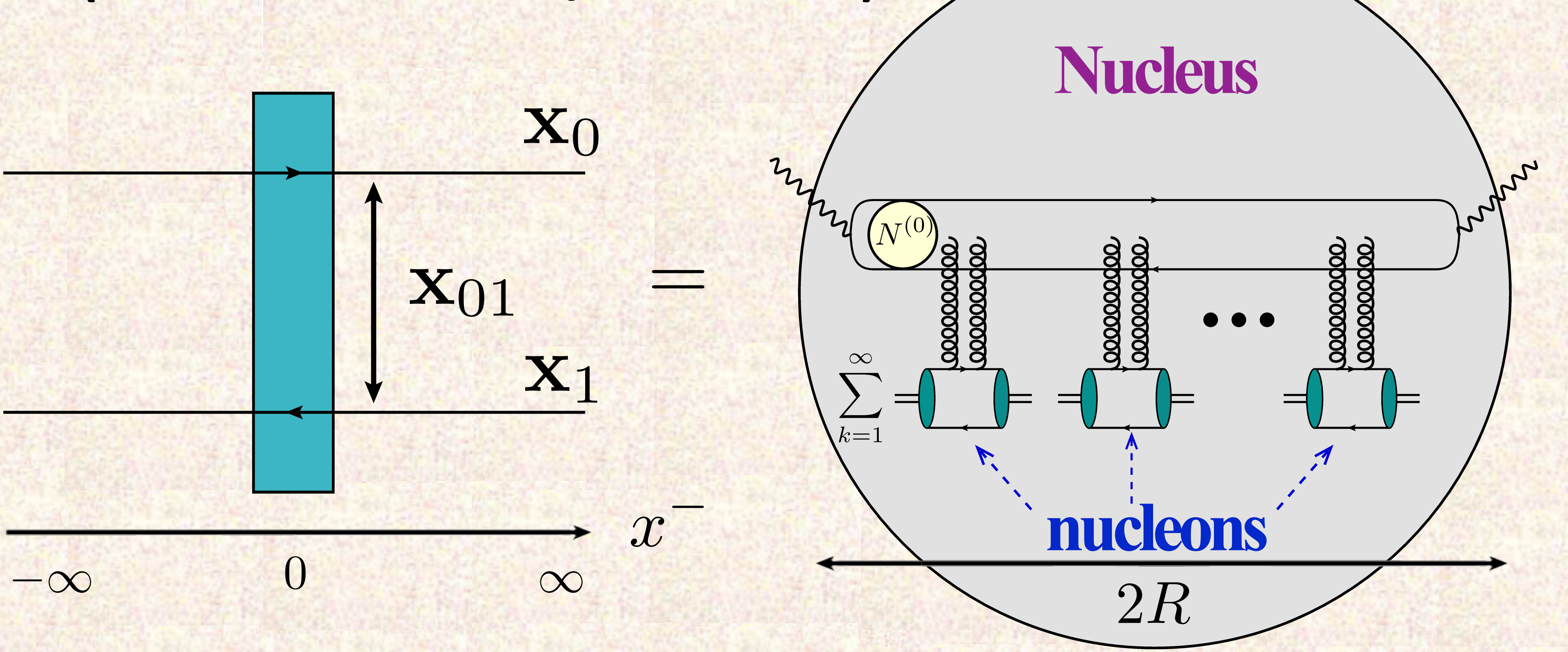
- **In QCD in the LLA, there is an equivalence between the pomeron theory of high energy strong interactions and the dipole approach to QCD (almost always work).**

Multiple Pomerons <-> Non-linear evolution

- **We learned how to write nonlinear corrections only when the amplitude is the sum of ‘fan’ diagrams, in which we have P-> 2 P vertex.**
- **In our treatment of the multiplicity distributions, we are going to explore again this equivalence.**

The scattering amplitude

- **For the calculation of the $q\bar{q}$ propagator through the nucleus, we make use of the quasi-classical approximation (the standard GGM/MV model):**

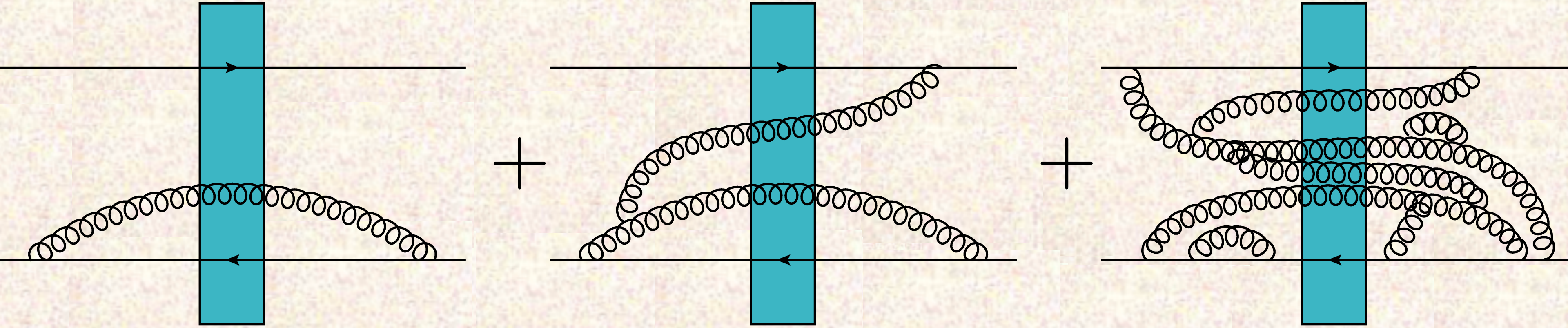


$$N\left(Y=Y_A,\mathbf{x}_{01},\mathbf{b}\right) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k!}\left(\frac{x_{01}^2\,Q_s^2(Y_A,\mathbf{b})}{4}\right)^k = 1-\exp\left(-\frac{1}{4}\,x_{01}^2\,Q_s^2(Y_A,\mathbf{b})\right)$$

where $Q_s(b)$ is the initial saturation scale.

The scattering amplitude

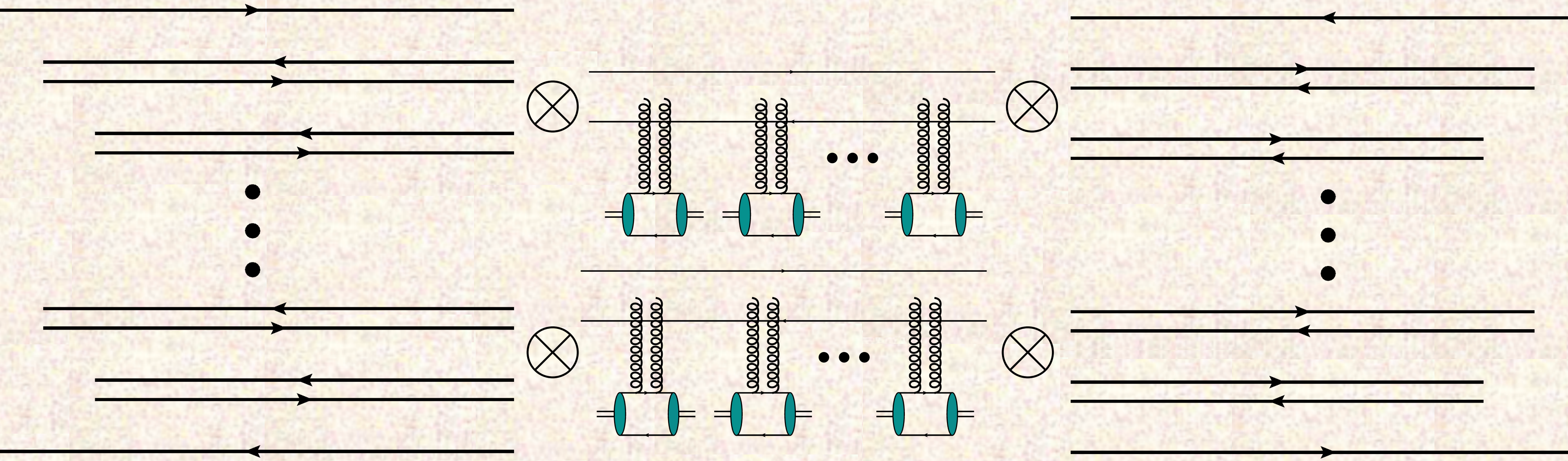
- To include the quantum evolution corrections to the quasi-classical scattering amplitude, we consider only long-lived gluons emissions and virtual gluons (as gluons emitted inside the shockwave are suppressed):



- The gluon cascade includes all possible emissions, both from the quark and the antiquark. Further, gluons emit gluons (recombination), etc.
- We can resume the cascade but only can do it at leading-log (and next-to-leading log), we cannot sum it exactly.

The scattering amplitude

- To further simplify it, we employ the 't Hooft's large-N_c limit, where only the planar diagrams survive and our gluon cascade turns into a cascade of color dipoles:



- Each dipole, that was developed through the QCD evolution, later interacts with the nucleus by a series of Glauber-type multiple rescatterings on the nucleons.
- Resuming this cascade, we arrive at the following evolution equation for the dipole S-matrix:

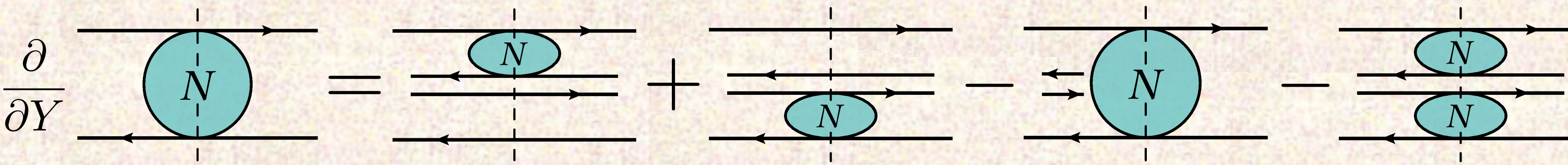
The scattering amplitude

$$\frac{\partial}{\partial Y} S\left(Y, \mathbf{x}_{01}, \mathbf{b}\right) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \, \frac{x_{01}^2}{x_{12}^2 \, x_{02}^2} \left[S\left(Y; \mathbf{x}_{12}, \mathbf{b}\right) \, S\left(Y; \mathbf{x}_{02}, \mathbf{b}\right) - S\left(Y; \mathbf{x}_{01}, \mathbf{b}\right) \right]$$

where $Y = \ln 1/x \sim \ln s$**,** $\bar{\alpha}_S = \alpha_S N_c/\pi$**. For N=1-S:**

$$\frac{\partial}{\partial Y} N\left(Y, \mathbf{x}_{01}, \mathbf{b}\right) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \, \frac{x_{01}^2}{x_{12}^2 \, x_{02}^2} \left[N\left(Y; \mathbf{x}_{12}, \mathbf{b}\right) + N\left(Y; \mathbf{x}_{02}, \mathbf{b}\right) - N\left(Y; \mathbf{x}_{01}, \mathbf{b}\right) - N\left(Y; \mathbf{x}_{12}, \mathbf{b}\right) \, N\left(Y; \mathbf{x}_{02}, \mathbf{b}\right) \right]$$

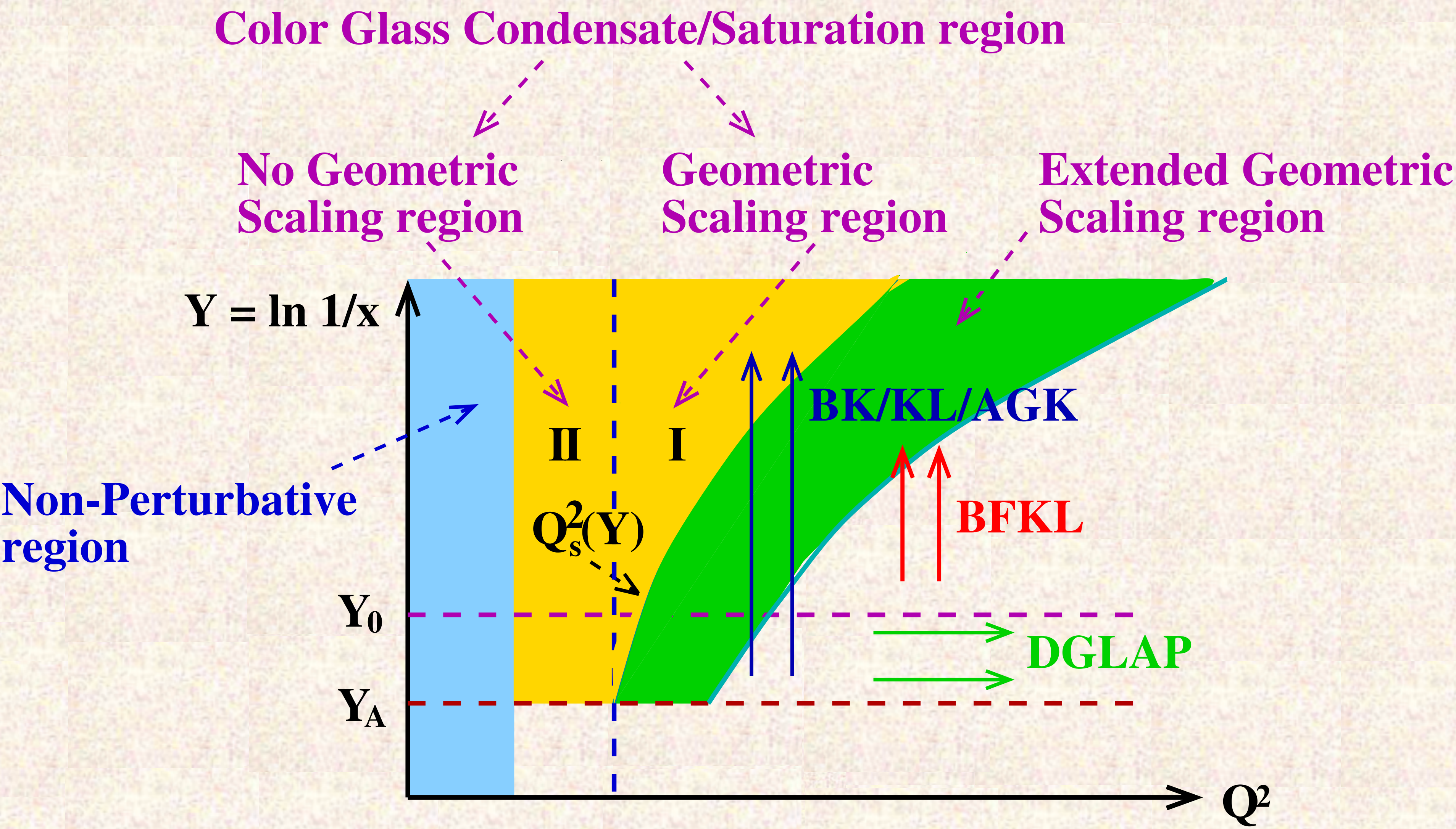
Balitsky and Kovchegov (BK), 1999



with initial condition $N\left(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}\right) = 1 - \exp\left(-\frac{1}{4} \, x_{01}^2 \, Q_s^2(Y_A, \mathbf{b})\right)$

- **the nonlinear BK evolution equation describe the transition into the saturation region, along with the physics inside that region. To date, there is no exact solution (but we have a number of known approximate analytical solutions).**

The scattering amplitude



- **Three regions: pQCD ($x_{01}^2\, Q_s^2 < 1$), vicinity of saturation region ($x_{01}^2\, Q_s^2 \sim 1$) and the saturation region ($x_{01}^2\, Q_s^2 > 1$)**

The AGK cutting rules

- **The Abramovsky-Gribov-Kancheli (AGK) cutting rules allows us to calculate the cross sections for production of n-cut Pomerons if we know F_k: the contribution of the exchange of k-Pomerons to the cross section**

$$\sigma_n^{\mathrm{AGK}}\left(Y;\mathbf{r},\mathbf{b}\right) \;=\; \sum_{k=n}^{\infty}\sigma_n^k\left(Y;\mathbf{r},\mathbf{b}\right)$$

where

$$\sigma_n^k\left(Y;\mathbf{r},\mathbf{b}\right) \;=\; \begin{cases} (-1)^k\left(2^k\;-\;2\right)F_k(Y;\mathbf{r},\mathbf{b}) & \text{for }\; n=0\\ (-1)^{k-n}\dfrac{k!}{(k-n)!\,n!}\,2^k\,F_k(Y;\mathbf{r},\mathbf{b}) & \text{for }\; n\geq 1\end{cases}$$

and

$$F_k(Y;\mathbf{x}_{01},\mathbf{b}) \;=\; C_k(\mathbf{x}_{01},\mathbf{b})\left(\mathrm{Im}\,G_{\mathrm{BFKL}}(Y;\mathbf{x}_{01},\mathbf{b})\right)^k$$

The AGK cutting rules

- **n=1**

$$\sigma_1^{\mathrm{AGK}} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k!}{(k-1)!\,1!} 2^k C_k \left(\mathrm{Im}\, G_{\mathrm{BFKL}}\right)^k$$

$$= C_1\, \sigma_{in}^{\mathrm{BFKL}} + C_2\, 2\, (2\, \mathrm{Im}\, G_{\mathrm{BFKL}})\, \sigma_{in}^{\mathrm{BFKL}}$$

$$+ C_3\, 3\, (2\, \mathrm{Im}\, G_{\mathrm{BFKL}})^2\, \sigma_{in}^{\mathrm{BFKL}} + \cdots$$

- **n=2**

$$\sigma_2^{\mathrm{AGK}} = \sum_{k=2}^{\infty} (-1)^{k-2} \frac{k!}{(k-2)!\,2!} 2^k C_k \left(\mathrm{Im}\, G_{\mathrm{BFKL}}\right)^k$$

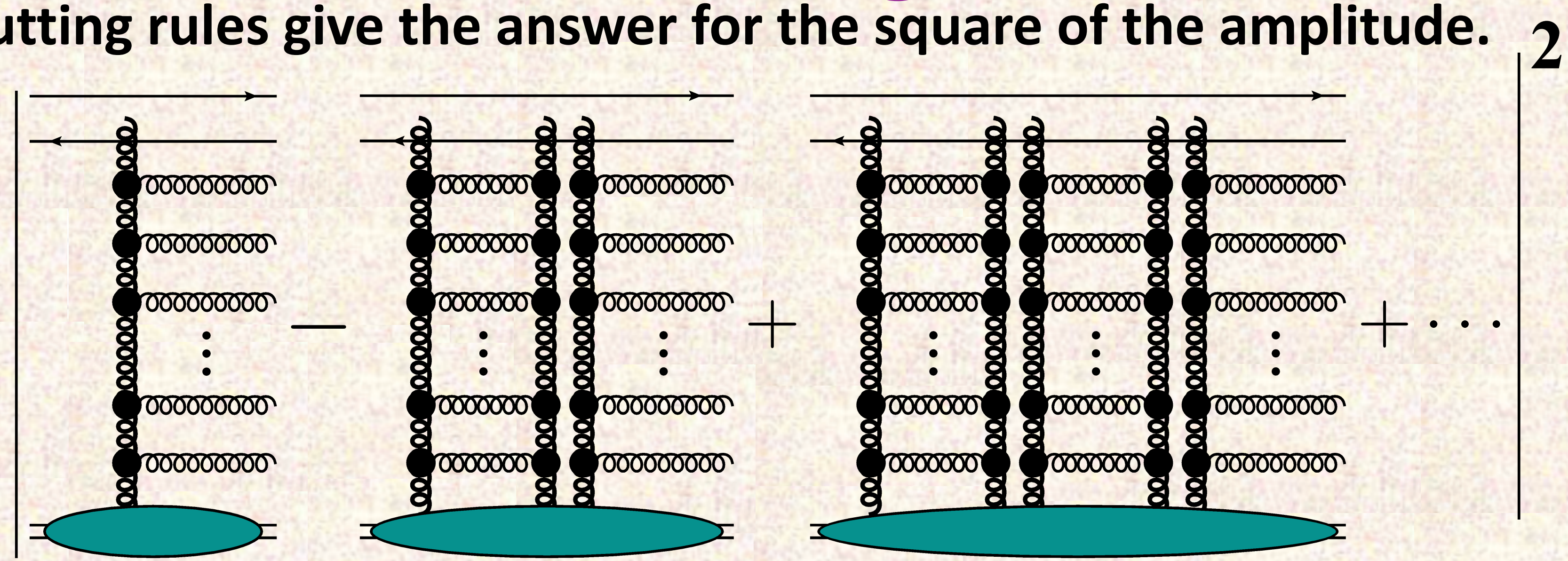
$$= C_2\, (\sigma_{in}^{\mathrm{BFKL}})^2 + C_3\, 3\, (2\, \mathrm{Im}\, G_{\mathrm{BFKL}})\, (\sigma_{in}^{\mathrm{BFKL}})^2$$

$$+ C_4\, 6\, (2\, \mathrm{Im}\, G_{\mathrm{BFKL}})^2\, (\sigma_{in}^{\mathrm{BFKL}})^2 + \cdots$$

The AGK cutting rules

- **AGK cutting rules give the answer for the square of the amplitude.**

$\sigma_1^{\rm AGK} =$



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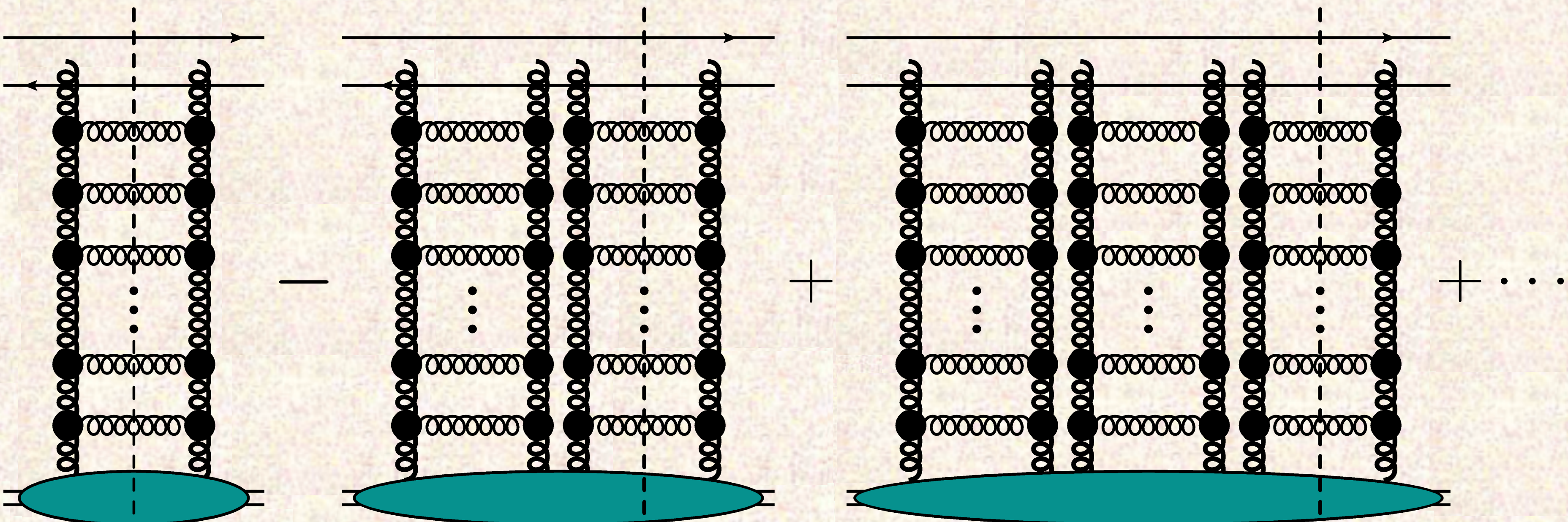


$C_1\,\sigma_{in}^{\rm BFKL}$ $\qquad C_2\,2\,(2\,{\rm Im}\,G_{\rm BFKL})\,\sigma_{in}^{\rm BFKL}$ $\qquad C_3\,2\,(2\,{\rm Im}\,G_{\rm BFKL})^2\,\sigma_{in}^{\rm BFKL}$

Produced particles

- **To find the multiplicity distribution of particles in the final states one needs to convolute σ_k with the distribution of particles inside k-cut Pomerons**

$$\sigma_n^{f.s.}(Y;\mathbf{x}_{01},\mathbf{b}) = \sum_{k=1}^{\infty} \underbrace{\sigma_k^{\rm AGK}(Y;\mathbf{x}_{01},\mathbf{b})}_{\propto\ \left(\sigma_{in}^{\rm BFKL}\right)^k} \underbrace{\mathcal{P}_n^{I\!P}(k\,\Delta_{\rm BFKL} Y)}_{\text{Poisson distribution}}$$

$$\xrightarrow{Y \gg 1} \sigma^{\rm AGK}_{k=n/(\Delta_{\rm BFKL}\,Y)}\,(Y,\mathbf{x}_{01},\mathbf{b})$$

The AGK cutting rules

- **We don't know how to sum this series except for the initial condition:**

$$F_k(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) \; = \; \frac{1}{k!} \left(\frac{x_{01}^2 \, Q_{s0}^2}{4} \right)^k$$

$$\sigma_{n,01}^{\rm AGK}(Y = Y_A) \; = \; \sum_{k=n}^{\infty} (-1)^{k-n} \, \frac{k!}{(k \, - \, n)! \, n!} \, 2^k \, \frac{1}{k!} \, \left(\frac{x_{01}^2 \, Q_{s0}^2}{4} \right)^k$$

$$= \; \frac{\left(\frac{1}{2} \, x_{01}^2 \, Q_{s0}^2 \right)^n}{n!} \, \exp \left\{ - \, \frac{1}{2} \, x_{01}^2 \, Q_{s0}^2 \right\}$$

- **For** $\xi \; = \; \ln \left(x_{01}^2 \, Q_{s0}^2 \right)$

$$\sigma_n^{\rm AGK}(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) \; = \; \frac{\left(\frac{1}{2} \, e^{\xi} \right)^n}{n!} \, \exp \left\{ - \, \frac{1}{2} \, e^{\xi} \right\}$$

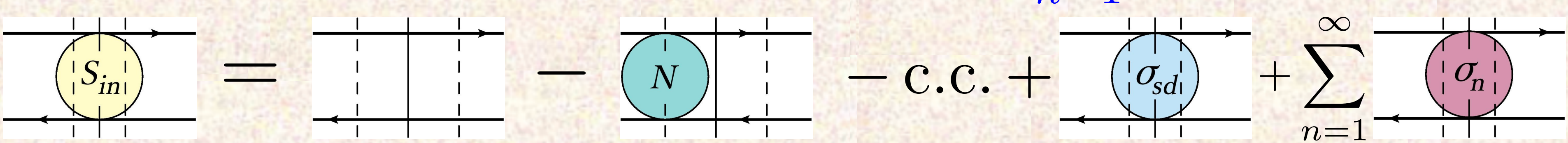
The particle production S-matrix

- Note that in the saturation region $|A_{el}|$^2 \sim σ_{in}.
- We can give a more simple derivation of the equations of σ_n from the s-channel unitarity (for an alternative derivation, see **arXiv:0807.3413 [hep-ph]**)

$$2\,N(Y;\mathbf{x}_{01},\mathbf{b}) \;=\; \sigma_{SD}(Y;\mathbf{x}_{01},\mathbf{b}) + \sigma_{in}(Y;\mathbf{x}_{01},\mathbf{b})$$

- We define a new quantity – the particle production S-matrix "S_{in}": it includes σ_n along with the no-interaction term (=1) and all the interaction terms (N and σ_{sd}) on either side (or both sides) of the final state cut:

$$S_{in} \;=\; 1 \;-\; 2\,N \;+\; \sigma_{SD} \;+\; \sum_{n=1}^{\infty} \sigma_n$$



The particle production S-matrix

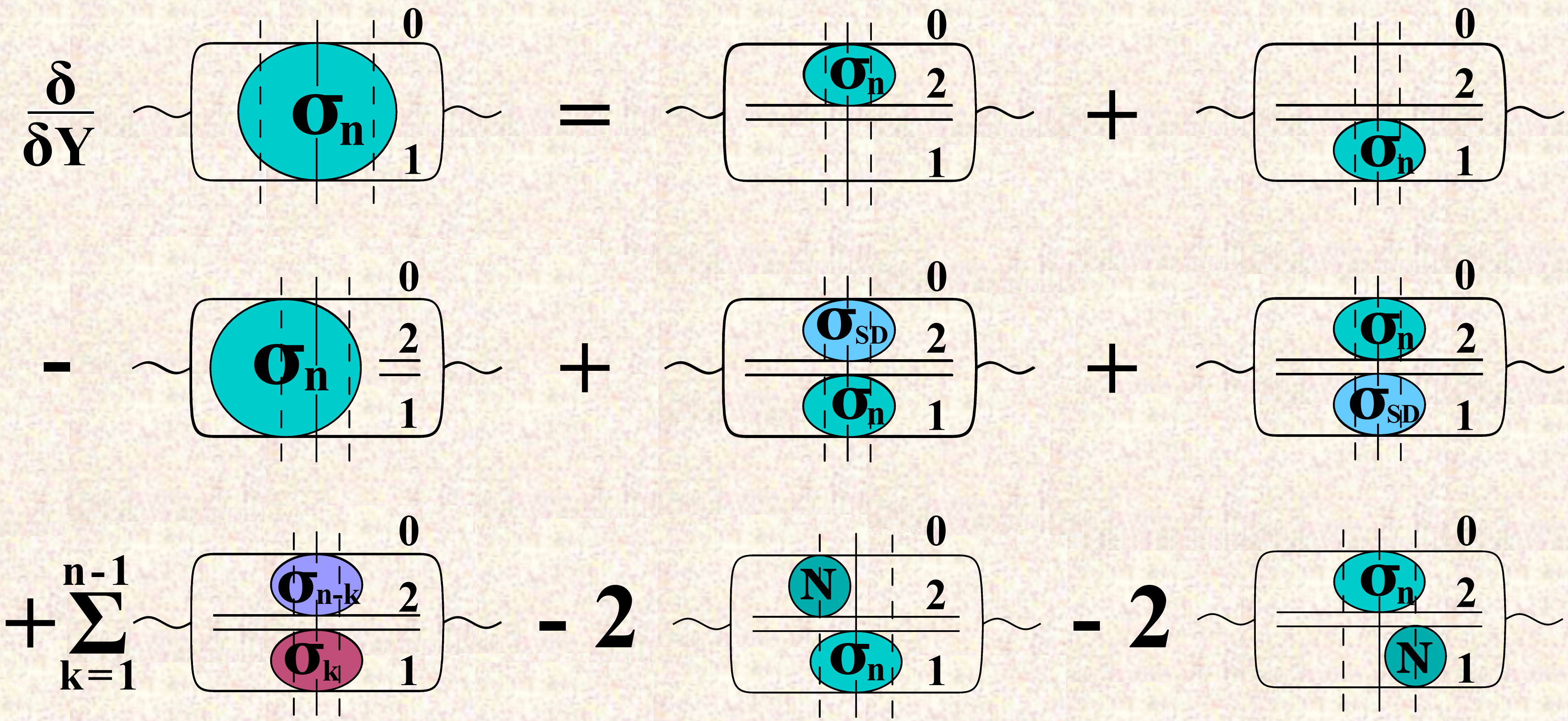
- **The evolution equation for S_{in} is**

$$\underbrace{\frac{\partial}{\partial Y} S_{in}(Y;\mathbf{x}_{01},\mathbf{b})}_{\text{select fixed multiplicity terms}} = \frac{\bar{\alpha}_S}{2\pi}\int d^2x_2\,\frac{x_{01}^2}{x_{12}^2\,x_{02}^2}\underbrace{\left[S_{in}(Y;\mathbf{x}_{12},\mathbf{b})\,S_{in}(Y;\mathbf{x}_{02},\mathbf{b}) - S_{in}(Y;\mathbf{x}_{01},\mathbf{b})\right]}_{\text{select fixed multiplicity terms}}$$

- **Replacing $S_{in}=1-2N+\sigma_{SD}+\Sigma_n\sigma_n$ in the equation for the S_{in}-matrix, we obtain the BK, the KL and the AGK/KLP evolution equations:**

$$\frac{\partial}{\partial Y}\sigma_n\left(Y;\mathbf{x}_{01},\mathbf{b}\right) = \frac{\bar{\alpha}_S}{2\pi}\int d^2x_2\,\frac{x_{01}^2}{x_{02}^2\,x_{12}^2}$$
$$\times\left[\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right) + \sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right) - \sigma_n\left(Y;\mathbf{x}_{01},\mathbf{b}\right)\right.$$
$$+\,\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right)\sigma_{SD}\left(Y;\mathbf{x}_{02},\mathbf{b}\right) + \sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\sigma_{SD}\left(Y;\mathbf{x}_{12},\mathbf{b}\right)$$
$$+\sum_{k=1}^{n-1}\sigma_{n-k}\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\sigma_k\left(Y;\mathbf{x}_{12},\mathbf{b}\right)$$
$$\left.-\,2\,\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right)N\left(Y;\mathbf{x}_{02},\mathbf{b}\right) - 2\,\sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right)N\left(Y;\mathbf{x}_{12},\mathbf{b}\right)\right]$$

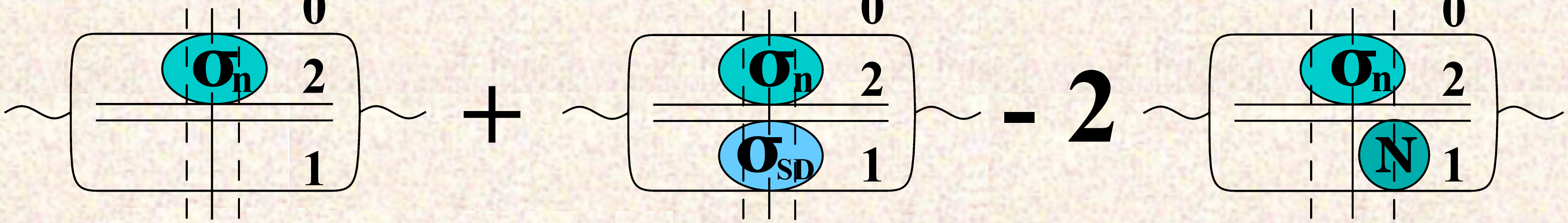
Evolution of σ_n at large N_c



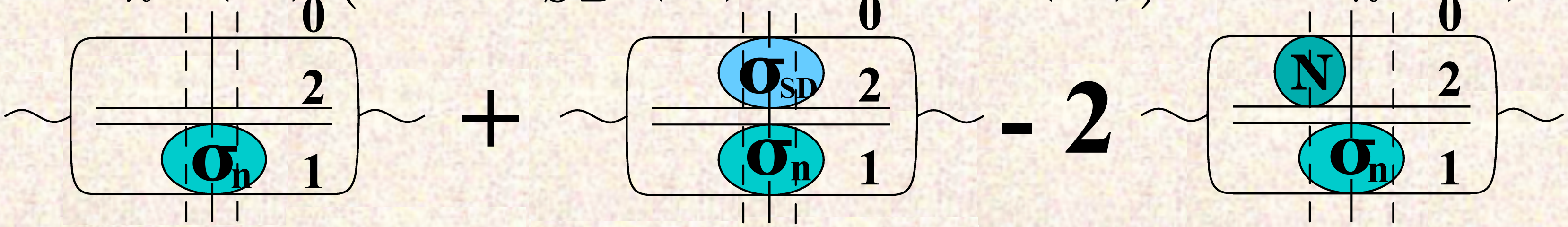
- We can see the equation is linear but with complicated kernel (K_{BFKL} + 2 - 4 N) and non-homogeneous

Evolution of σ_n at large N_c

- **Lets note that**



$$= \sigma_n^{02}(Y)\left(1 + \sigma_{SD}^{12}(Y) - 2\,N_{12}(Y)\right) = \sigma_n^{02}(Y)\,\Delta_{12}(Y)$$



$$= \sigma_n^{12}(Y)\left(1 + \sigma_{SD}^{02}(Y) - 2\,N_{02}(Y)\right) = \sigma_n^{12}(Y)\,\Delta_{02}(Y)$$

- **we reduce to**

$$\frac{\partial}{\partial Y}\sigma_n(Y, \mathbf{x}_{01}, \mathbf{b}) = \frac{\bar{\alpha}_S}{2\pi}\int d^2x_2\,\frac{x_{01}^2}{x_{12}^2\,x_{02}^2}$$
$$\times\,\Big[-\sigma_n(Y, \mathbf{x}_{01}, \mathbf{b}) + \sigma_n(Y, \mathbf{x}_{12}, \mathbf{b})\,\Delta(Y, \mathbf{x}_{02}, \mathbf{b})$$
$$+\,\Delta(Y, \mathbf{x}_{12}, \mathbf{b})\,\sigma_n(Y, \mathbf{x}_{02}, \mathbf{b}) + \sum_{k=1}^{n-1}\sigma_{n-k}(Y, \mathbf{x}_{02}, \mathbf{b})\,\sigma_k(Y, \mathbf{x}_{12}, \mathbf{b})\Big]$$

Evolution of σ_n at large N_c

- **Now we take the evolution kernel with leading logarithmic terms only (** $\begin{matrix} x_{01} \approx x_{02} \gg x_{12} \gg 1/Q_s \\ x_{01} \approx x_{12} \gg x_{02} \gg 1/Q_s \end{matrix}$ **)**

$$\frac{\bar{\alpha}_S}{2\pi}\int d^2x_2\,\frac{x^2_{01}}{x^2_{12}\,x^2_{02}}\;\rightarrow\;\frac{\bar{\alpha}_S}{2}\int\limits_{Q_s^{-2}(Y,b)}^{x^2_{01}}\frac{dx^2_{02}}{x^2_{02}}\;+\;\frac{\bar{\alpha}_S}{2}\int\limits_{Q_s^{-2}(Y,b)}^{x^2_{01}}\frac{dx^2_{12}}{x^2_{12}}\;\equiv\;\int\limits_{-\xi_s}^{\xi}K_{LT}\left(\xi,\xi'\right)d\xi'$$

- **The equations for σ_n for the leading twist BFKL kernel take the form:**

$$\frac{\partial}{\partial Y}\,\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\;=\;\bar{\alpha}_S\Big\{-\ln\left(x^2_{01}\,Q^2_s(Y,\mathbf{b})\right)\,\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right)$$
$$+\,\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\int\limits_{\ln\left(Q_s^{-2}(Y,\mathbf{b})\right)}^{\ln\left(x^2_{01}\right)}d\ln\left(x^2_{02}\right)\,\Delta\left(Y,\mathbf{x}_{02},\mathbf{b}\right)\;+\;\Delta\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\int\limits_{\ln\left(Q_s^{-2}(Y,\mathbf{b})\right)}^{\ln\left(x^2_{01}\right)}d\ln\left(x^2_{02}\right)\,\sigma_n\left(Y,\mathbf{x}_{02},\mathbf{b}\right)$$
$$+\sum_{k=1}^{n-1}\;\int\limits_{\ln\left(Q_s^{-2}(Y,\mathbf{b})\right)}^{\ln\left(x^2_{01}\right)}d\ln\left(x^2_{02}\right)\,\sigma_{n-k}\left(Y,\mathbf{x}_{02},\mathbf{b}\right)\,\sigma_k\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\Big\}$$

Evolution of σ_n at large N_c

- **Defining the geometric scaling variables**

$$z \,=\, \ln\left(x_{01}^2\, Q_s^2\left(Y,\mathbf{b}\right)\right);\;\; z' \,=\, \ln\left(x_{02}^2\, Q_s^2\left(Y,\mathbf{b}\right)\right)$$

and the rescaled variable $\delta\tilde{Y} \,=\, \bar{\alpha}_S\left(Y \,-\, Y_A\right)$**, we have**

$$\frac{\partial}{\partial \delta\tilde{Y}}\,\sigma_n\left(\delta\tilde{Y}, z\right) \,+\, \kappa\,\frac{\partial}{\partial z}\,\sigma_n\left(\delta\tilde{Y}, z\right) \,=\, -\,z\,\sigma_n\left(\delta\tilde{Y}, z\right)$$

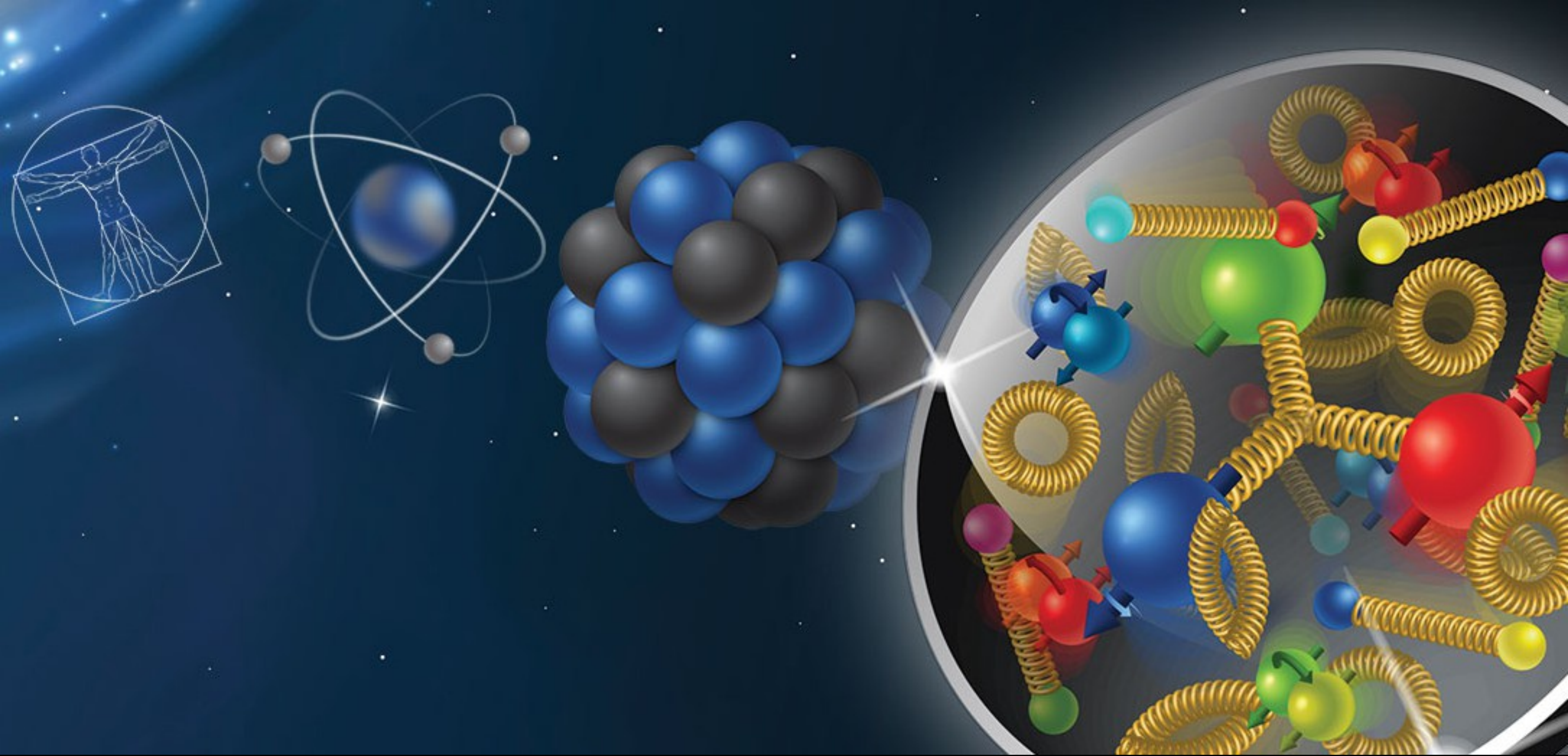
$$+\,\sigma_n\left(\delta\tilde{Y}, z\right)\int\limits_{\xi_0^A}^{z} dz'\,\Delta\left(\delta\tilde{Y}, z'\right) \,+\, \Delta\left(\delta\tilde{Y}, z\right)\int\limits_{\xi_0^A}^{z} dz'\,\sigma_n\left(\delta\tilde{Y}, z'\right)$$

$$+\,\sum_{k=1}^{n-1}\int\limits_{\xi_0^A}^{z} dz'\,\sigma_{n-k}\left(\delta\tilde{Y}, z'\right)\,\sigma_k\left(\delta\tilde{Y}, z\right)$$

where we used $Q_s^2(Y,\mathbf{b}) \,=\, Q_{s0}\,e^{\bar{\alpha}_S\,\kappa\,(Y\,-\,Y_A)}$ **with** $\kappa \,\equiv\, \dfrac{\chi\left(\gamma_{cr}\right)}{1-\gamma_{cr}}$

- **The minimum z is at $Y=Y_A$ ($\delta\tilde{Y}=0$), i.e.**

$$z \,=\, \kappa\,\delta\tilde{Y} \,+\, \xi;\;\; z \overset{\delta\tilde{Y}\,\to\,0}{=} \xi$$ **or for GS case** $z = \xi_0^A$



$$\sigma_n(Y; \boldsymbol{x}_{01}, \boldsymbol{b})$$

σ_1: region I (GS)

$$\kappa \frac{d\sigma_1(z)}{dz} = -z\,\sigma_1(z) + \sigma_1(z)\int_{\xi_0^A}^{z} dz'\,\Delta(z') + \Delta(z)\int_{\xi_0^A}^{z} dz'\,\sigma_1(z')$$

$$= -\underbrace{\left(z - \int_{\xi_0^A}^{z} dz'\,\Delta(z')\right)}_{T(z)}\sigma_1(z) + \Delta(z)\underbrace{\int_{\xi_0^A}^{z} dz'\,\sigma_1(z')}$$

$$\int_{\xi_0^A}^{z} dz'\,\sigma_1(z') = \underbrace{\int_{\xi_0^A}^{\infty} dz'\,\sigma_1(z')}_{=\sigma_{0,1}} - \underbrace{\int_{z}^{\infty} dz'\,\sigma_1(z')}_{=\tilde{\Sigma}_1(z)}$$

$$= -T(z)\,\sigma_1(z) + \sigma_{0,1}\,\Delta(z) - \Delta(z)\,\tilde{\Sigma}_1(z)$$

$$\Rightarrow \underbrace{\kappa \frac{d\sigma_1(z)}{dz} + T(z)\,\sigma_1(z) - \sigma_{0,1}\,\Delta(z)}_{\mathcal{L}[\sigma_1]} + \underbrace{\Delta(z)\,\tilde{\Sigma}_1(z)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} = 0$$

σ_1: region I (GS)

- **We have divided the equation in two parts**

$$\mathcal{L}[\sigma_1] \;=\; \kappa \,\frac{d\sigma_1\,(z)}{dz} \;+\; T(z)\,\sigma_1\,(z) \;-\; \sigma_{0,1}\,\Delta\,(z)$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_1] \;=\; \Delta\,(z)\,\tilde{\Sigma}_1\,(z)$$

- **As a solution, we introduce**

$$\sigma_1 \;=\; \sigma_1^{(0)} \;+\; p\,\sigma_1^{(1)} \;+\; p^2\,\sigma_1^{(2)} \;+\; \cdots$$

such that $\mathcal{L}[\sigma_1^{(p)}] \;+\; p\,\mathcal{N}_{\mathcal{L}}[\sigma_1^{(p)}] \;=\; 0$

- **The 0th iteration is**

$$\mathcal{L}[\sigma_1^{(0)}];\;\; \kappa\,\frac{d\sigma_1^{(0)}(z)}{dz} \;=\; -T(z)\,\sigma_1^{(0)}(z) \;+\; \sigma_{0,1}\,\Delta(z)$$

σ_1: region I (GS)

- **The solution is a sum of the solution to the homogeneous part of the equation and the particular solution of the non-homogeneous one. We obtain**

$$\sigma_1^{(0,I)}(z) = \underbrace{\exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T(z')\right)}_{\tilde{\sigma}_1^{(0)}} \underbrace{\left\{\frac{\sigma_{0,1}}{\kappa}\int_{\xi_0^A}^{z} dz'\,\Delta(z')\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T(z'')\right) + C_\phi^{(1)}\right\}}_{\tilde{\sigma}_1^{(0,I)}}$$

where $C_\phi^{(1)} = \frac{1}{2}e^{\xi_0^A}\exp\left(-\frac{1}{2}e^{\xi_0^A}\right)$

- **For the next homotopy iteration we need to account for the linear terms in p:** $\mathcal{L}[\sigma_1^{(0)} + p\,\sigma_1^{(1)}] + p\mathcal{N}_{\mathcal{L}}[\sigma_1^{(0)}] = 0$
- **The equation takes the form**

$$\kappa\,\frac{d\sigma_1^{(1,I)}(z)}{dz} = -T(z)\,\sigma_1^{(1,I)}(z) - \Delta(z)\,\tilde{\Sigma}_1^{(0)}(z)$$

σ_1: region I (GS)

where $\tilde{\Sigma}_1^{(0)}(z) = \int_z^\infty dz' \sigma_1^{(0)}(z')$. The general solution is

$$\sigma_1^{(1,I)}(z) = -\frac{1}{\kappa}\exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T(z')\right)\int_{\xi_0^A}^{z} dz'\Delta(z')\,\tilde{\Sigma}_1^{(0)}(z')\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T(z'')\right)$$

- In general, the equation for the p-iteration (p≥1) in region I takes the form

$$\kappa\,\frac{d\sigma_1^{(p,I)}(z)}{dz} = -T(z)\,\sigma_1^{(p,I)}(z) - \Delta(z)\,\tilde{\Sigma}_1^{(p-1)}(z)$$

with solution

$$\sigma_1^{(p,I)}(z) = -\frac{1}{\kappa}\exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T(z')\right)\int_{\xi_0^A}^{z} dz'\,\Delta(z')\,\tilde{\Sigma}_1^{(p-1)}(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T(z'')\right)$$

σ_1: region II (no GS)

$$\frac{\partial \sigma_1\left(\delta\tilde{Y}, z\right)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1\left(\delta\tilde{Y}, z\right)}{\partial z} =$$

$$- \Bigg(\underbrace{z - \int_{\xi_0^A}^{z} dz' \, \Delta\left(\delta\tilde{Y}, z'\right)}_{T(\delta\tilde{Y}, z)} \Bigg) \sigma_1\left(\delta\tilde{Y}, z\right) + \Delta\left(\delta\tilde{Y}, z\right) \underbrace{\int_{\xi_0^A}^{z} dz' \, \sigma_1\left(\delta\tilde{Y}, z'\right)}$$

$$\int_{\xi_0^A}^{z} dz' \, \sigma_1\left(\delta\tilde{Y}, z'\right) = \underbrace{\int_{\xi_0}^{\infty} dz' \, \sigma_1\left(z'\right)}_{=\sigma_{0,1}} + \underbrace{\int_{\xi_0^A}^{\infty} dz' \left(\sigma_1\left(\delta\tilde{Y}, z'\right) - \sigma_1\left(z'\right)\right)}_{=\delta\Sigma_1\left(\delta\tilde{Y}\right)} - \underbrace{\int_{z}^{\infty} dz' \sigma_1\left(\delta\tilde{Y}, z'\right)}_{=\tilde{\Sigma}_1\left(\delta\tilde{Y}, z\right)}$$

$$\underbrace{\frac{\partial \sigma_1\left(\delta\tilde{Y}, z\right)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1\left(\delta\tilde{Y}, z\right)}{\partial z} + T\left(\delta\tilde{Y}, z\right) \sigma_1\left(\delta\tilde{Y}, z\right) - \sigma_{0,1} \Delta\left(\delta\tilde{Y}, z\right)}_{\mathcal{L}[\sigma_1]} + \Delta\left(\delta\tilde{Y}, z\right) \delta\Sigma_1\left(\delta\tilde{Y}\right)$$

$$\underbrace{- \Delta\left(\delta\tilde{Y}, z\right) \tilde{\Sigma}_1\left(\delta\tilde{Y}, z\right)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} = 0$$

σ_1: region II (no GS)

- In this case, the solution is more complex. After some algebra, the general solution takes the form

$$\sigma_1^{(0,II)}\left(\delta\tilde{Y},z\right) \;=\; \Phi_1\left(-\kappa\,\delta\tilde{Y}\;+\;z\right)\sigma_1^{0'}\left(\delta\tilde{Y},z\right)\;+\;\sigma_1^{0'}\left(\delta\tilde{Y},z\right)\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y},z\right)$$

where

$$\sigma_1^{0'}\left(\delta\tilde{Y},z\right) \;=\; \exp\left(-\int\limits_0^{\delta\tilde{Y}} d\delta\tilde{Y}'\,T\left(\delta\tilde{Y}',-\kappa\left(\delta\tilde{Y}\;-\;\delta\tilde{Y}'\right)+\;z\right)\right)$$

$$\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y},z\right) \;=\; \frac{1}{\kappa}\int\limits_{\xi_0^A}^{z} dz'\,\frac{\sigma_{0,1}}{\sigma_1^{0'}\left(\delta\tilde{Y}-\frac{z-z'}{\kappa},z'\right)}\Delta\left(\delta\tilde{Y}-\frac{z-z'}{\kappa},z'\right)\;+\;C_{\sigma_1}$$

$$\Phi_1\left(\xi\right) \;=\; \frac{\frac{1}{2}e^{\xi}\exp\left(-\frac{1}{2}e^{\xi}\right)}{\sigma_1^{0'}\left(\delta\tilde{Y}=0,z=\xi\right)}\;-\;\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y}=0,z=\xi\right)$$

- For other iterations, we just modify our definition of the non-homogeneous term by adding the contribution from N_L

σ_n

- **Now we need to add to our definition of L[σ_1] the non-homogeneous term**

$$\mathcal{L}[\sigma_n] \;=\; \kappa \frac{d\sigma_n(z)}{dz} \;+\; T(z)\,\sigma_n(z) \;-\; \underbrace{\sigma_{0,n}\,\Delta(z) \;-\; \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\sigma_k(z)}_{-\,U_n(z)}$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_n] \;=\; \Delta(z)\;\tilde{\Sigma}_n(z)$$

- **We solve the equation in a similar way to σ_1, yielding**

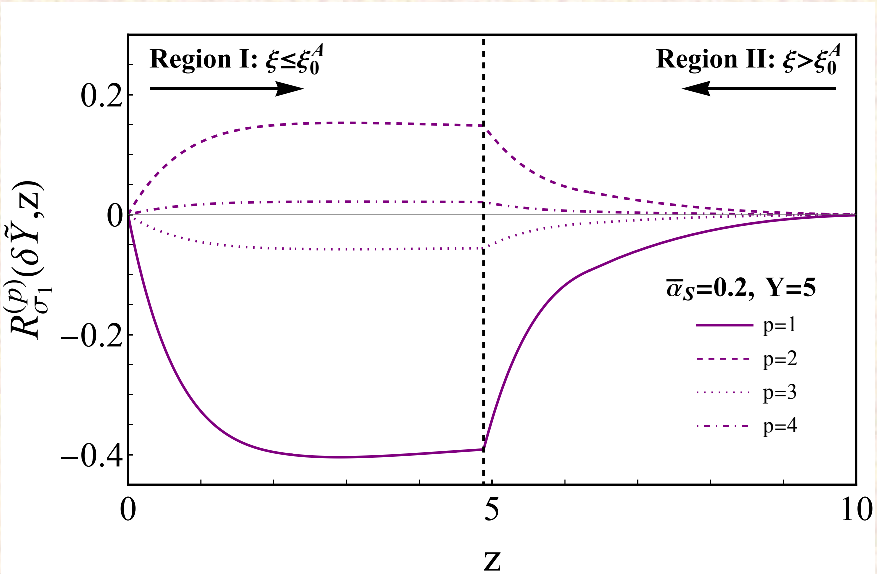
$$\sigma_n^{(0,I)}(z) \;=\; \exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T(z')\right)\left\{\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,U_n(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T(z'')\right) \;+\; C_\phi^{(n)}\right\}$$

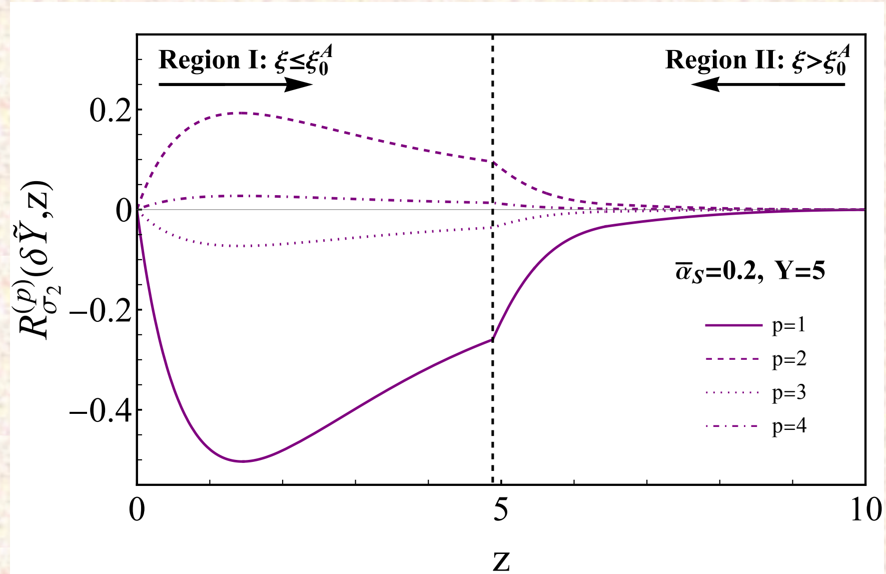
where $C_\phi^{(n)} \;=\; \dfrac{\left(\frac{1}{2}e^{\xi_0^A}\right)^n}{n!}\,e^{-\frac{1}{2}e^{\xi_0^A}}$**. The formula for the p-iteration is**

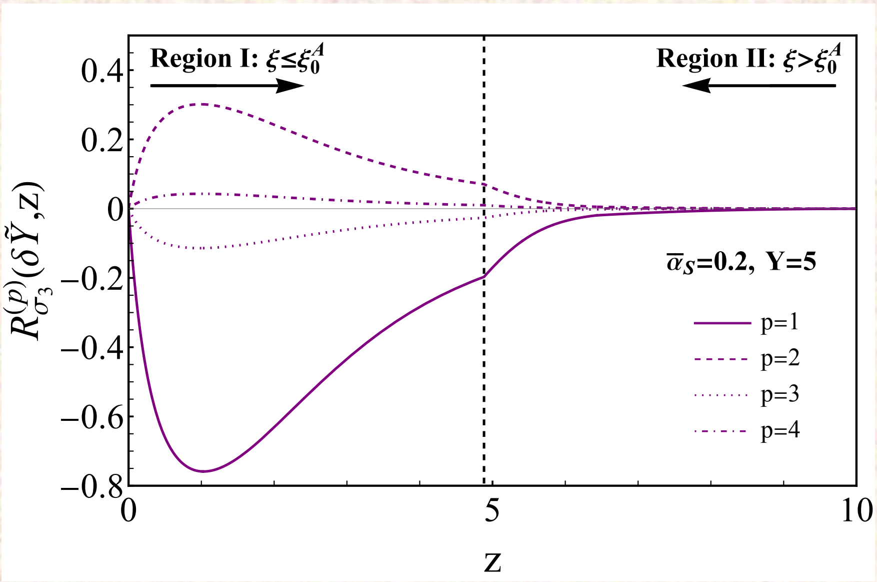
$$\sigma_n^{(p,I)}(z) \;=\; \exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T(z')\right)\left\{\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,U_n(z')\,\tilde{\Sigma}_n^{(p-1)}(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T(z'')\right) \;+\; C_\phi^{(n)}\right\}$$

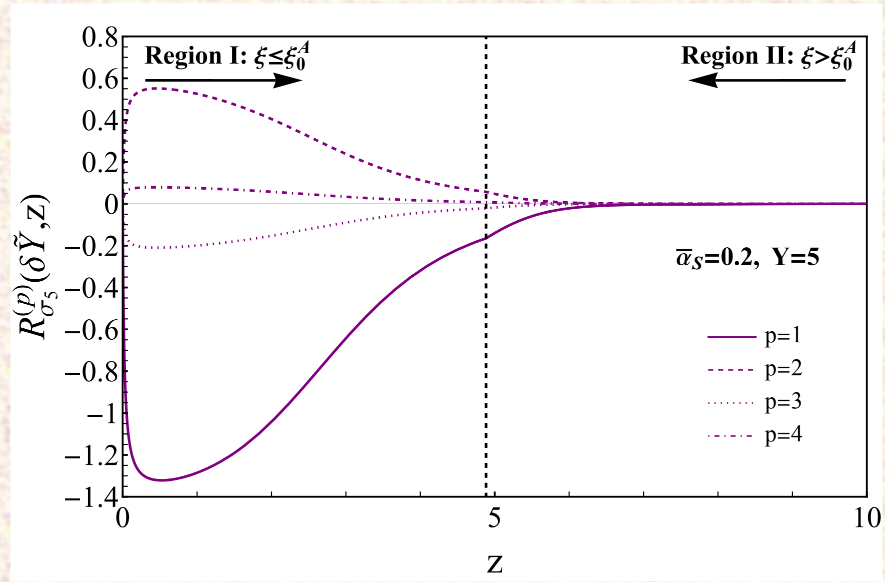
Numerical estimates

- **Ratios of σ_1, σ_2, σ_3 and σ_5 for \xi_A=0, \alpha_S=0.2, z_0=2.**



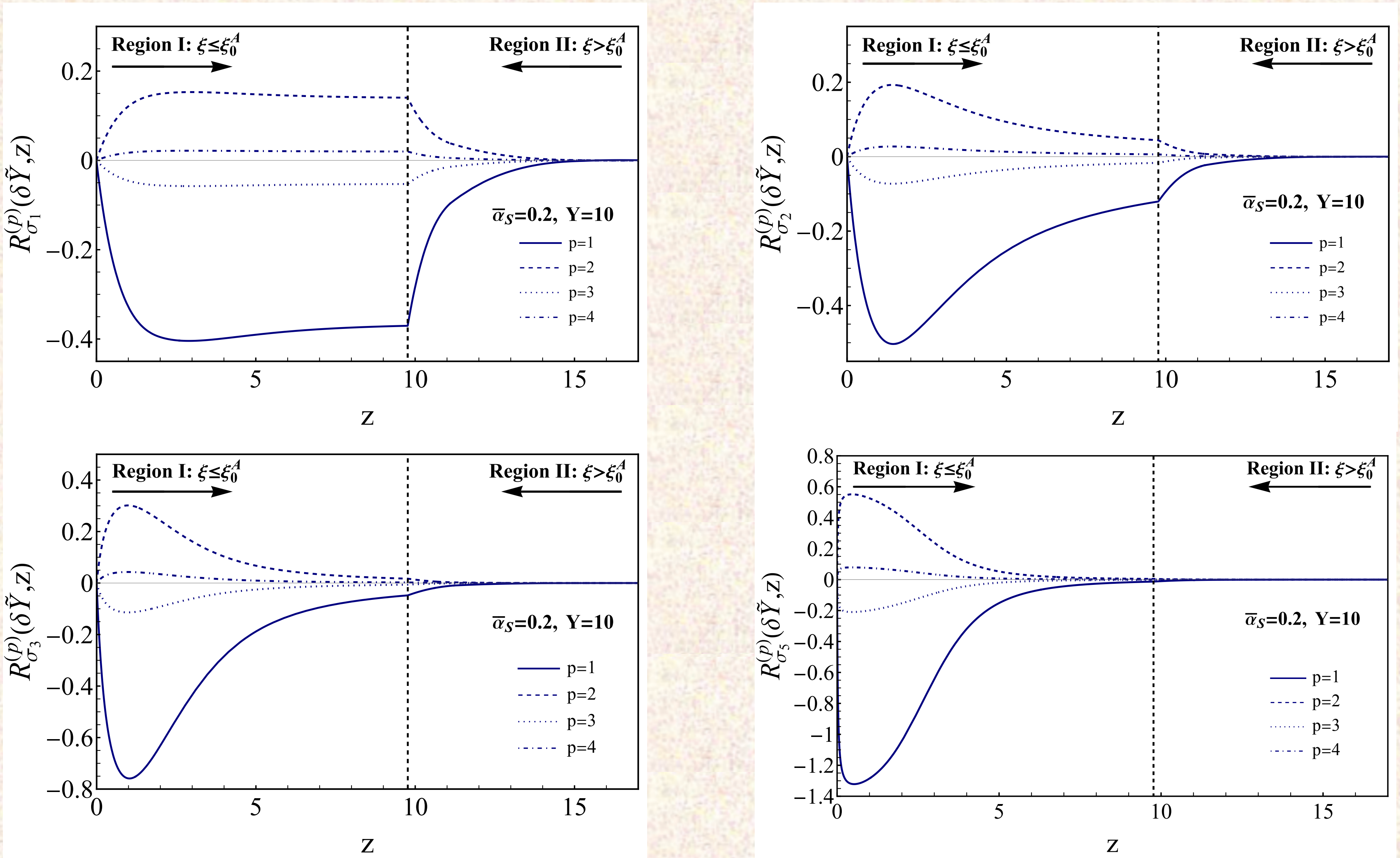






Numerical estimates

- **Ratios of σ_1, σ_2, σ_3 and σ_5 for \xi_A=0, \alpha_S=0.2 , z_0=2.**



Numerical estimates

- **One can see that the first iteration turns out to be rather large in the small-z region which corresponds to region I.**
- **Despite the fact that after an appropriate number of iterations the corrections become small, the approach may break down for a sufficiently large n. For this reason, we use this method only for small values of n.**
- **We solve the equation in the large-n limit, and study the possible matching with this solution.**

The large n≳N Solution

- **We start with our equation**

$$\kappa \frac{d\sigma_n(z)}{dz} = -\left(z - \Sigma_\Delta(z)\right)\sigma_n(z) + \Delta(z)\,\Sigma_n(z) + \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\sigma_k(z)$$

where $\Sigma_n(z) = \int^z dz'\sigma_n(z')$ **and** $\Sigma_\Delta(z) = \int^z dz'\Delta(z')$

- **We suggest that the solution has the following form**

$$\Sigma_n(z) = \phi(z)\,\exp\left(-n\,\Phi(z)\right)$$

- **Functions Φ (z) and φ (z) we will find from the equation. Noticing that**

$$\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\sigma_k(z) = \frac{1}{2}\frac{d}{dz}\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\Sigma_k(z) = \frac{n-1}{2}\frac{d}{dz}\phi^2(z)\exp\left(-n\Phi(z)\right)$$

- **The equation can be rewritten as**

$$\kappa \frac{d^2\Sigma_n}{dz^2} = -\frac{d}{dz}\left(\left(z - \Sigma_\Delta(z)\right)\Sigma_n(z)\right) + \Sigma_n(z) + \frac{n-1}{2}\frac{d}{dz}\left(\phi(z)\Sigma_n(z)\right)$$

The large n≳N Solution

- **Introducing**

$$\mathcal{S}_n(z) = \int^z dz' \, \Sigma_n(z')$$

and assuming that all functions decreases at large z, we have

$$\kappa \frac{d\Sigma_n}{dz} = -(z - \Sigma_\Delta(z))\Sigma_n(z) + \mathcal{S}_n(z) + \frac{n-1}{2}\phi(z)\,\Sigma_n(z)$$

- **Using** $\frac{d\Sigma_n}{dz} = (\phi'(z) - n\,\phi(z)\Phi'(z))\exp(-n\,\Phi(z))$

we can find now all unknown functions Φ, φ and S.

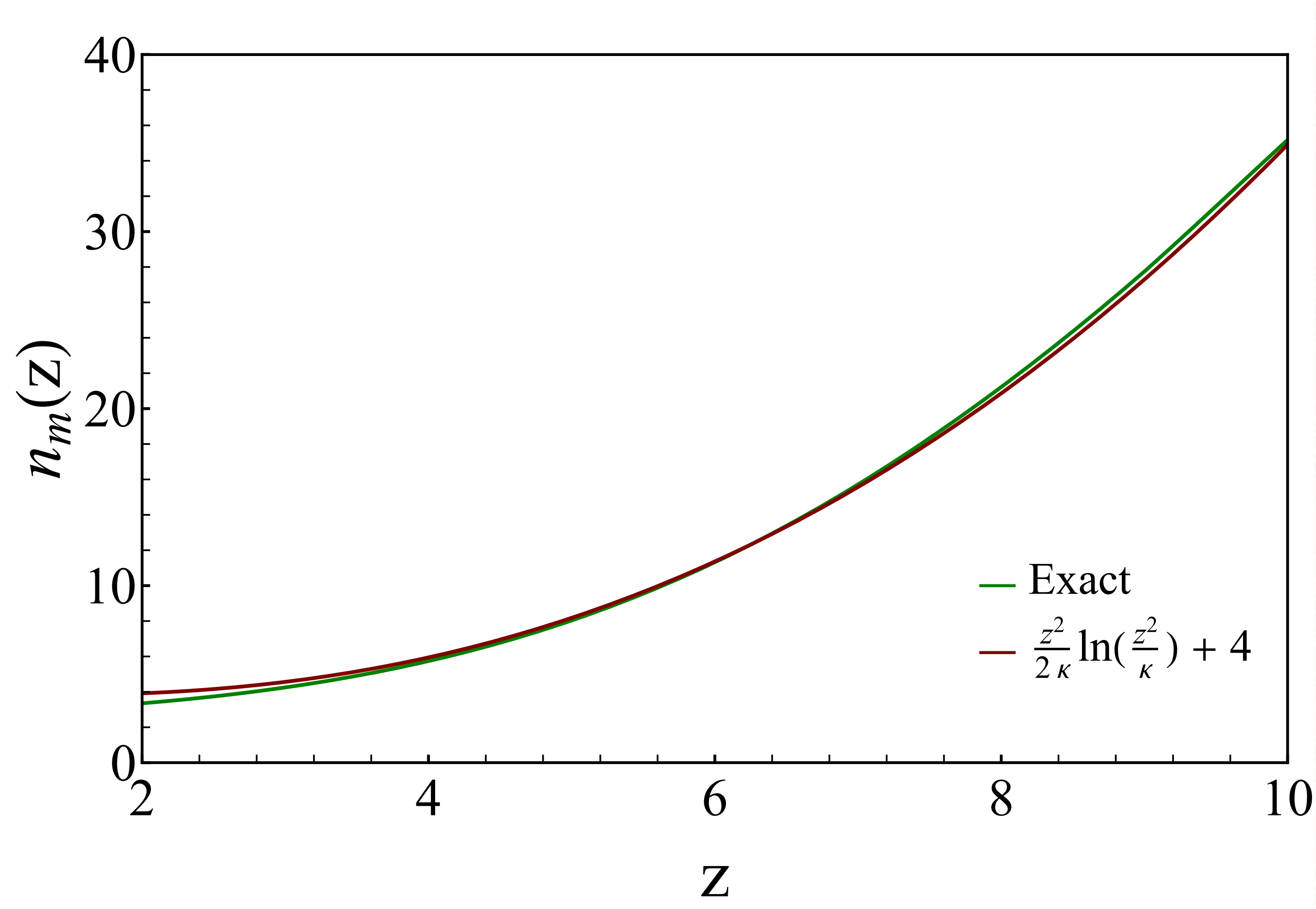
- **Taking the large z and n limit, we find for DIS:**

$$\sigma_n(z) = \left(\phi'(z) - n\phi(z)\,\Phi'(z)\right)\exp\left(-n\Phi(z)\right) \xrightarrow{n \geq N(z)} \frac{2}{\kappa}\frac{z^2}{N(z)}\Psi\left(\xi = \frac{n}{N(z)}\right)$$

where $\Psi(\xi) = \xi\, e^{-\xi}$ **and** $N(z) = \frac{2\,z}{C_\phi}\, e^{\frac{(z-z_\Delta)^2}{2\,\kappa}}$

Matching of the solutions

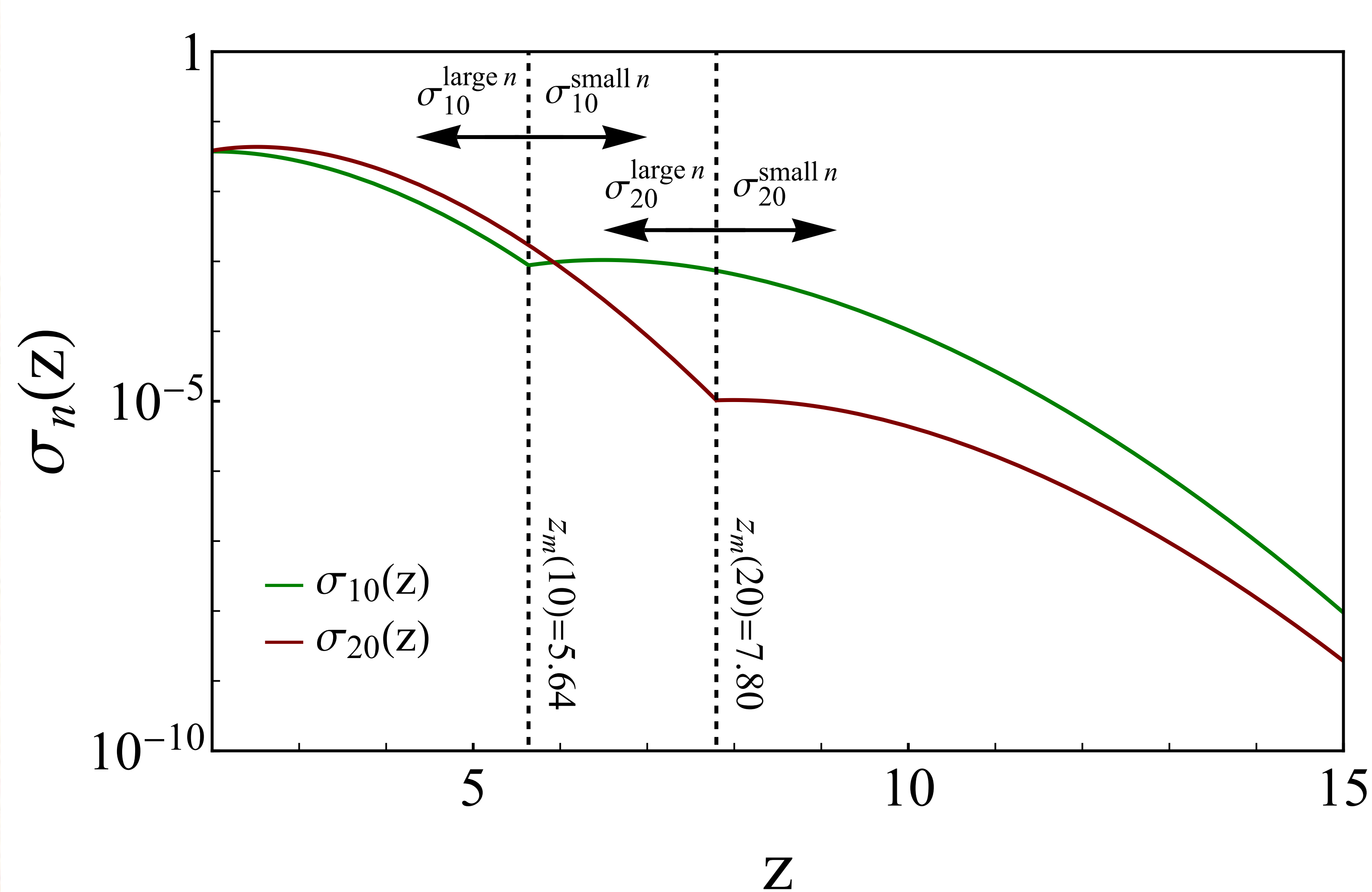
$$\sigma_{\text{small n}} = \sigma^{(0)}_{n_m}(z)\left(1 + \sum_{p=1}^{p_{\max}} \frac{\sigma^{(p)}_{n_m}(z)}{\sigma^{(0)}_{n_m}(z)}\right) = \sigma_{\text{large n}} = \frac{2}{\kappa}\frac{z^2}{N(z)}\Psi\left(\frac{n_m}{N(z)}\right)$$



Matching of the solutions

- **From the previous plot, we can invert n to obtain the matching z**

$$\sigma_n(z) = \sigma_{\text{large n}}(z)\,\Theta(z_m - z) + \sigma_{\text{small n}}(z)\,\Theta(z - z_m)$$




Quantum entanglement in
high-energy collisions

Entropy of produced gluons

- **The probability to have n-cut BFKL Pomerons in the final sate is equal to**

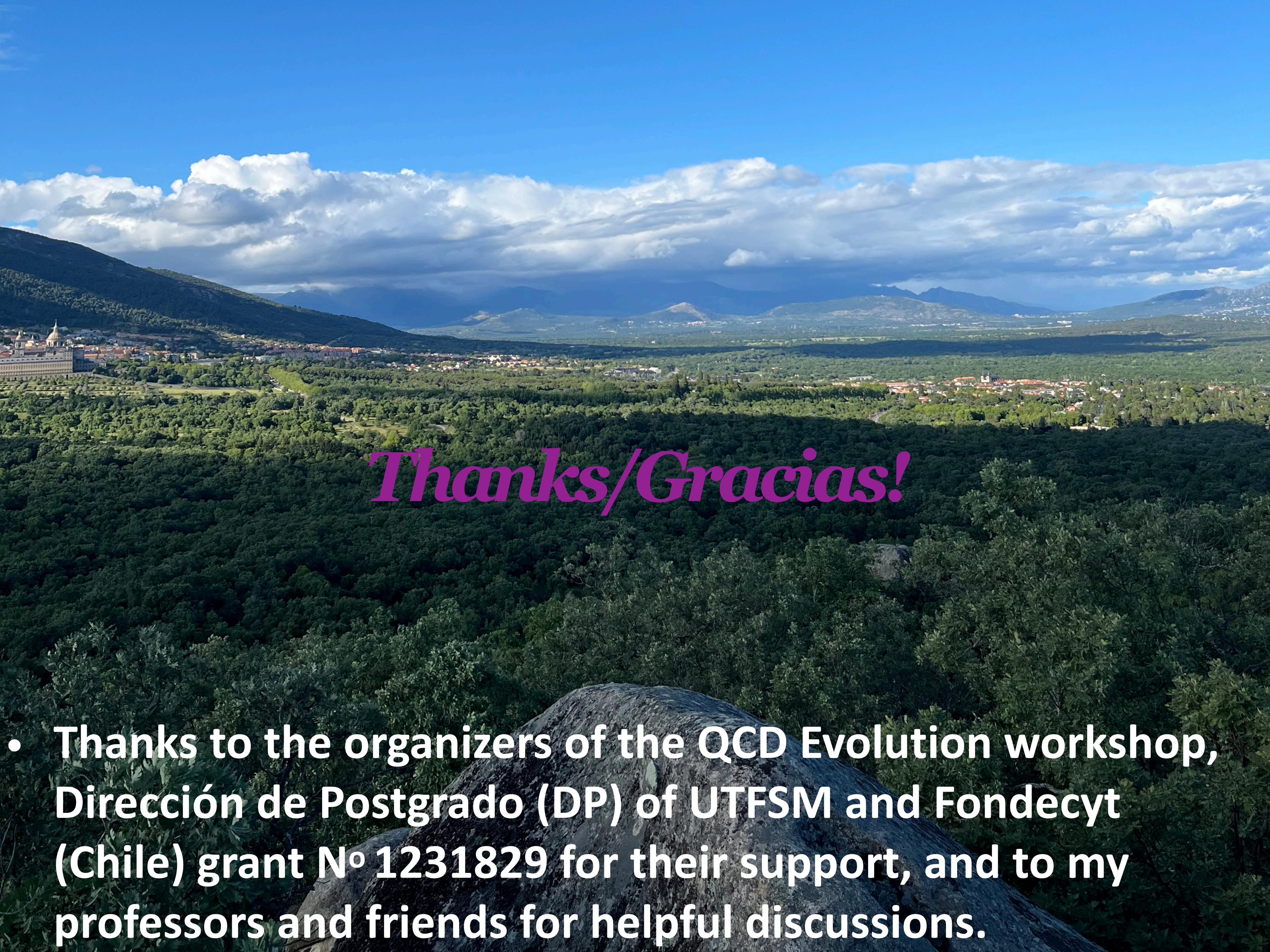
$$\mathcal{P}_n^{\mathrm{AGK}}(z) \equiv \frac{\sigma_n^{\mathrm{AGK}}(z)}{\sigma_{in}^{\mathrm{AGK}}(z)} = \frac{2}{\kappa}\,\frac{z^2}{N(z)}\,\Psi\left(\frac{n}{N(z)}\right)$$

- **The entropy content of multiplicity distributions is defined by**

$$\begin{aligned} S_E(z) &= -\sum_n \mathcal{P}_n^{\mathrm{AGK}}(z)\,\ln\left(\mathcal{P}_n^{\mathrm{AGK}}(z)\right) \\ &= \frac{(z - z_\Delta)^2}{2\,\kappa} - \ln z - \ln\left(\frac{C_\phi}{\kappa}\right) + \frac{2}{\kappa}\ln 2 \\ &\xrightarrow{z \gg 1} \frac{z^2}{2\,\kappa} \end{aligned}$$

Conclusions and outlook

- We calculated the entropy and the multiplicity distributions directly from QCD evolution equations and the s-channel unitarity constraints. We obtained a much larger entropy that other QCD models ($\propto z^2$). We plan to clarify this discrepancy in our further publications.
- σ_1, σ_2 and σ_3 are cross sections for 1, 2 and 3 cut Pomeron production which are closely related to <n>, <n^2-n>, <n^3 - 3 n^2+3 n> moments of the multiplicity distributions for DIS experiment. Calculations of the moments <n^k> for small k will be subject of next papers.



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