A FINITE ELEMENT METHOD ANALYSIS OF SHADOW AMBIGUITIES IN GENERALIZED PARTON DISTRIBUTIONS

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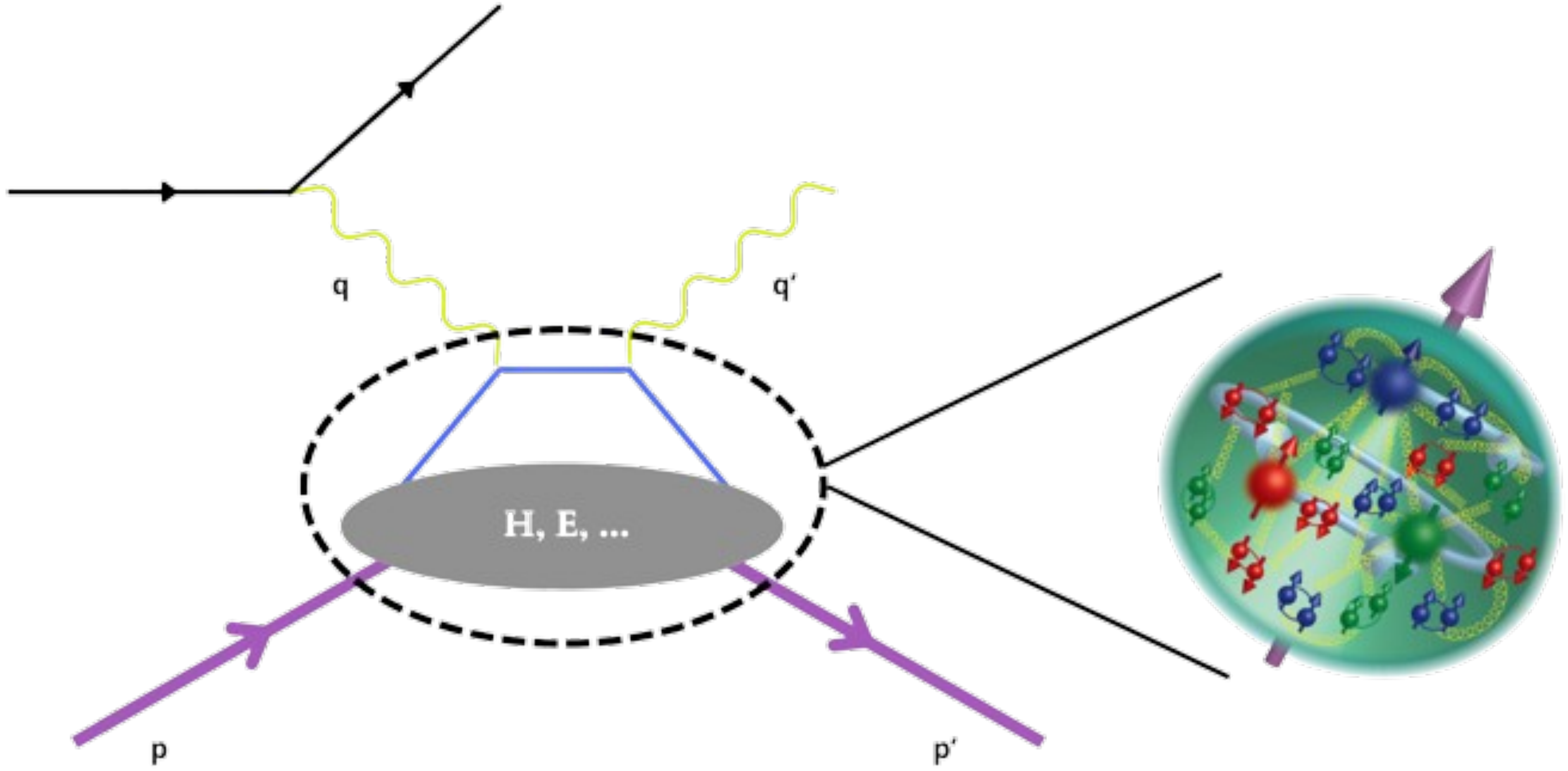
INVERSE PROBLEM

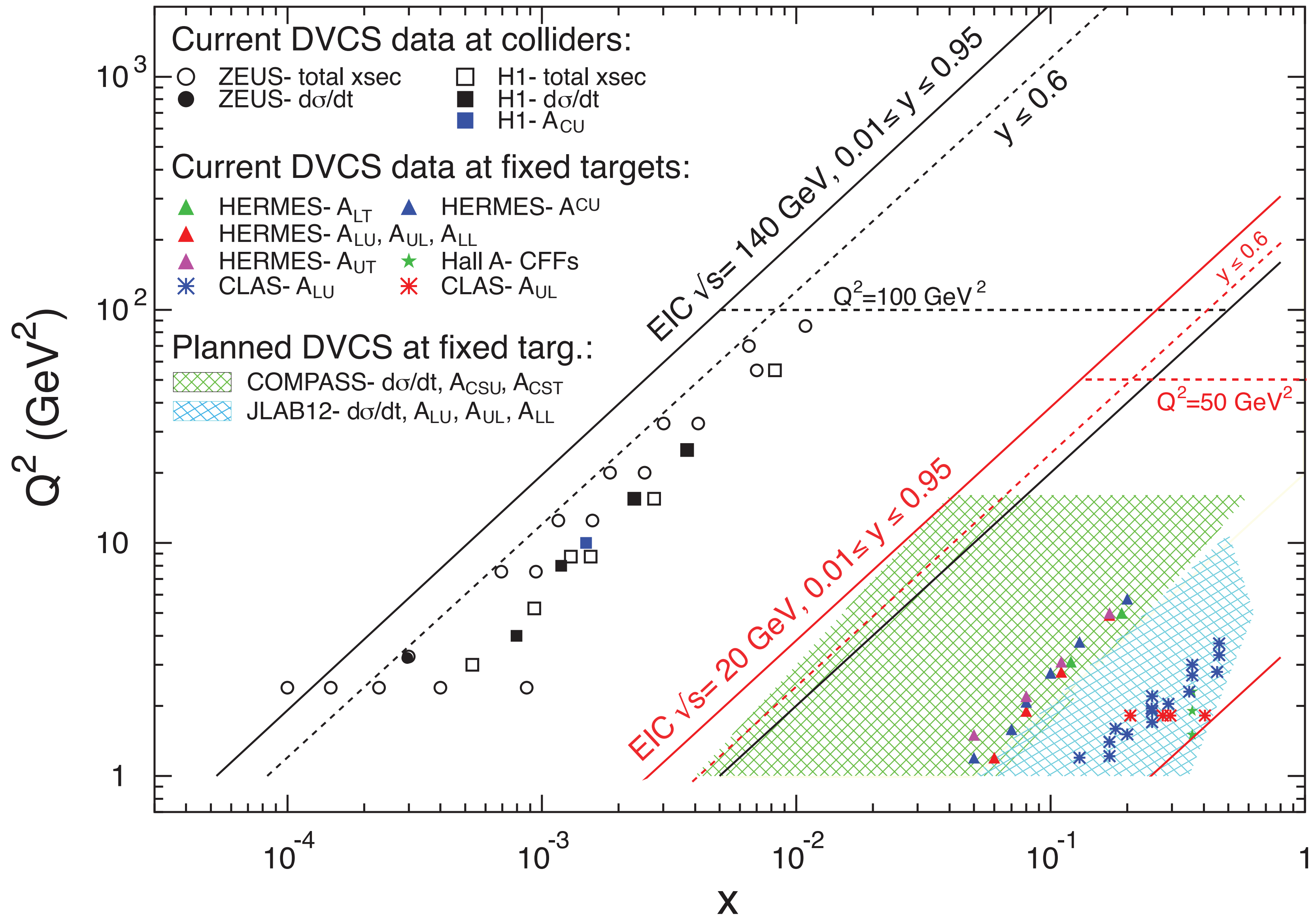
DVCS: $\ell + p \to \ell + \gamma + p$



$$\frac{d^5\sigma_{DVCS}}{dx_B\,dy\,dt\,d\phi\,d\varphi} \propto 4\left(1-x_B\right)\left(|\mathcal{H}|^2 + |\widetilde{\mathcal{H}}|^2\right) + \dots$$

$$\mathcal{H}^A\left(\xi,\Delta^2,Q^2\right) = \int_{-1}^{1}\frac{dx}{2\xi}\,\underbrace{{}^AT\left(x,\xi \,|\, \alpha_s\left(\mu_R\right),\left\{\frac{Q^2}{\mu^2}\right\}\right)}_{\text{hard scale}}\underbrace{H^A\left(x,\xi,\Delta^2,\mu_F^2\right)}_{\text{soft scale}}$$

Limited experimental coverage:



Inverse problem: observables → GPDs inside convolution
→ ill-posed problem!

Additionally, experiment dominantly probes the DGLAP region, specifically at LO

$$\mathfrak{Im}\mathcal{H}\left(\xi,t,Q^2\right) = \pi \sum_{q=u,d,\dots} e_q^2\left[H^q\left(x=\xi,\xi,t,\mu_F^2\right) - H^q\left(x=-\xi,\xi,t,\mu_F^2\right)\right].$$

Any extraction beyond that is unreliable. **How do we access the ERBL region?**

DOUBLE DISTRIBUTION REPRESENTATION OF GPDS

[arXiv:hep-ph/9704207]

GPD definition: $F^q_{\Lambda_2,\Lambda_1} = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P_2, \Lambda_2 \right| \bar{\psi}^q \left(-\frac{z}{2}\right) \gamma^+ \psi^q \left(\frac{z}{2}\right) \left| P_1, \Lambda_1 \right\rangle \Big|_{\substack{z^+_\perp = 0 \\ z_\perp = 0}}$

$$= \frac{1}{2P^+}\left[H^q(x,\xi,t)\bar{u}\left(P_2,\Lambda_2\right)\gamma^+ u\left(P_1,\Lambda_1\right) + E^q(x,\xi,t)\bar{u}\left(P_2,\Lambda_2\right)\frac{i\sigma^{+\mu}\Delta_\mu}{2M_H}u\left(P_1,\Lambda_1\right)\right]$$

Alternative parametrization of the matrix element:

$$\left\langle P+\frac{\Delta}{2}\right| \bar{\psi}^q\left(-\frac{z^-}{2}\right)\gamma^+\psi^q\left(\frac{z^-}{2}\right)\left|P-\frac{\Delta}{2}\right\rangle = \iint_\Omega d\beta\, d\alpha e^{-i\beta P^+z^- + i\alpha\frac{1}{2}\Delta^+z^-}\left(2P^+ f(\beta,\alpha) - \Delta^+ g(\beta,\alpha)\right)$$

double distributions

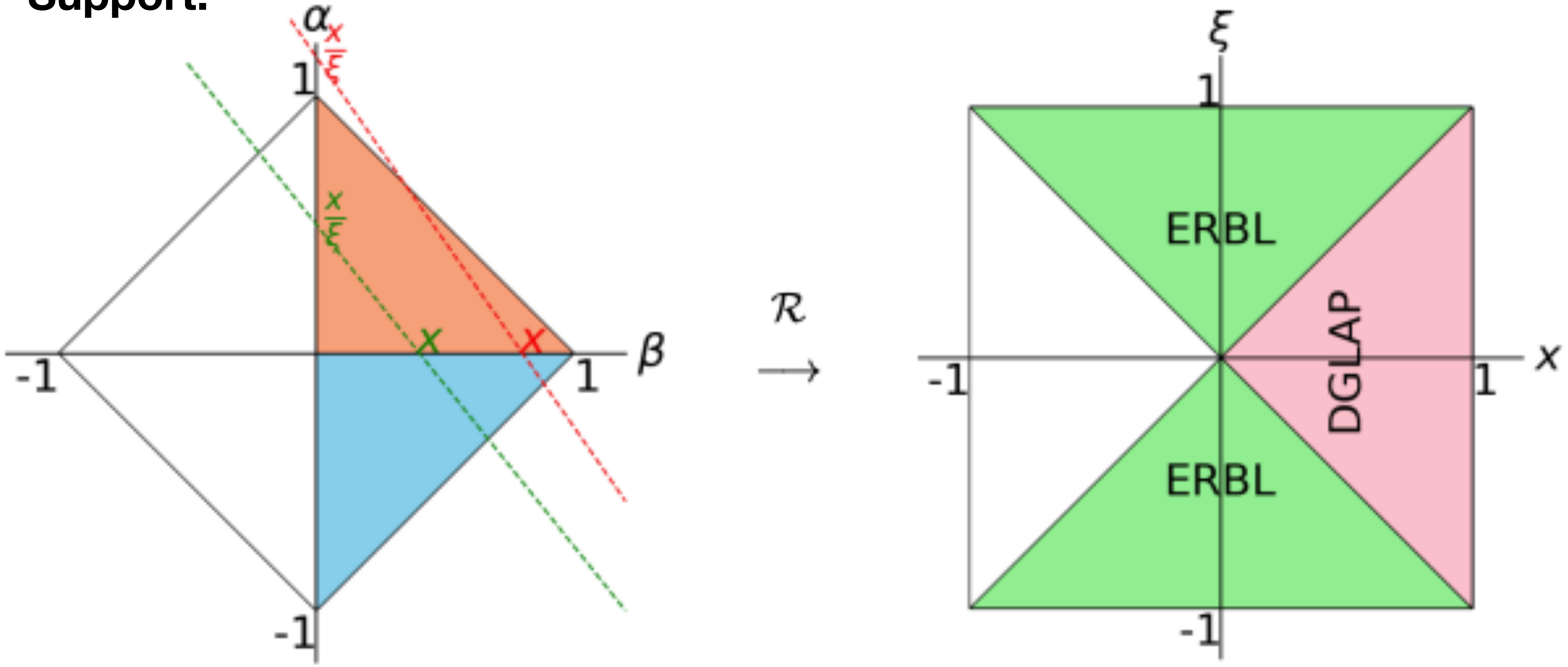
Radon transform: $H(x,\xi) = \iint_\Omega d\beta\, d\alpha \delta(x-\beta-\xi\alpha)(f(\beta,\alpha)+\xi g(\beta,\alpha))$

Useful property - polynomiality:

$$H^m(\xi) = \int_{-1}^{1} x^m H(x,\xi)dx = \sum_{\substack{k=0 \\ k \text{ even}}}^{m+1} H^m_k \xi^k \Rightarrow \int_{-1}^{1} x^m H(x,\xi)dx = \iint_\Omega d\beta\, d\alpha(\beta+\xi\alpha)^m (f(\beta,\alpha)+\xi g(\beta,\alpha))$$

RADON TRANSFORM

Support:



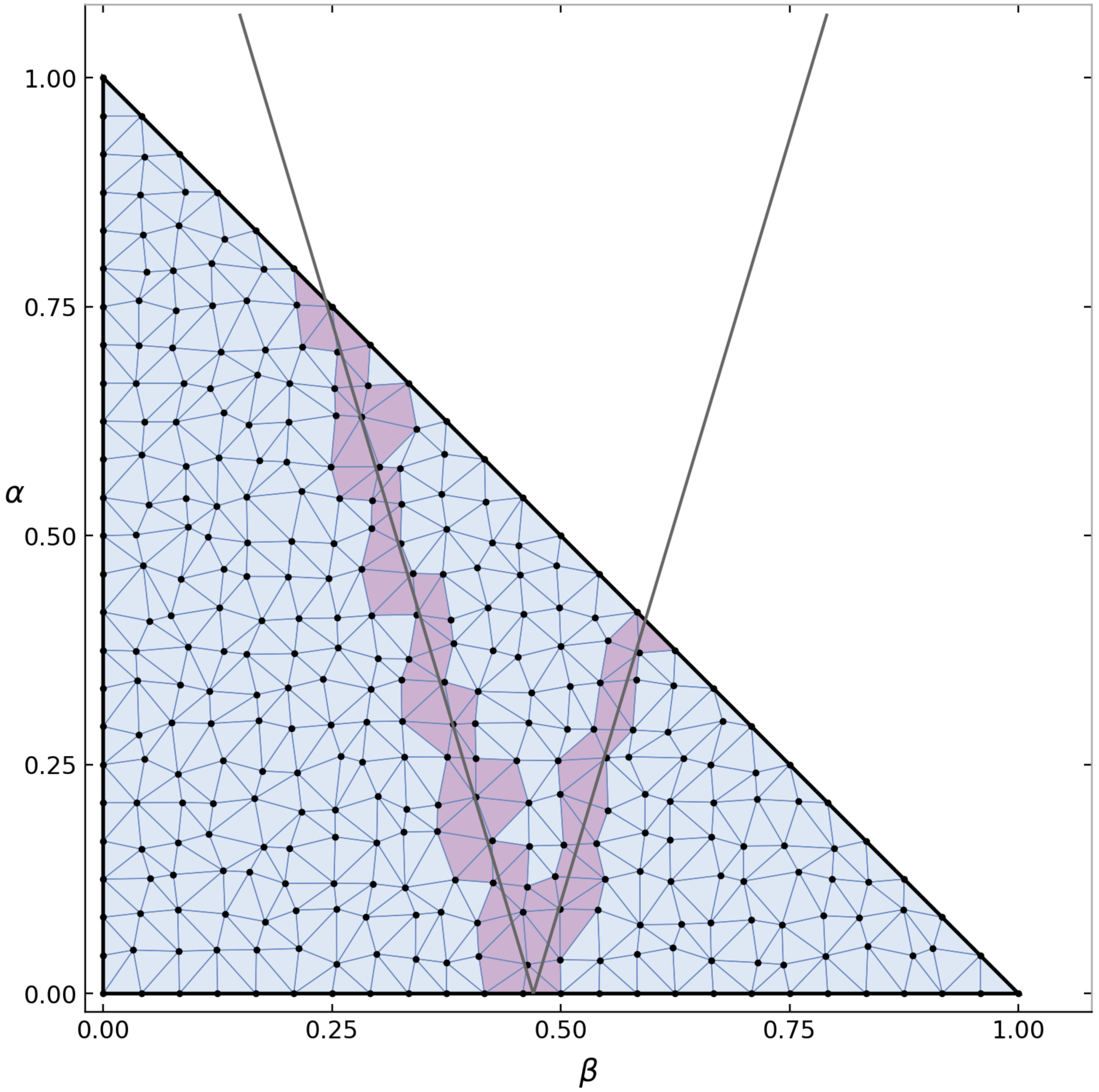
We obtain a 2D function by performing integrals over the lines $x - \beta - \xi\alpha = 0$. The negative α domain is equivalent to sampling the positive domain with the reflected lines $x - \beta + \xi\alpha = 0$. The $\beta < 0$ is probed by the antiquark distributions, which we do not consider at this point.

Symmetries: $H(x, -\xi, t) = H(x, \xi, t) \Rightarrow f(\beta, -\alpha) = f(\beta, \alpha), \quad g(\beta, -\alpha) = -g(\beta, \alpha)$

Goal: extract GPDs in the DGLAP region, obtain the corresponding double distribution, extend it to the ERBL region, invert the Radon transform and obtain the GPD over the whole domain. Analytically hard to perform, we rely on numerics. [arXiv:2401.12013]

FINITE ELEMENTS METHOD

$$H(x,\xi) = \mathscr{R}[h(\beta,\alpha)] \Rightarrow \begin{pmatrix} \vdots \\ H_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \cdots & \\ \vdots & R_{ij} \end{pmatrix} \begin{pmatrix} \vdots \\ h_j \\ \vdots \end{pmatrix}$$



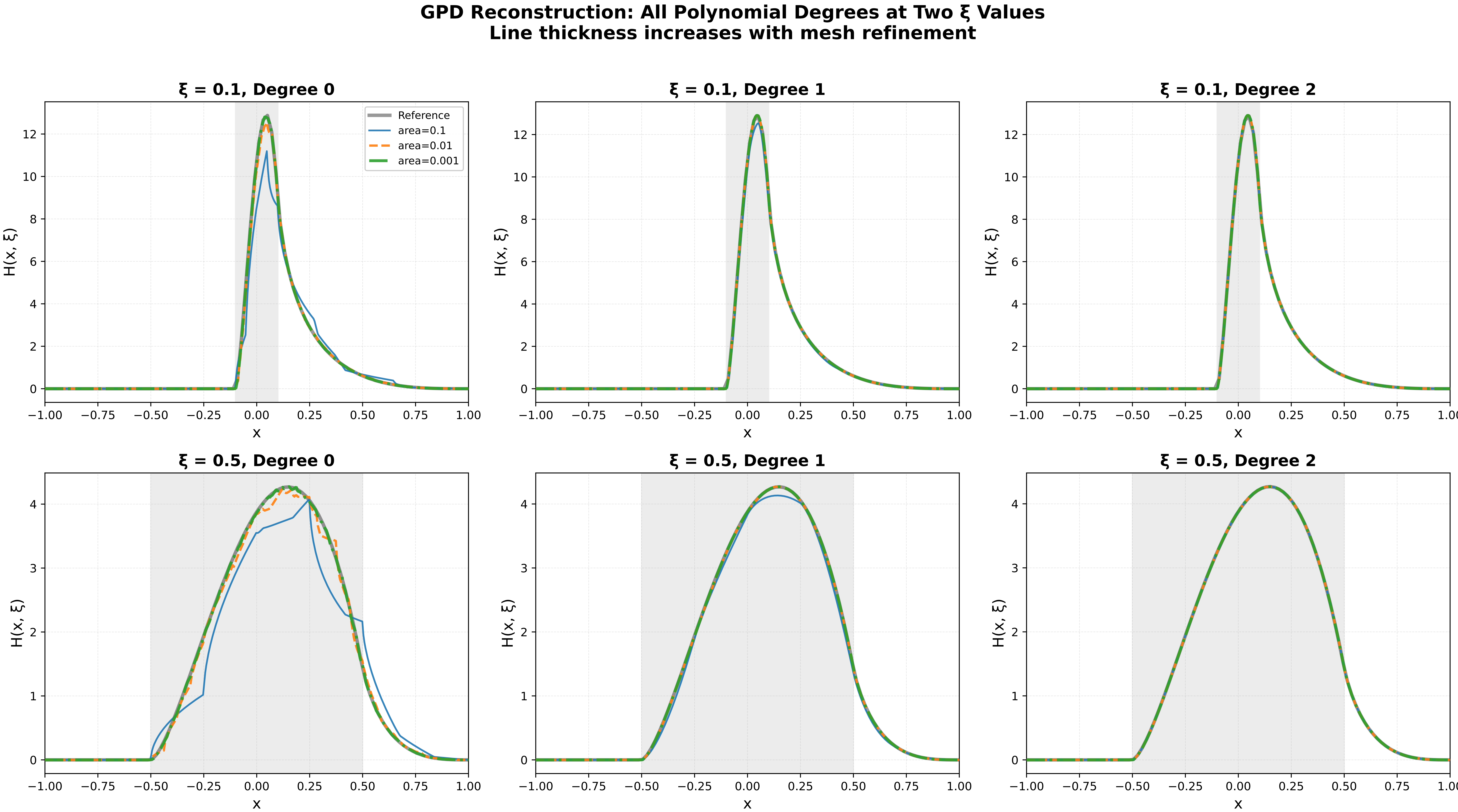
Radon matrix is a sparse matrix, we need to oversample to reach full rank!

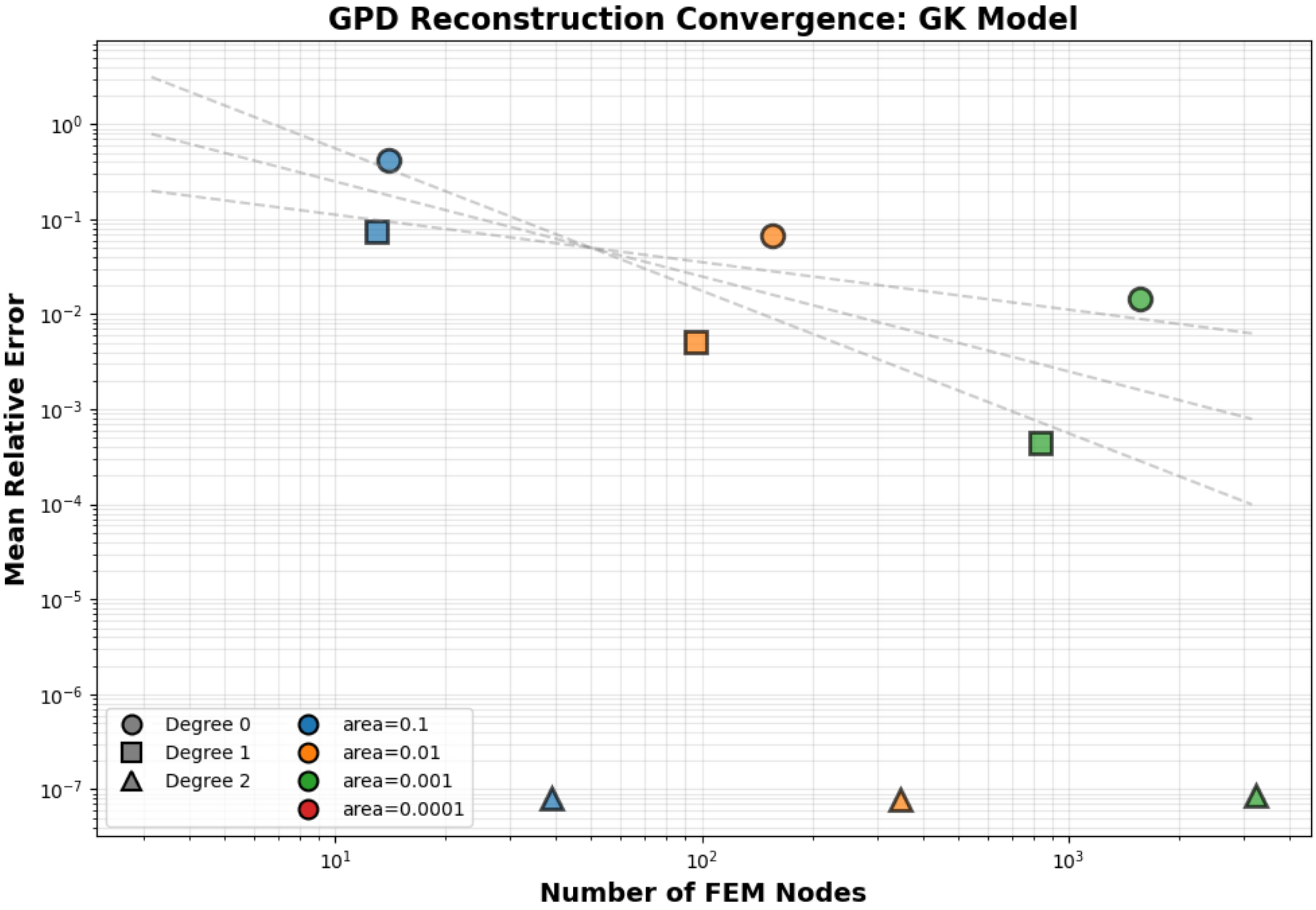
1. Discretize the domain with Delaunay triangulation
$$\Omega^+ = \bigcup_e \Omega_e^+$$
2. Interpolate the discrete double distribution
$$h(\beta,\alpha) \to P(\beta,\alpha) = \sum_e P_e(\beta,\alpha)\theta\left(\Omega_e^+\right)$$
   A. On each element of the triangulation define a simple function, such as linear polynomial
   $$P_e(\beta,\alpha) = c_e + b_e\beta + a_e\alpha$$
   B. Calculate the polynomials using barycentric coordinates
   $$P_e(\beta,\alpha) = \begin{vmatrix} \beta & \alpha & 1 \\ \beta_2 & \alpha_2 & 1 \\ \beta_3 & \alpha_3 & 1 \end{vmatrix} h_1^e + \begin{vmatrix} \beta_1 & \alpha_1 & 1 \\ \beta & \alpha & 1 \\ \beta_3 & \alpha_3 & 1 \end{vmatrix} h_2^e + \begin{vmatrix} \beta_1 & \alpha_2 & 1 \\ \beta_2 & \alpha_2 & 1 \\ \beta & \alpha & 1 \end{vmatrix} h_3^e$$
3. Choose sampling lines (x_i, ξ_i), usually oversample
4. Find the intersection points of each line in each mesh element
5. Calculate the Radon integral for each node in the mesh
6. Add up all sampled elements

$$H(x_i,\xi_i) = \sum_j h_j \int_0^1 \mathrm{d}\beta \int_0^1 \mathrm{d}\alpha \delta(x_i - \beta - \xi_i\alpha) v_j(\beta,\alpha)\Theta(1-\alpha-\beta) + \text{reflected}$$

$$= \sum_{\text{sampled } \Omega_e^+} \sum_{j\in\Omega_e^+} h_j \left\{ \left(c_j + b_j x_i\right)(\alpha_{in,e} - \alpha_{out,e}) + \left(a_j - b_j\xi_i\right) \frac{\alpha_{in,e}^2 - \alpha_{out,e}^2}{2} \right\} + \text{reflected}$$

RESULTS FOR GK MODEL




GPD Reconstruction Convergence: GK Model
Mean Relative Error
Number of FEM Nodes
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^1
10^2
10^3
Degree 0
Degree 1
Degree 2
area=0.1
area=0.01
area=0.001
area=0.0001

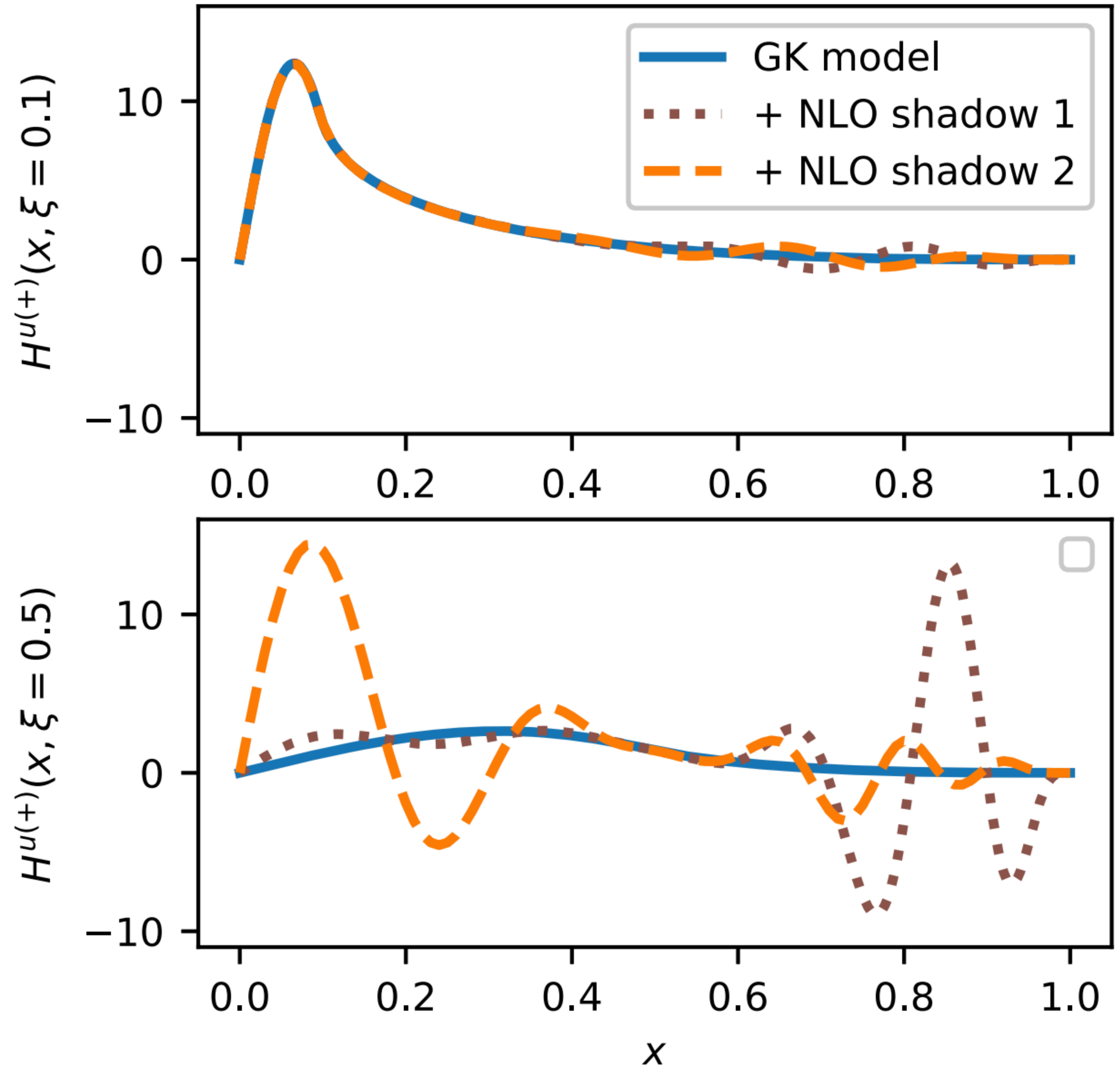
WHAT IS THE SHADOW GPD CONTRIBUTION?

SHADOW GPDS

Result of the inverse problem: $F_S^A(x,\xi,t) = F_E^A(x,\xi,t) - F_T^A(x,\xi,t)$

Must satisfy all properties of the GPD and vanish in physical observables:

- **Polynomiality**
- **Zero contribution to CFF:** $\sum_A {}^A T\left(x, \xi \mid \alpha_s(\mu_R), \left\{\frac{Q^2}{\mu^2}\right\}\right) \otimes H^A\left(x, \xi, \Delta^2, \mu_F^2\right)$
- **Zero contribution to PDF:** $H_S^A(x,0,0) = 0$



We test polynomial shadow DDs:

- **LO shadow DD** [arXiv:2104.03836v1]**:**

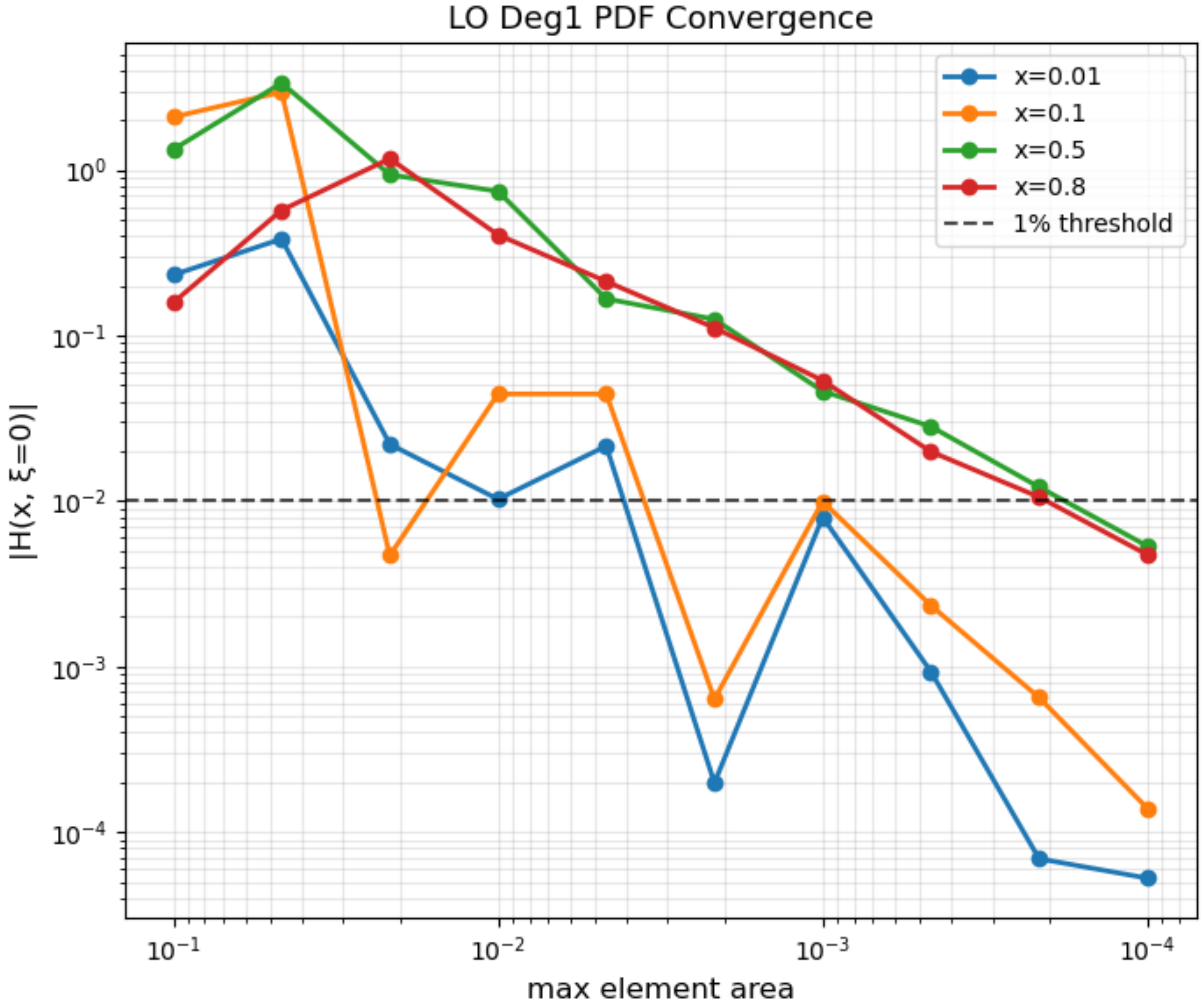
$$F_N^{q(+)}(\beta,\alpha) = \beta^{N-8}\left[\alpha^8 - \frac{28}{9}\alpha^6\left(\frac{N^2-3N+20}{(N+1)N}+\beta^2\right) + \frac{10}{3}\alpha^4\left(\frac{N^2-7N+40}{(N+1)N} + \frac{2\left(N^2-3N+44\right)}{3(N+1)N}\beta^2+\beta^4\right)\right.$$

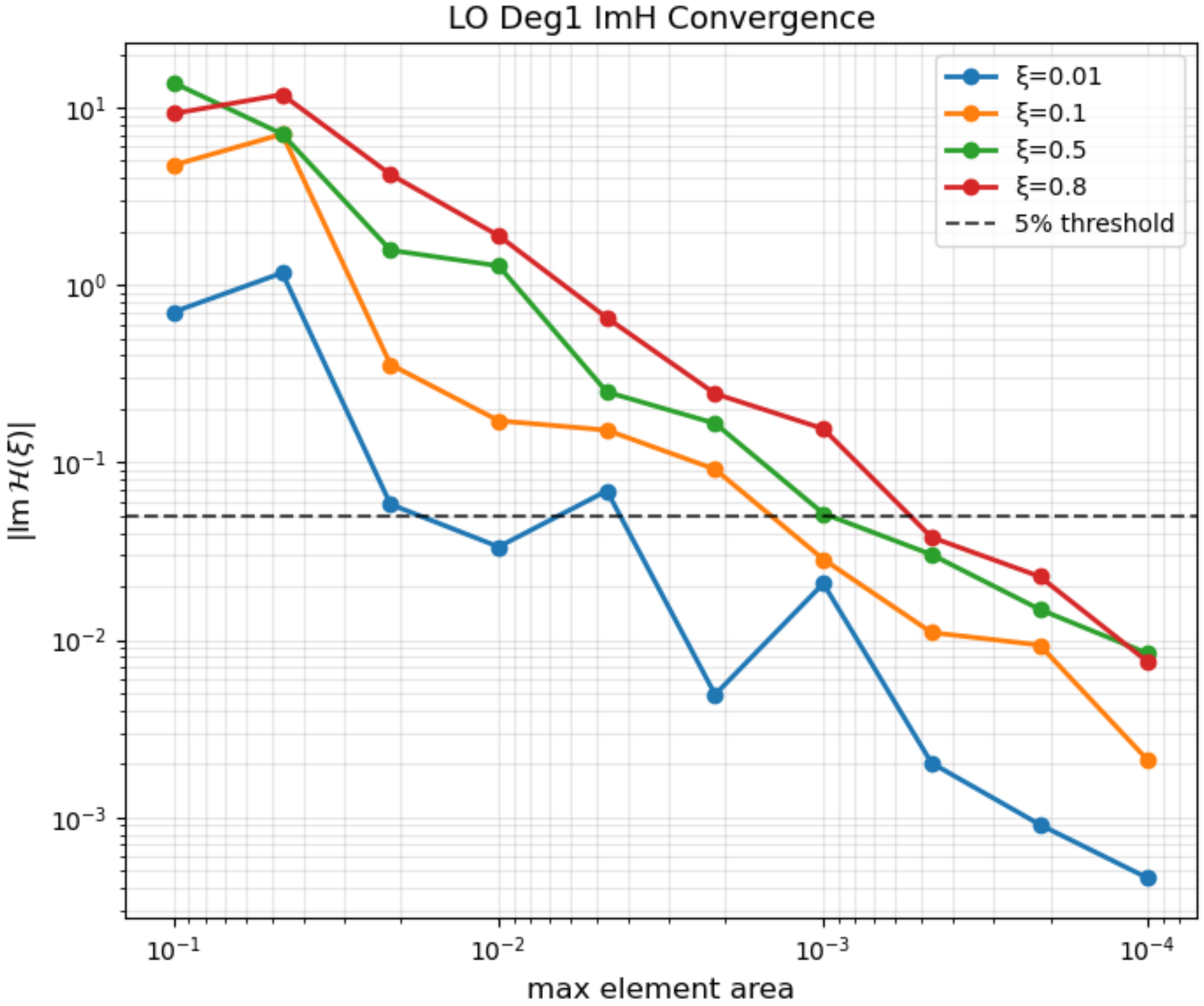
$$\left. -\frac{4}{3}\alpha^2\left(\frac{N^2-11N+60}{(N+1)N} - \frac{N-8}{N}\beta^2 - \frac{N^2-3N-28}{(N+1)N}\beta^4+\beta^6\right) + \frac{1}{9}\left(1-\beta^2\right)^2\left(\frac{N^2-15N+80}{(N+1)N} - \frac{2(N-8)}{N}\beta^2+\beta^4\right)\right], N \geq 9, \textbf{odd}$$

- **NLO shadow DD: order 27 polynomial**

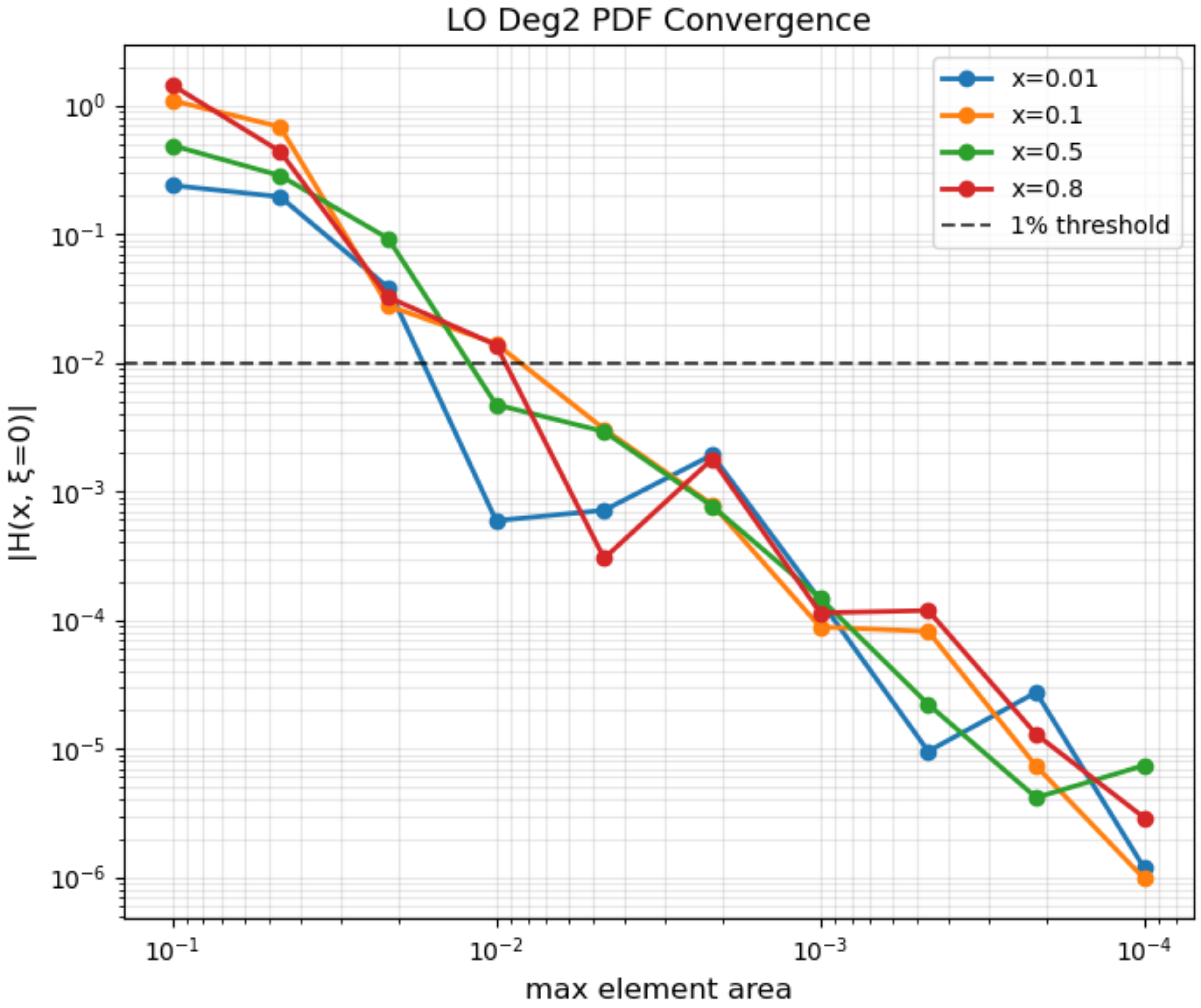
LO POLYNOMIAL SHADOW DD

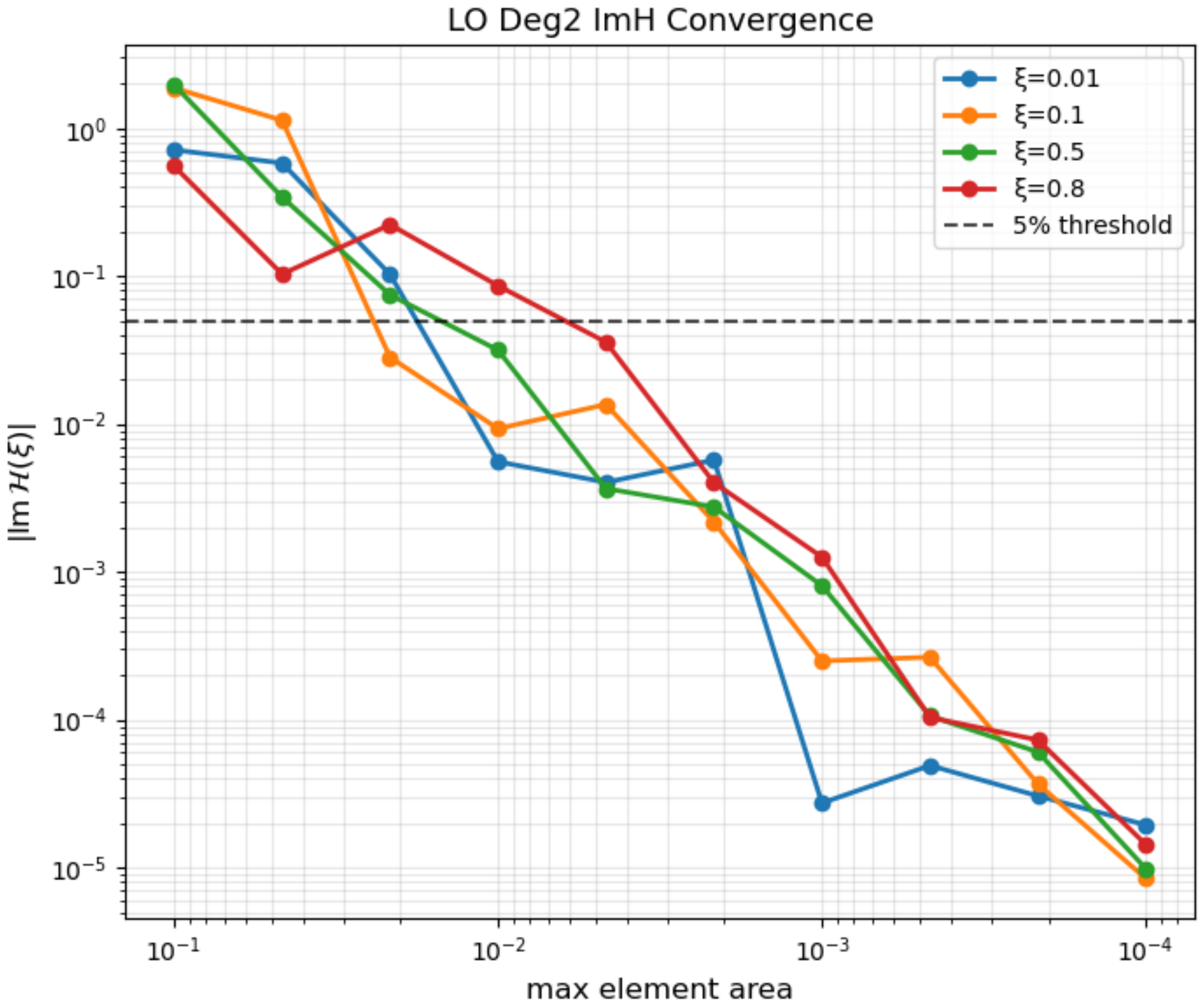
LO shadow DD, degree 1 polynomial interpolation





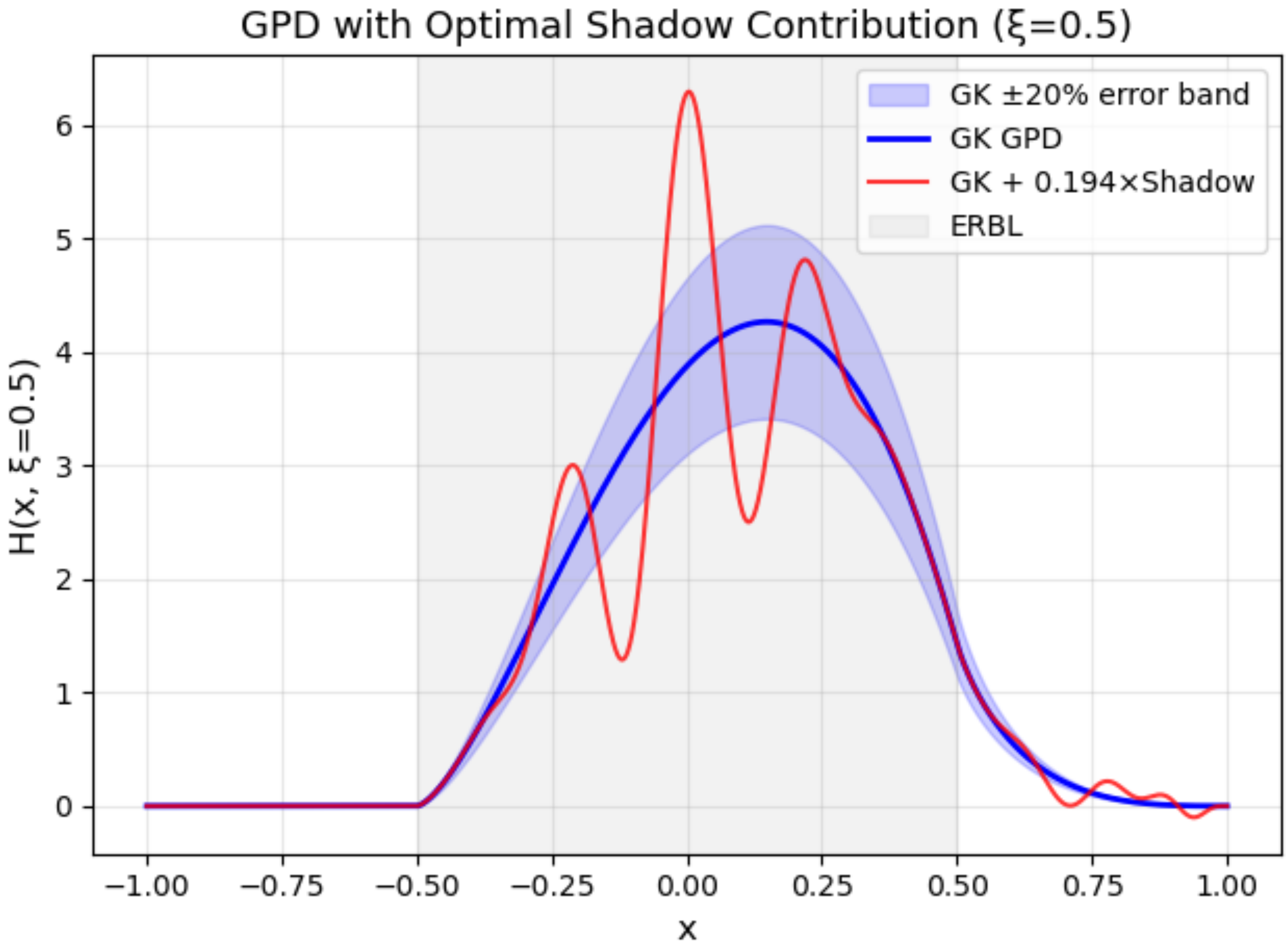
LO shadow DD, degree 2 polynomial interpolation



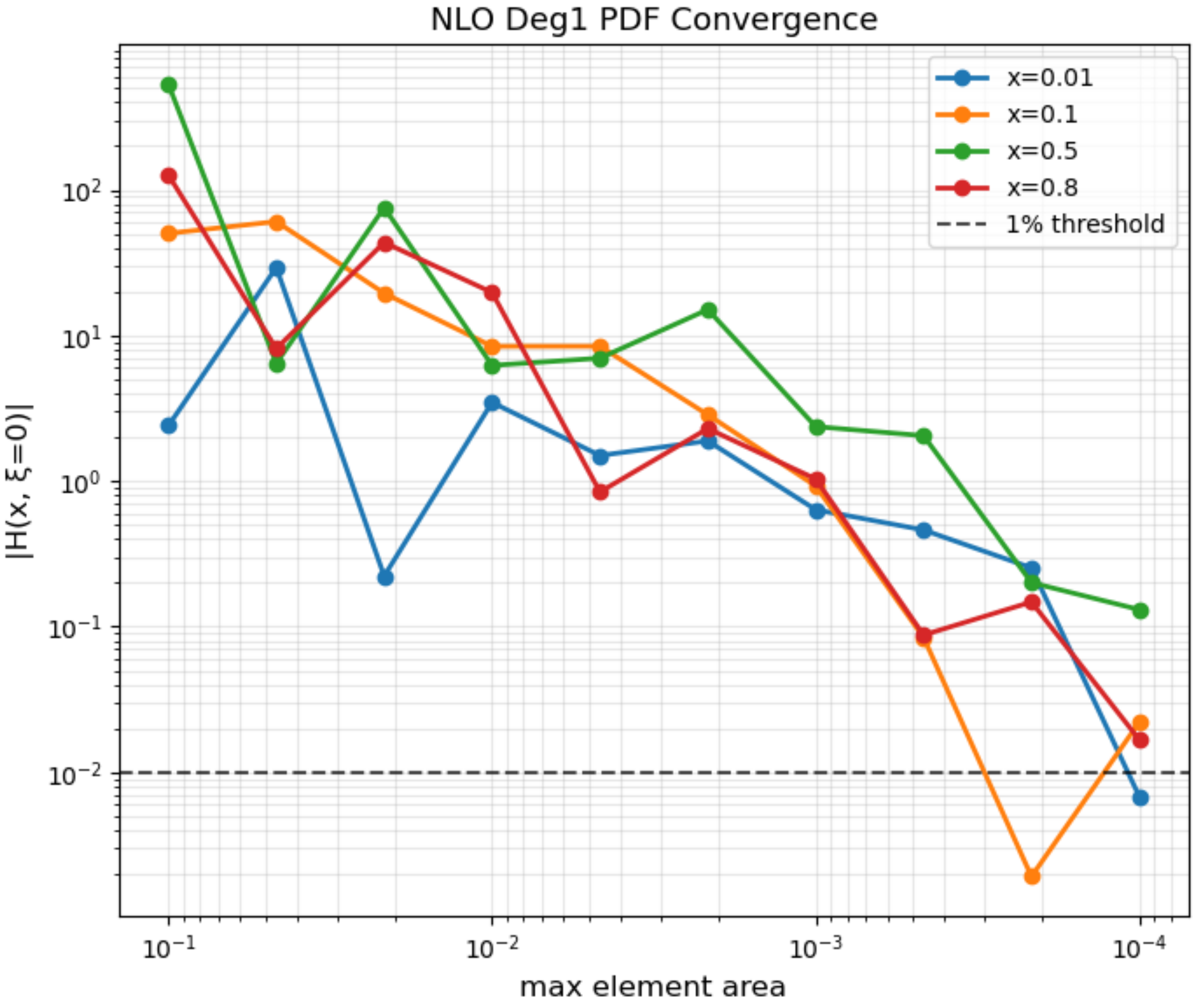


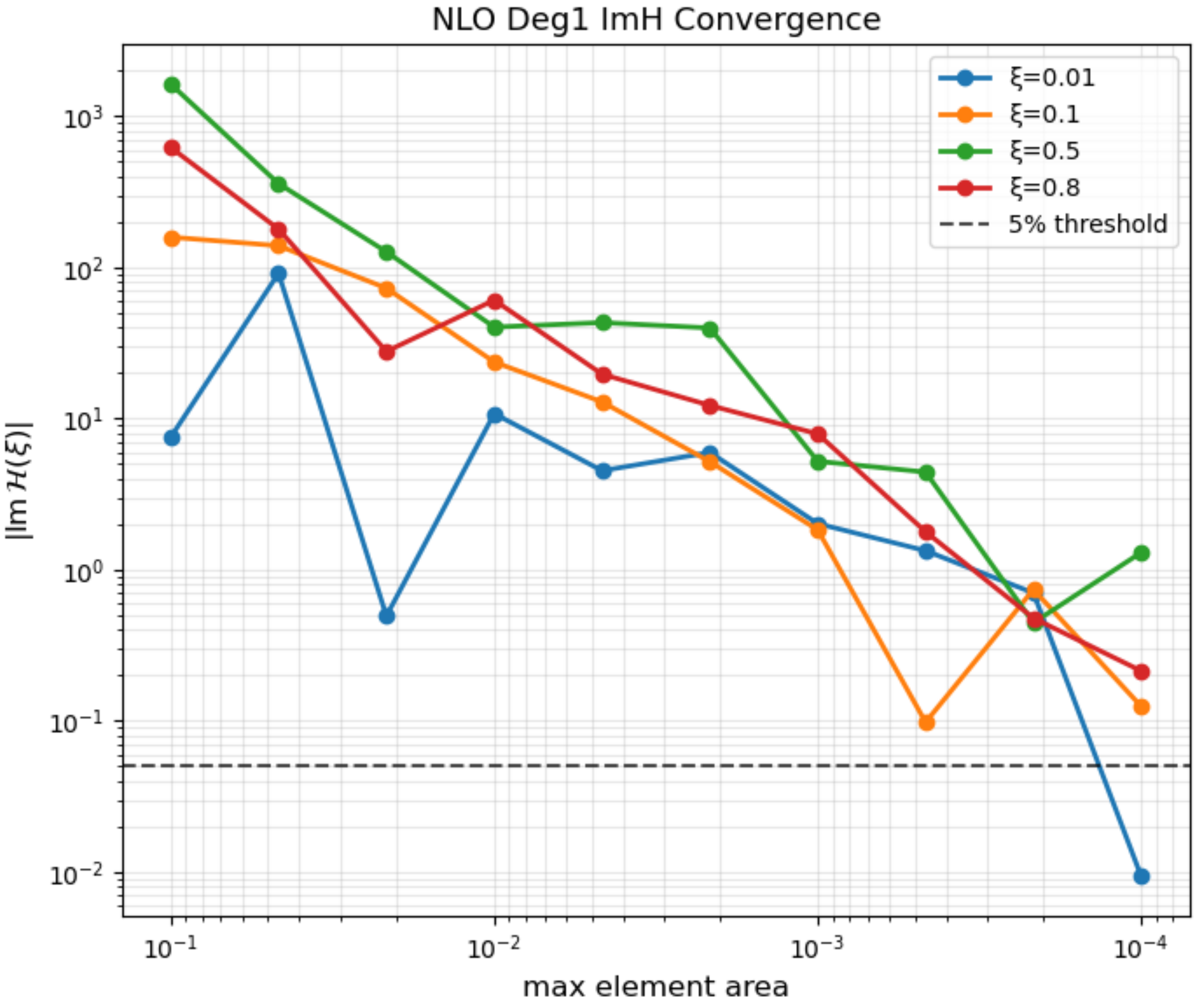
NLO POLYNOMIAL SHADOW DD

Shadow DD defined so that GK+shadow GPD is 67% of the time inside 20% GK error.

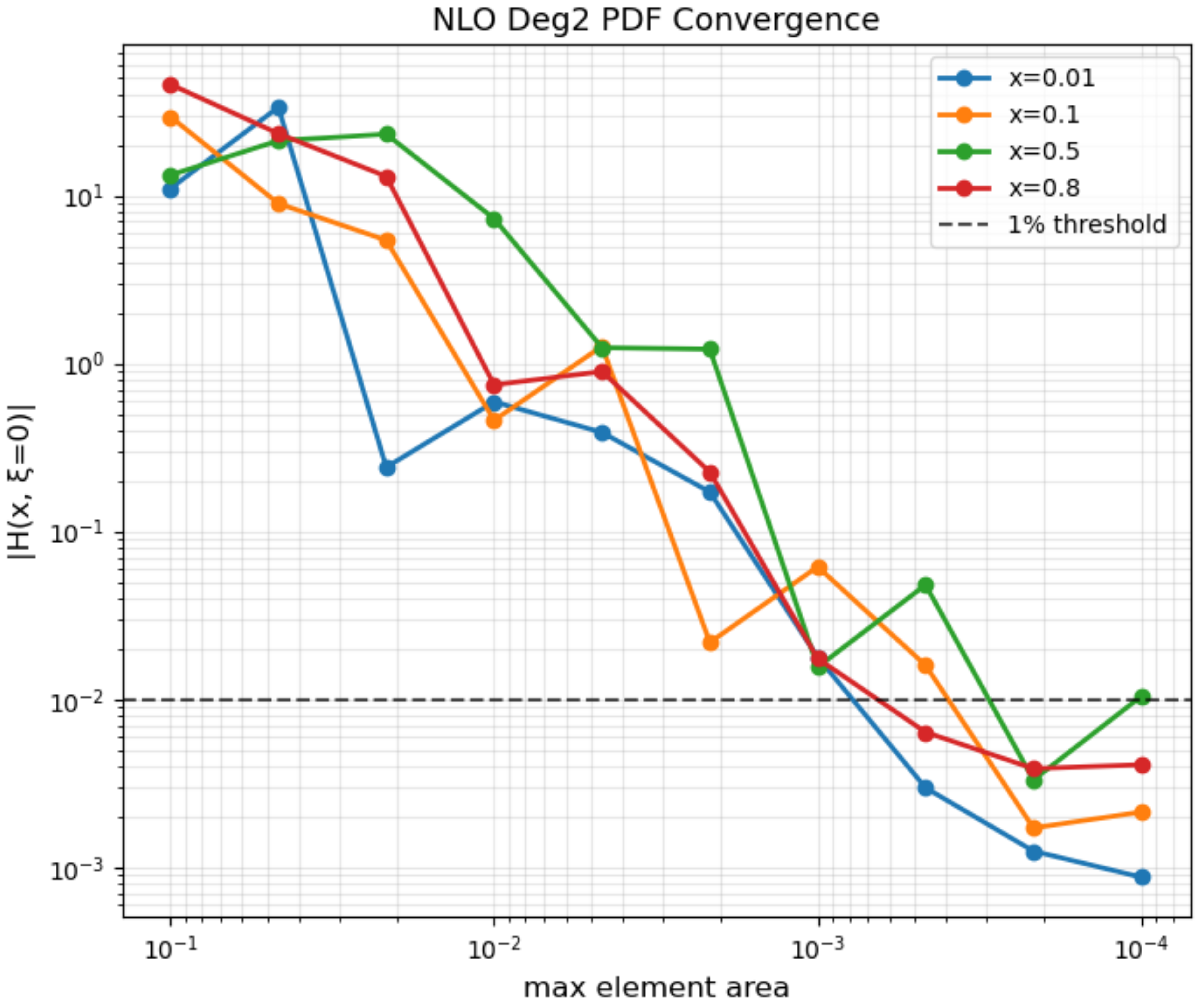


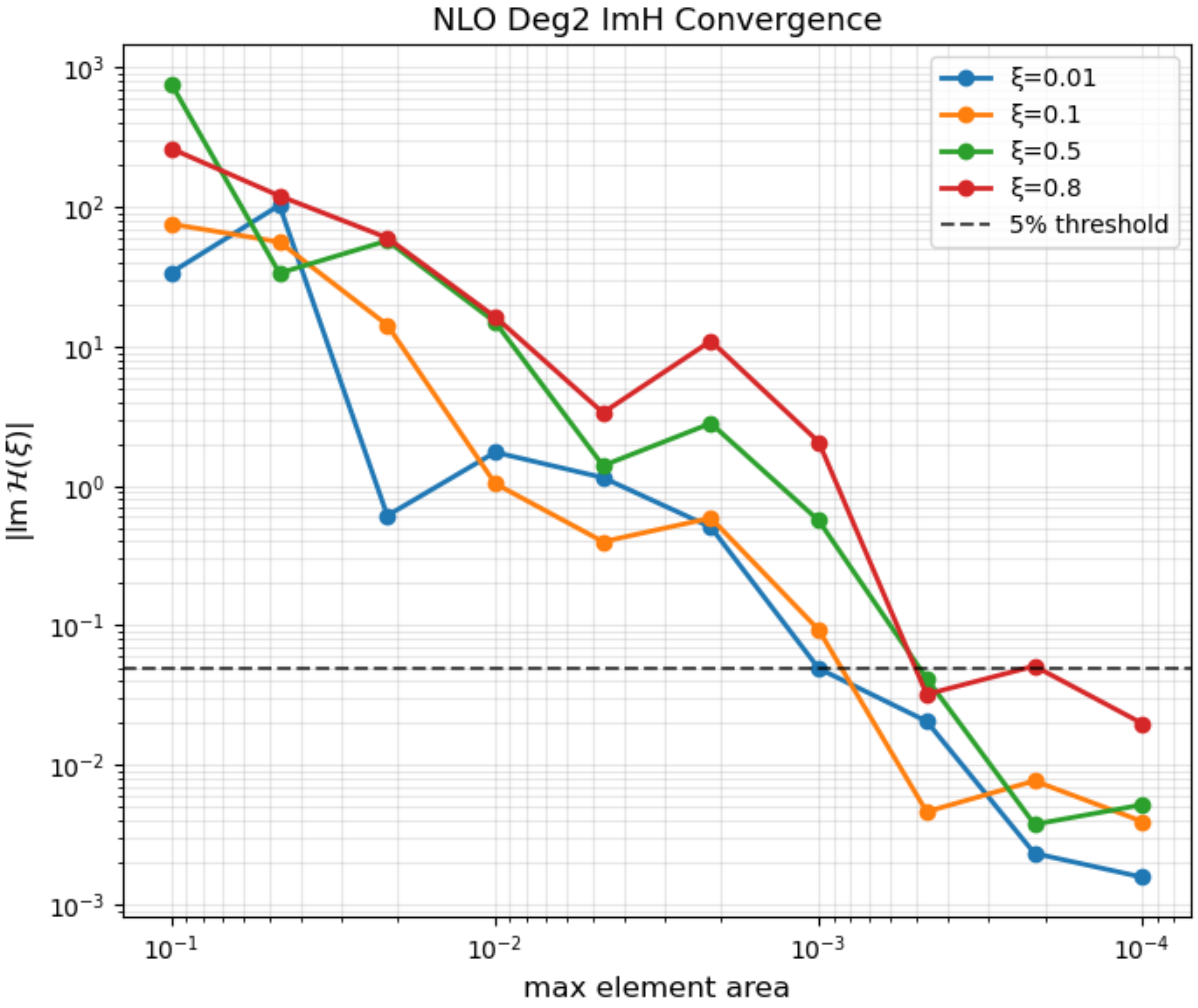
NLO shadow DD, degree 1 polynomial interpolation





NLO shadow DD, degree 2 polynomial interpolation

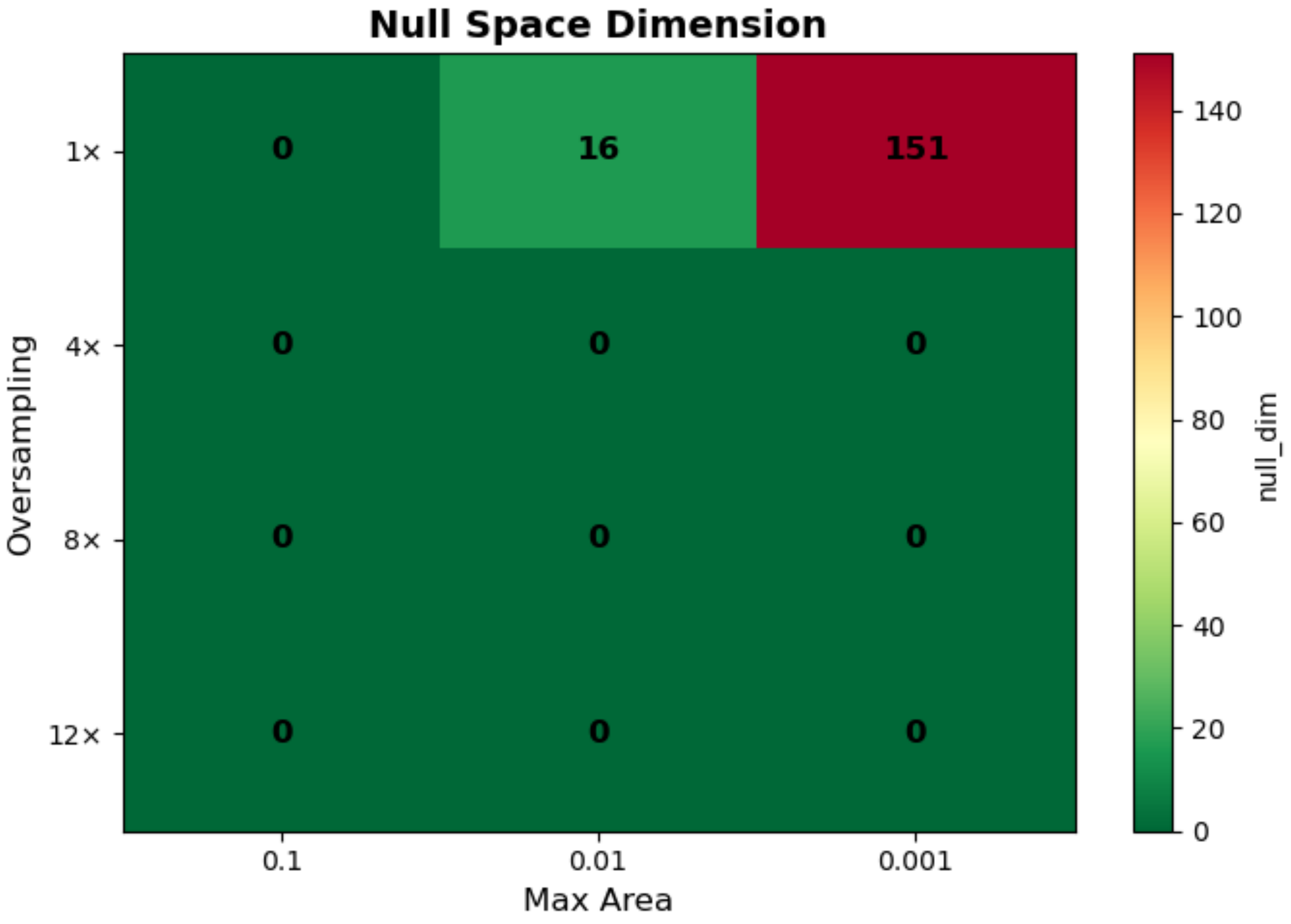


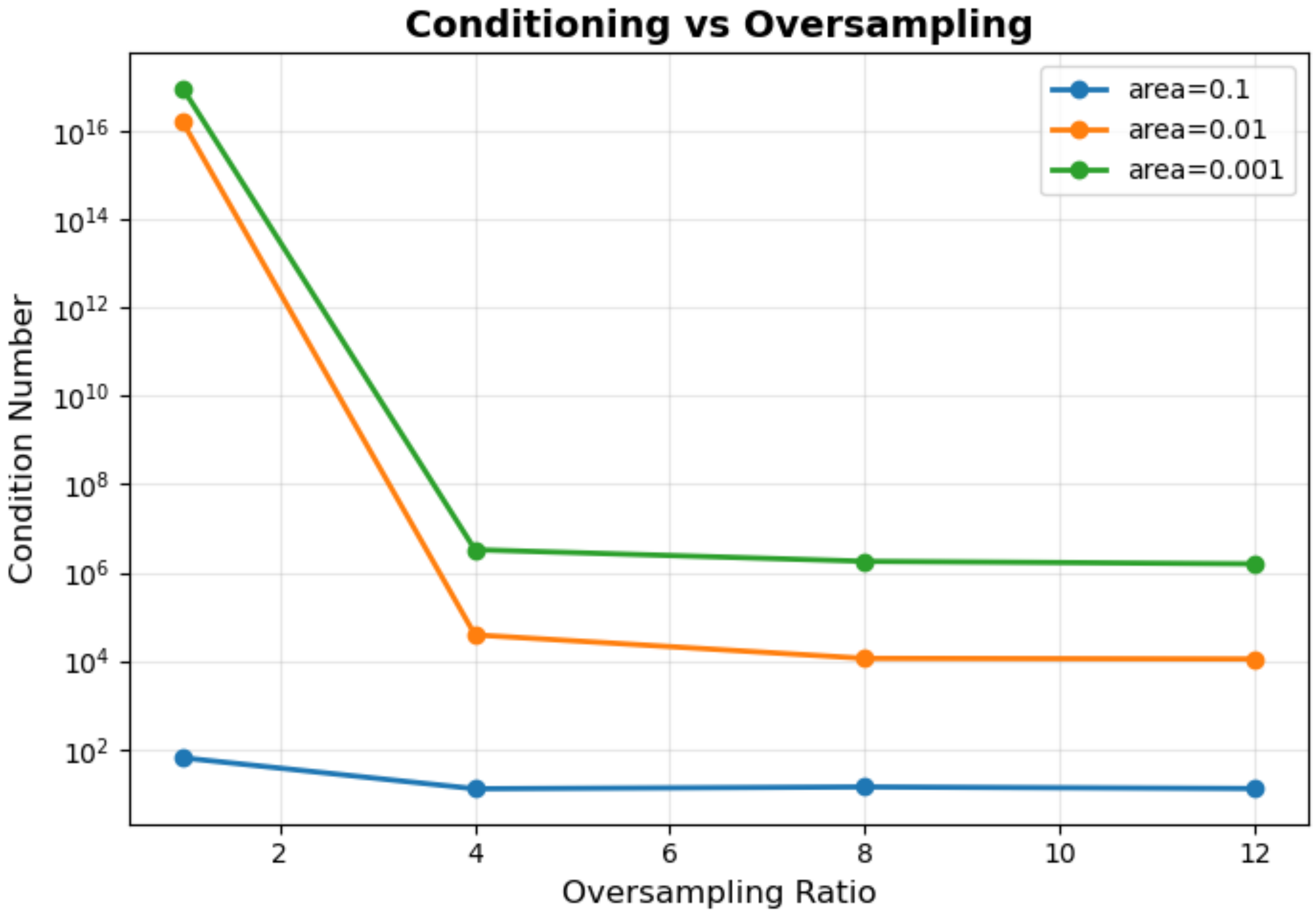


The FEM allows us to see shadow contributions and understand how they behave with the precision of the FEM, i.e. it quantifies the relationship between experimental precision in terms of x and ξ and the theoretical precision of accessing the GPD without shadow contaminations!

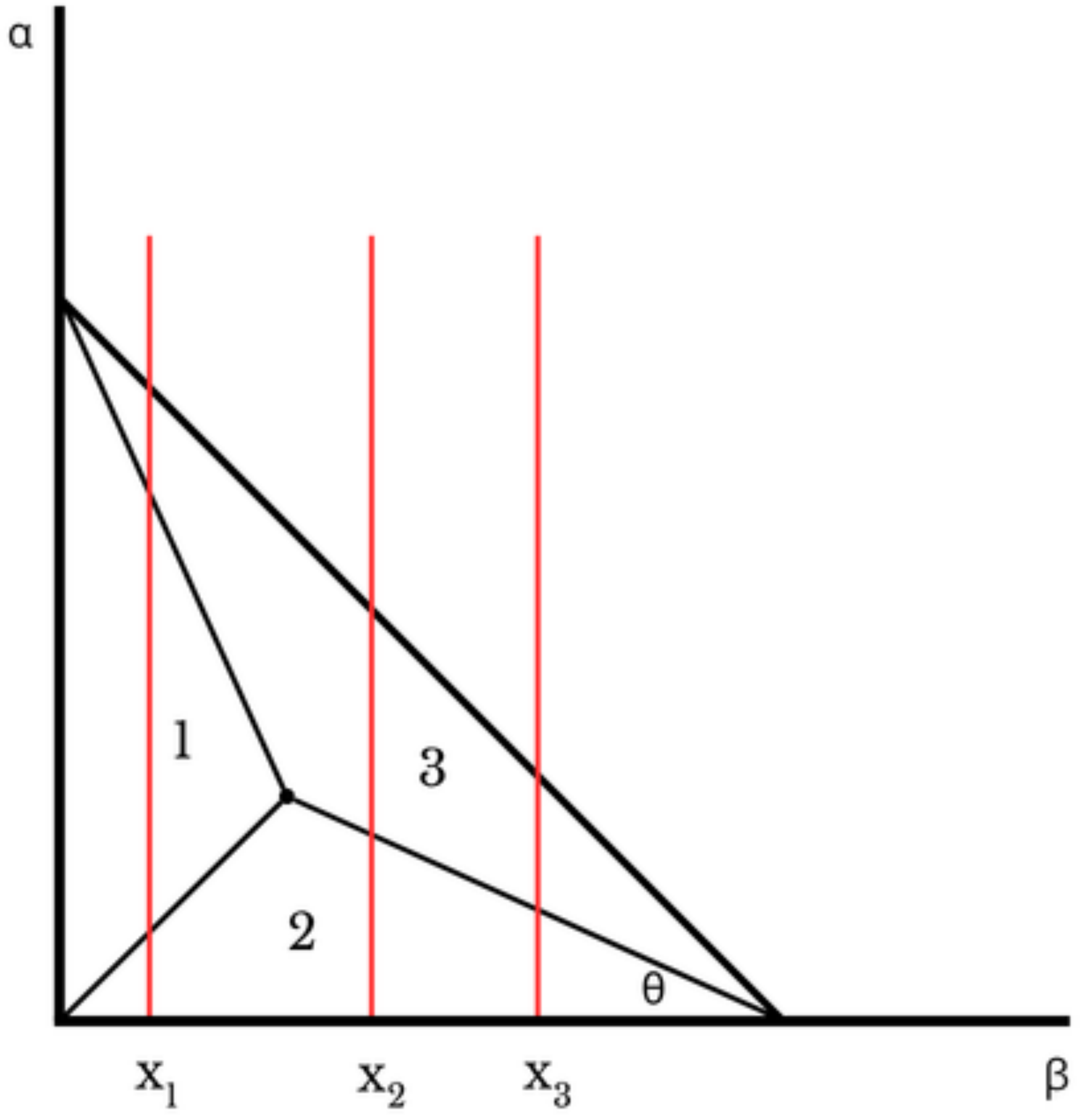
DO WE HAVE DISCRETE SHADOW DDS?

$$\left.\begin{aligned} H_S(x,0) = Rh = 0 \rightarrow h = 0, \forall x \\ H_S(\xi,\xi) = Rh = 0 \rightarrow h = 0, \forall \xi \end{aligned}\right\} \quad R \textbf{ has to be invertible}$$

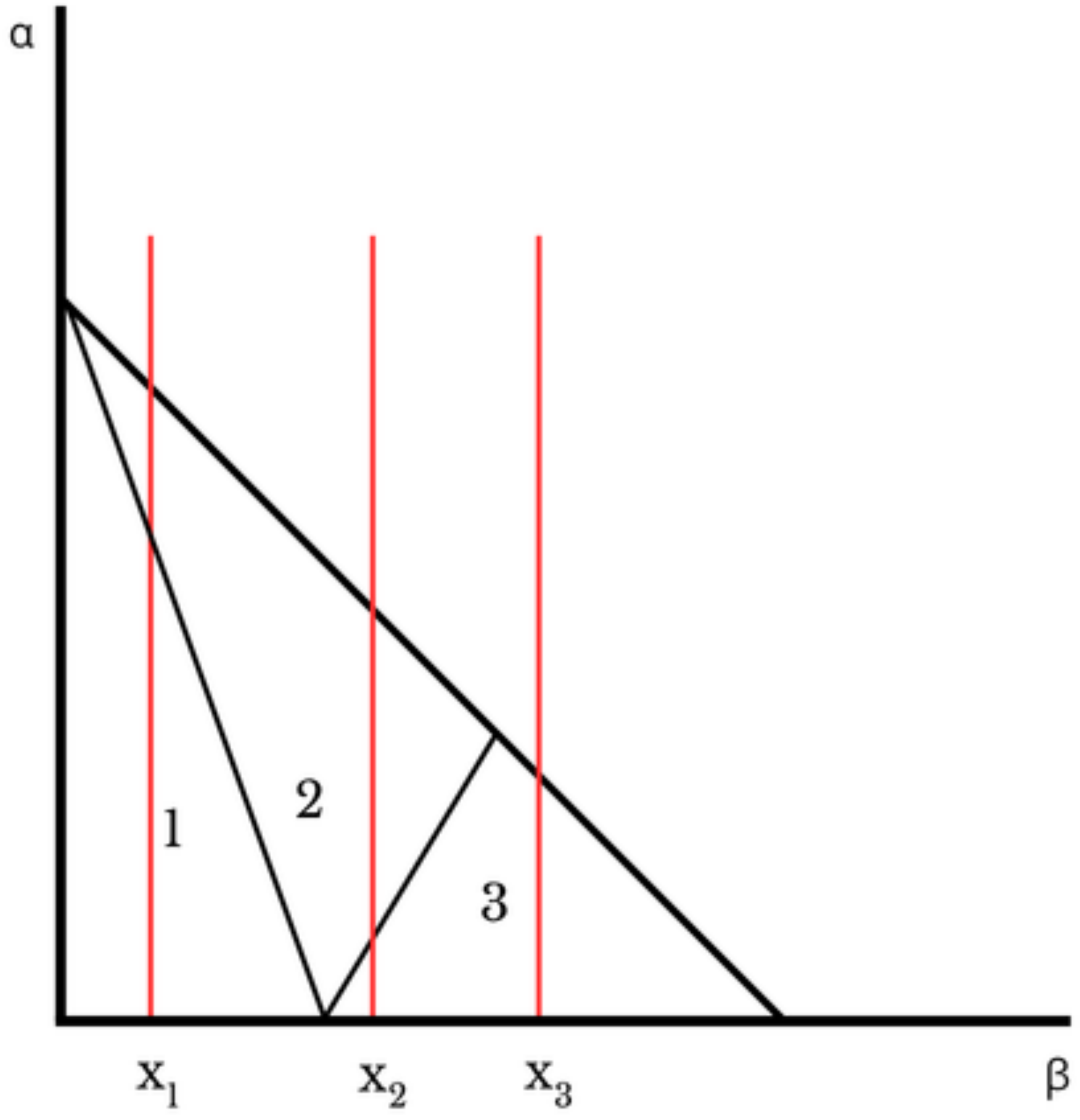




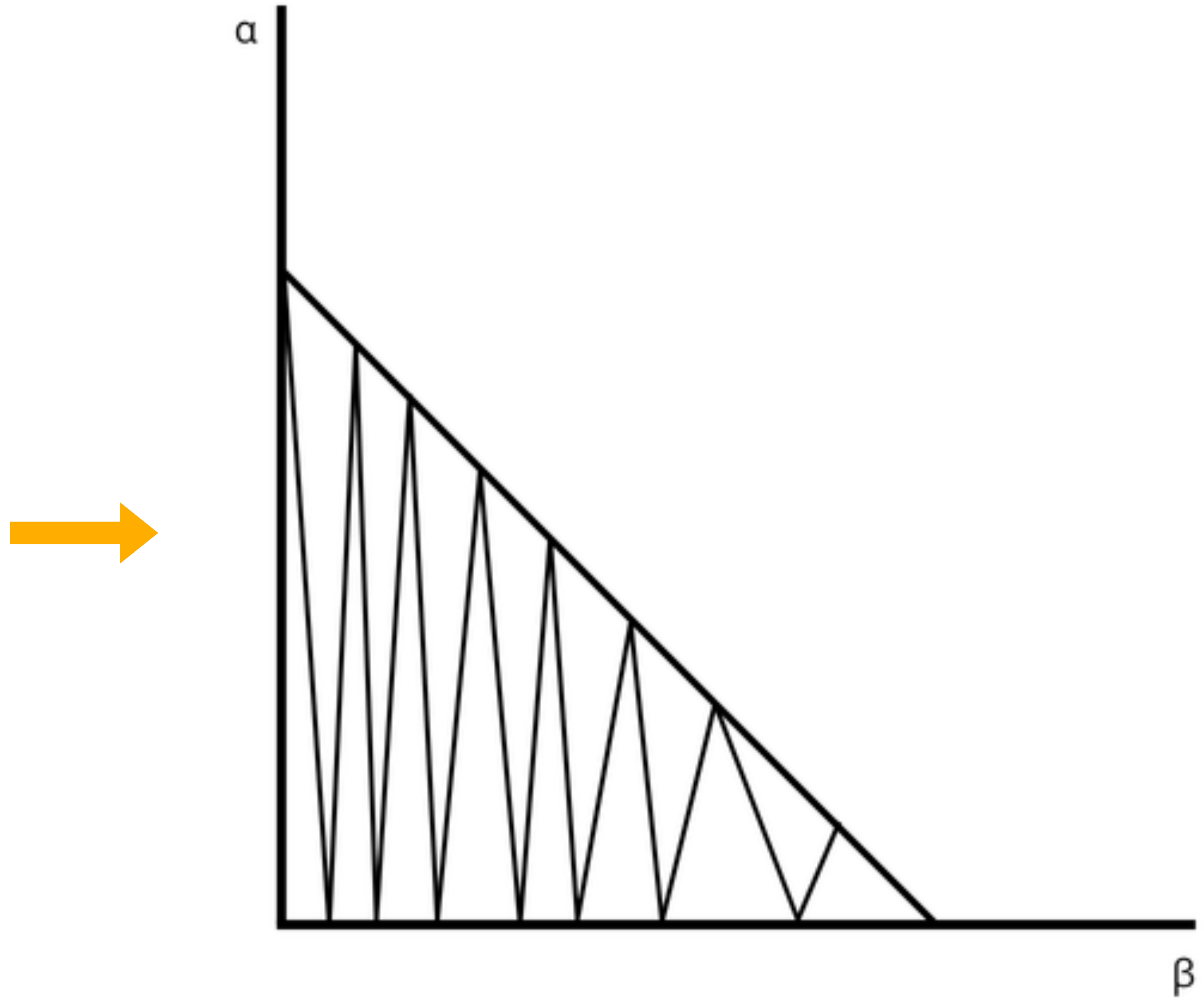
GEOMETRY OF PDF SAMPLING



$$\det R = \begin{vmatrix} l_1^1 & l_1^2 & l_1^3 \\ 0 & \tan\theta(1-x_2) & \frac{1}{\sqrt{2}}(1-x_2) \\ 0 & \tan\theta(1-x_3) & \frac{1}{\sqrt{2}}(1-x_3) \end{vmatrix} = 0$$



$$\det R = \begin{vmatrix} l_1^1 & l_1^2 & 0 \\ 0 & l_2^2 & l_2^3 \\ 0 & 0 & l_3^3 \end{vmatrix} \neq 0$$



Sliver triangles, ill-conditioned, erasure of α information

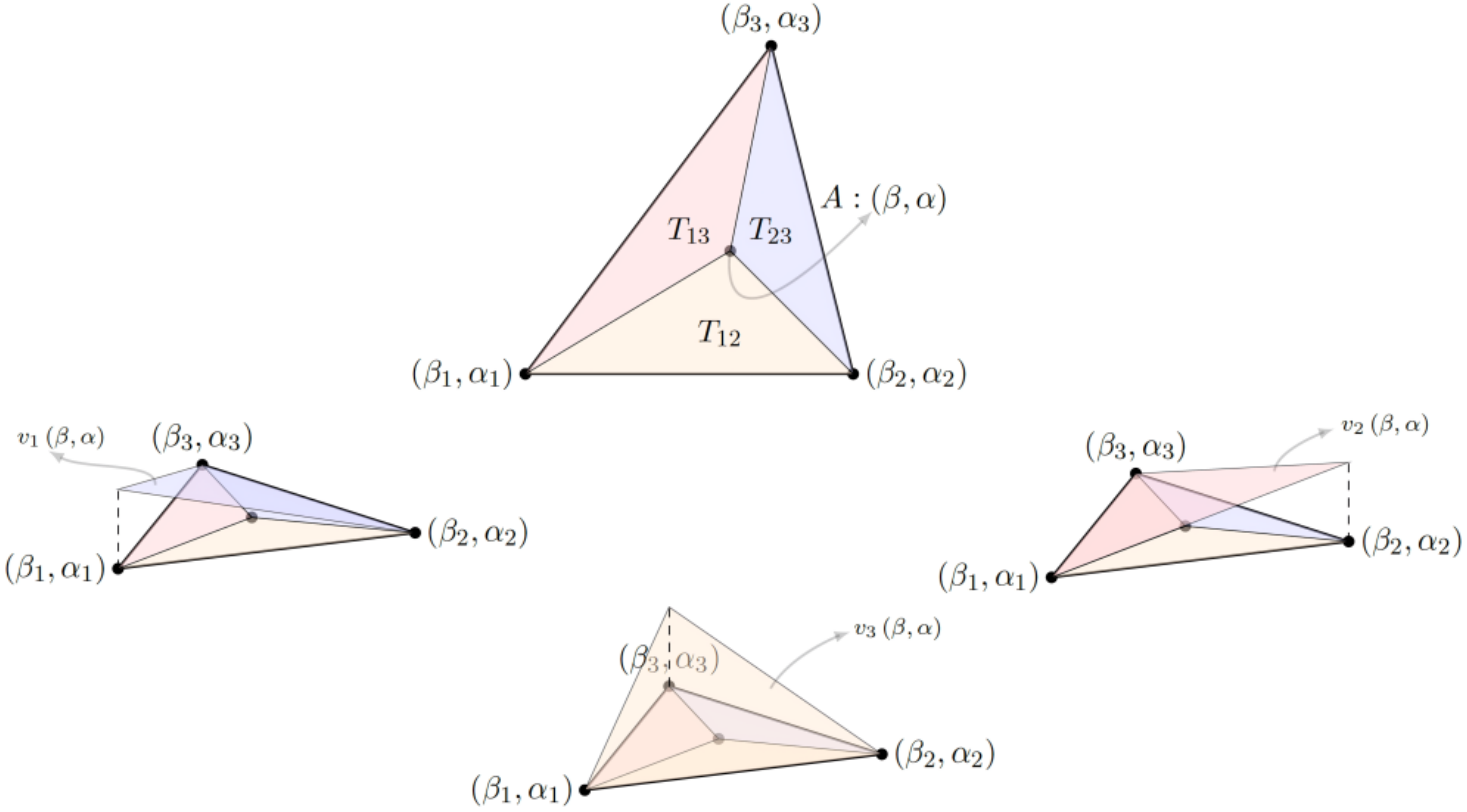
Proof of no discrete DDs highly depends on the geometry and ordering of elements and sampling lines, random geometries do not reach full rank!

CONCLUSIONS

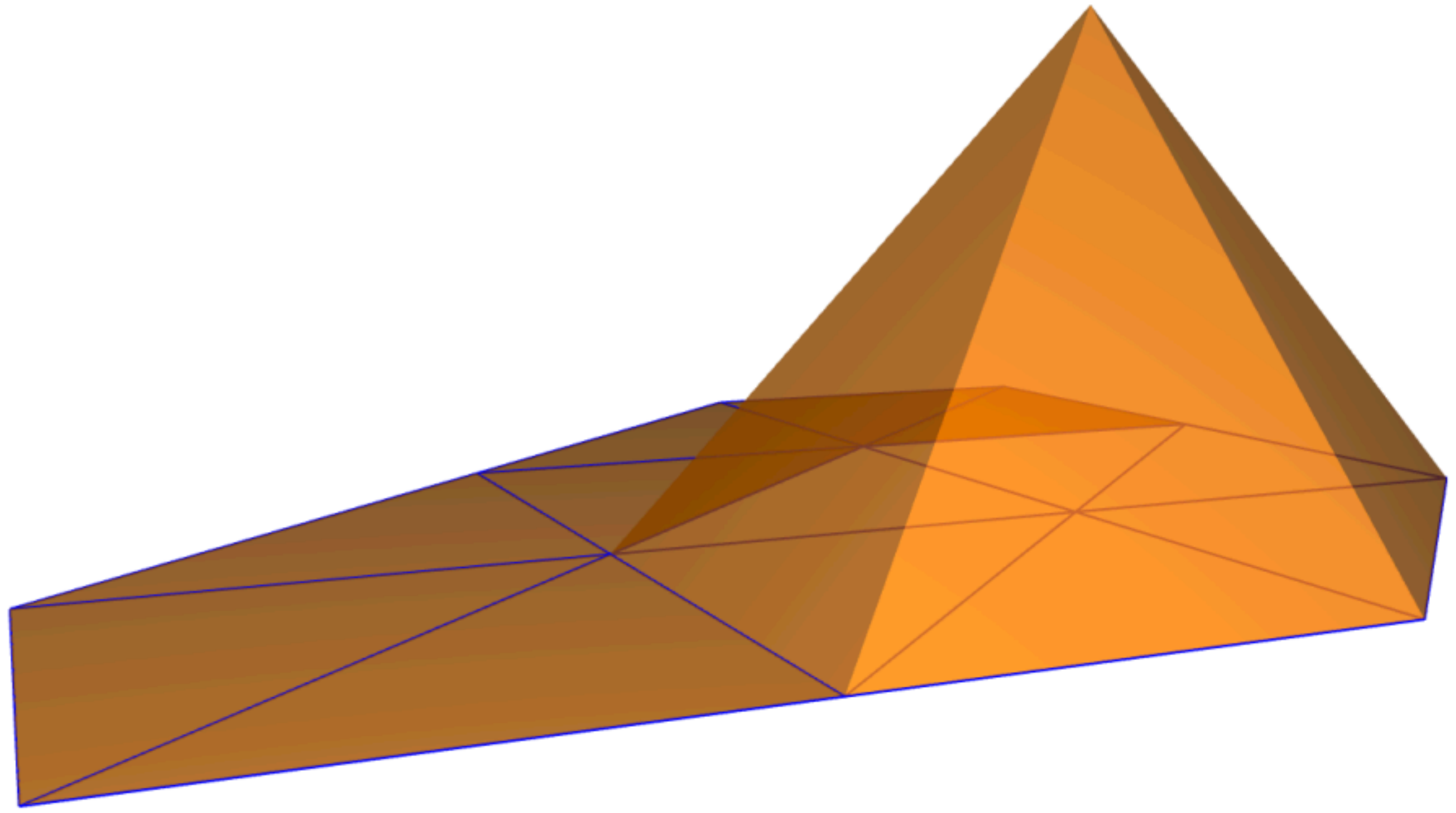
- Double distributions obtained from DGLAP GPDs contain the ERBL region
- FEM precision in terms of x and ξ connected to theoretical precision in terms of shadow contamination
- Degree-2 polynomial interpolation sufficient for reasonable precision for LO and NLO studies
- Exclusion of discrete DDs depends on the geometry of the mesh

SUPPLEMENTAL SLIDES

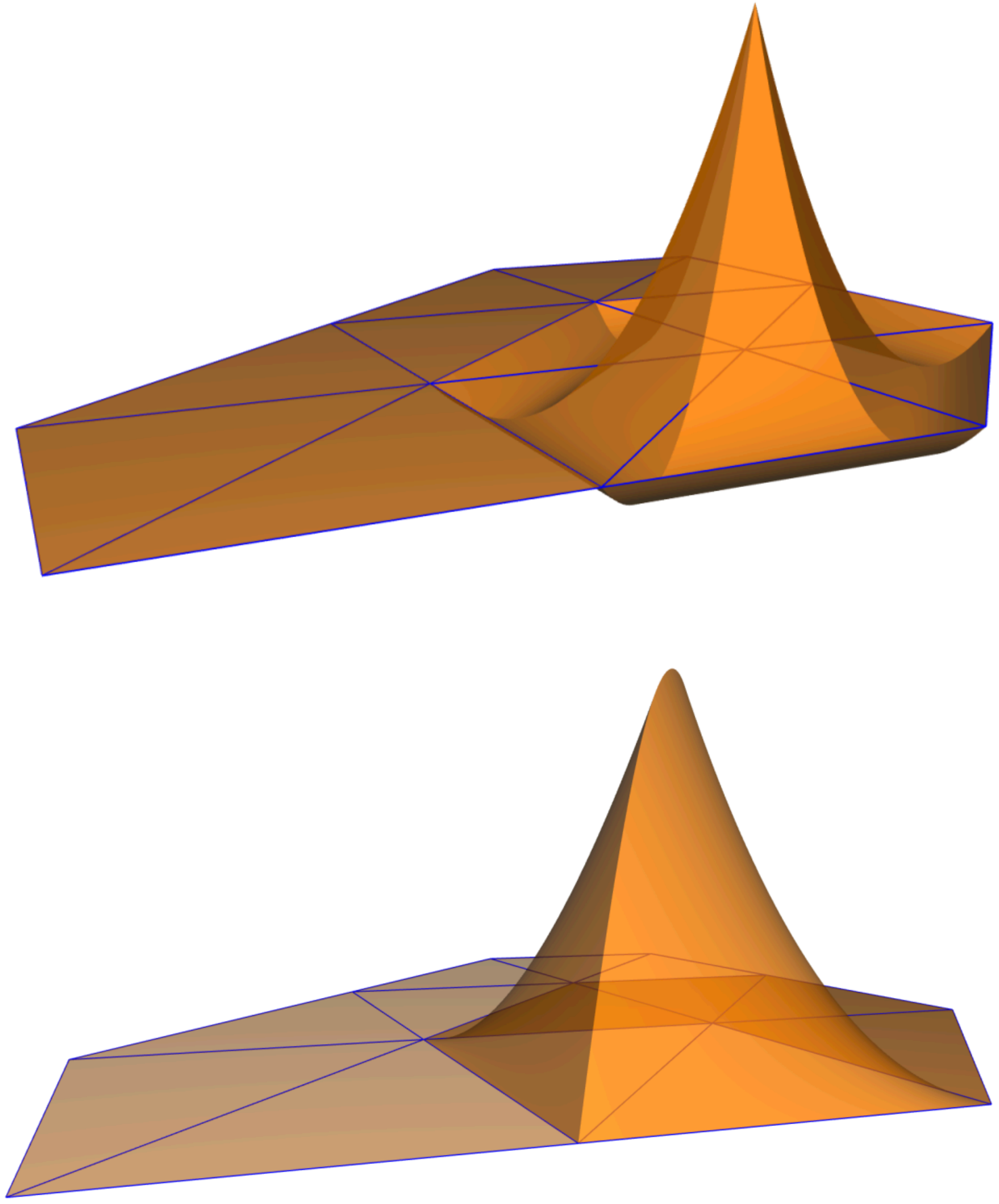
Linear polynomials on each element:



P1 Lagrange polynomials



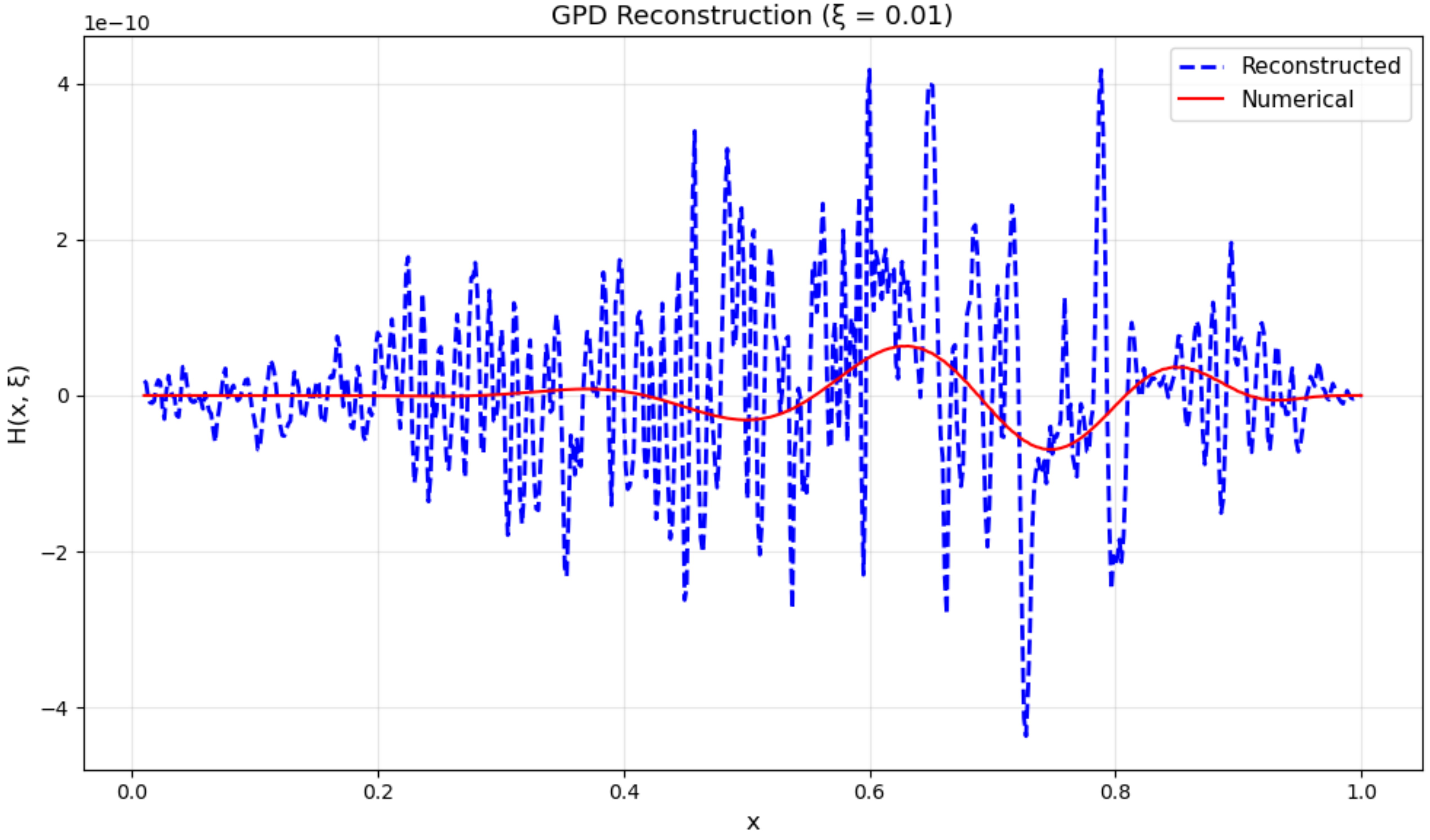
P2 Lagrange polynomials



DGALP PRECISION AT LOW ξ

NLO shadow GPD

Max area = 10^{-4}



Max area = 5×10^{-5}