High-Energy DIS Beyond the Eikonal Approximation

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QCD evolution 2026

Madrid, Spain

11–15 May, 2026

Motivation

- EIC physics requires a unified description of DIS across the full kinematic range: from the finite-x_B partonic regime to the small-x_B saturation/dipole regime.
- Two well-established but separate descriptions:
  - Finite x_B: non-local light-ray operators, collinear PDFs, helicity distributions, DGLAP.
  - Small x_B: Wilson lines, dipole picture, BK/B-JIMWLK evolution.
- Sub-eikonal corrections are the natural bridge: they encode the first energy-suppressed interactions beyond the eikonal dipole approximation and carry the quark/helicity operator content absent at leading power.
- Central question: How do the standard light-ray operators $q_f(x_B)$, $\Delta q_f(x_B)$ emerge explicitly from the Wilson-line framework at first sub-eikonal order?

Two complementary papers

Paper 1 G.A.C., arXiv:2603.23428 [hep-ph]

- Sub-eikonal DIS in shock-wave formalism $\Rightarrow$ standard quark and helicity operators at $x_B \neq 0$.
- *Non-commutativity* of small-x_B limit and phase-space integration.
- Independent derivation from high-energy limit of non-local OPE.
- High-energy evolution of Q_1^f, Q_5^S, $Q_5^{\rm NS}$ in dipole form; double-logarithmic analysis; Kirschner–Lipatov exponent with full C_F.

Paper 2 G.A.C., arXiv:2603.30000 [hep-ph]

- Mixed-space (momentum/coordinate) formulation of sub-eikonal DIS.
- Sub-eikonal Feynman rules from LSZ reduction in the shock-wave background.
- Full sub-eikonal corrections to F_L, F_T, and the helicity asymmetry related to g_1, organized in dipole form.
- Divergence structure: F_L finite; F_T and g_1 carry single-log divergences absorbed by the one-loop operator evolution.

References

In recent years, there has been a lot of activity in calculating and/or implementing sub-eikonal corrections:

Balitsky-Tarasov, (2014-2016) Armesto et.al (2014-2026); Altinoluk, Beuf et.al (2020-2026); Kovchegov et.al (2012-2025)

Notation

$n_1 \cdot n_2 = 1$, $n_1^2 = n_2^2 = 0$; $\not{n}_1 = \gamma^-$, $\not{n}_2 = \gamma^+$.

Finite gauge link along n_1:

$$[x^+, y^+]_x = \mathrm{P}\exp\left\{ig\int_{y^+}^{x^+} dw^+ A^-(w^+ n_1 + x_\perp)\right\}$$

Infinite Wilson line: $U_x \equiv [\infty n_1, -\infty n_1]_x$

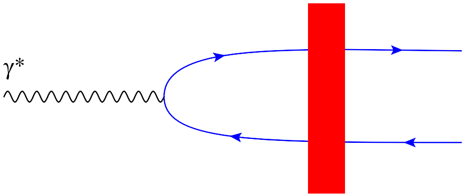
Dipole operator (vanishes as $x_\perp \to y_\perp$):

$$\mathcal{U}_{xy} \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{U_x U_y^\dagger\}$$

High-energy power counting under longitudinal boost $\lambda \to \infty$ ($P^- \to \infty$):

$$\bar\psi t^a \gamma^- \psi \sim \lambda, \quad \bar\psi t^a \gamma^\perp \psi \sim \lambda^0, \quad \bar\psi t^a \gamma^+ \psi \sim \lambda^{-1}$$

DIS at eikonal approximation



$\mathcal{U}_{xy} = 1 - \frac{1}{N_c}\mathrm{Tr}\{U_x U_y^\dagger\}$

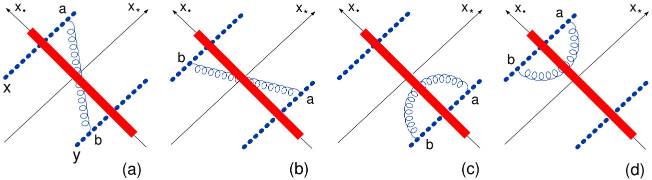
$$F_L = \frac{4Q^2 N_c \alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\, z^2\, \bar{z}^2 \int d^2x d^2y\, \left|K_0(\bar{Q}|x-y|)\right|^2 \mathcal{U}_{xy}$$

$$F_T = \frac{Q^2 N_c \alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\, z\, \bar{z}\, (z^2 + \bar{z}^2) \int d^2x d^2y\, \left|K_1(\bar{Q}|x-y|)\right|^2 \mathcal{U}_{xy}$$

DIS at eikonal approximation

BK evolution equation

$\mathcal{U}_{xy} = 1 - \frac{1}{N_c}\mathrm{Tr}\{U_x U_y^\dagger\}$



$U_x = \mathrm{Pex}\{ig\int A^-(x^+ + x_\perp)\}$ rapidity cut-off $\int \frac{dk^+}{k^+}$

$$\frac{d}{d\eta}\mathcal{U}_{xy} = \frac{\alpha_s N_c}{2\pi^2}\int d^2z \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (z-y)_\perp^2}\left[\mathcal{U}_{xz} + \mathcal{U}_{zy} - \mathcal{U}_{xy} - \mathcal{U}_{xz}\mathcal{U}_{zy}\right]$$

- NLO F_L and F_T: 2 contributions
  - α_s pure NLO contribution;
  - $\alpha_s \ln s$ contributions resummed by BK eq.;

Physics question

Small-x_B / high-energy DIS

$$\gamma^* \to q\bar{q}$$

$$U(x_\perp) = P\exp\left[ig\int dx^+ A^-(x^+, x_\perp)\right]$$

Wilson lines, dipoles, BK/JIMWLK evolution.

Finite-x_B partonic DIS

$$\bar{\psi}(0)\,\Gamma\,[0, z^+]\,\psi(z^+)$$

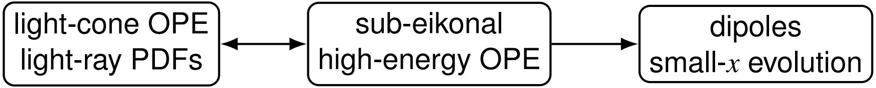
Light-ray operators, PDFs, helicity distributions, DGLAP.

Question

Where do the finite-x_B quark and helicity light-ray operators appear in the high-energy expansion?

The bridge is sub-eikonal

Answer: Sub-eikonal corrections



- At eikonal accuracy the Wilson line is spin blind
- Quark and helicity information first appears through sub-eikonal insertions
- The same operator sector controls finite-x_B light-ray distributions and the $x_B = 0$ high-energy evolution equation

Sub-eikonal diagrams

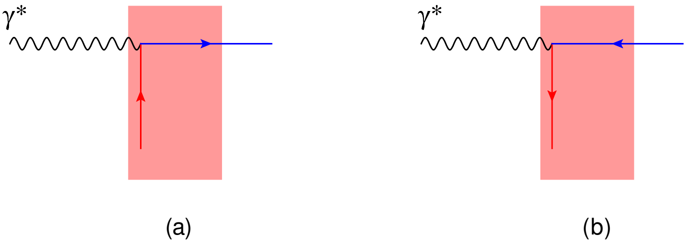
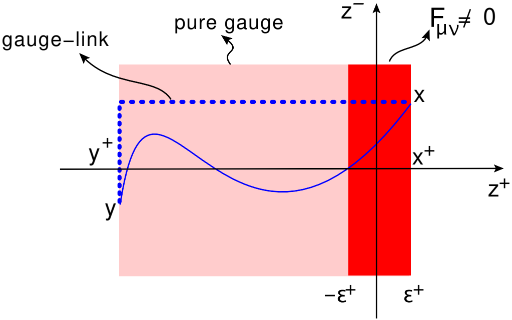


Figure: Transition amplitude $\gamma^*(q) \to q(k)$ (right) and $\gamma^*(q) \to \bar{q}(k)$ (left). Blue: quantum fields; red: classical background.

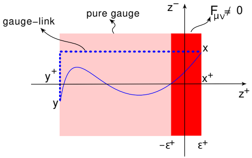
One endpoint of the quark propagator inside the shock-wave, one outside. This is the minimal configuration in which quark operator content enters beyond the strict eikonal dipole approximation.

Quark propagator with one point in the shock-wave



G.A.C. (2019), JHEP 01, 118

Quark propagator with one point in the shock-wave



$$\langle x|\frac{i}{\not{P}+i\epsilon}|y\rangle$$

$$= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i\not{k}}{k^+(k^2+i\epsilon)}e^{-ik\cdot(x-y)}$$

$$\times\Big[\theta(x^+-y^+)\Big([x^+,-\infty]_x\,\not{k}\gamma^+ - g\int_{-\infty}^{x^+}dz^+\gamma^i\gamma^+[x^+,z^+]_x F_i^{\ -}(z^+,x_\perp)[z^+,-\infty]_x\Big)$$

$$+\theta(y^+-x^+)\Big([x^+,+\infty]_x\,\not{k}\gamma^+ + g\int_{x^+}^{+\infty}dz^+\gamma^i\gamma^+[x^+,z^+]_x F_i^{\ -}(z^+,x_\perp)[z^+,+\infty]_x\Big)\Big]$$

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Differential cross-section: quark TMD

Transverse photon polarization

$$\frac{d^2\sigma^{\gamma_T^*\to q}}{d^2k} = \frac{\alpha_{\rm em}}{s}\, e_f^2\, q_{1,f}(x_B, k_\perp).$$

Identifying the quark TMD at $x_B \neq 0$:

$$q_{1,f}(x_B,k_\perp) \equiv \frac{1}{4\pi}\int d\Delta^+ d^2r\, e^{-i(k,r)_\perp} e^{i\left(x_B+\frac{k_\perp^2}{2q^+P^-}\right)P^-\Delta^+}$$
$$\times\langle N(P)|\bar{\psi}_f(0^+,0_\perp)[0,\infty n_1]_0\gamma^-[\infty n_1,\Delta^+]_r\psi_f(\Delta^+,r_\perp)|N(P)\rangle$$

G. A. C. (2026)

Note: the kinematic factor $\exp\left(i\frac{k_\perp^2}{2q^+}\Delta^+\right)$ drops if the high-energy limit is taken *before* identifying the operator.

Non-commutativity: small-x_B vs. phase-space integration

Naive: *before* integrating over $k_\perp$, set

$$e^{-i(q^- - k_\perp^2/2q^+)(x^+ - y^+)} \to 1$$

$$\frac{1}{2\pi\delta(0)} \int \frac{d^4k}{(2\pi)^3} \delta(k^2)\theta(k^+) \left[\left| \langle q|\gamma_T^* \rangle \right|^2 + \left| \langle \bar{q}|\gamma_T^* \rangle \right|^2 \right] \Big|_{\text{naive}}$$
$$= \sum_f e^2 e_f^2 \int d^2x\, dx^+ dy^+ \langle \bar{\psi}(y^+, x_\perp)\gamma^- [y^+, x^+]_x \psi(x^+, x_\perp) \rangle + O(\lambda^{-2})$$

$\Rightarrow$ Naive collinear pdf at $x_B = 0$ only:

$$q_f(x_B = 0) = \frac{1}{4\pi} \int dx^+ \langle N(P)|\bar{\psi}_f(0^+, 0_\perp)\gamma^- [0^+, x^+]\psi_f(x^+, 0_\perp)|N(P)\rangle$$

Altinoluk, Beuf, Mulani (2025);
at NLO: Altinoluk, Beuf, Favrel, Fucilla (2025)

Non-commutativity: small-x_B vs. phase-space integration

We can complete the phase-space integration without making any small-x_B approximation in the integrand

$$
\begin{aligned}
\mathcal{M}^{T,LO}_{\rm Quark} \equiv & \quad \frac{1}{2\pi\delta(0)}\int \bar{d}^4k\,\delta(k^2)\theta(k^+)\left(\left|\langle q(k)|\gamma^*_T(q)\rangle_{\rm Fig.1a}\right|^2 + \left|\langle q(k)|\gamma^*_T(q)\rangle_{\rm Fig.1b}\right|^2\right) \\
= & \quad \sum_f e^2 e_f^2 \int d^2x d^2y\, e^{i(q,\Delta)_\perp} \int dx^+ dy^+\, e^{ix_B P^-\Delta^+}\, \frac{iq^+}{2\pi\Delta^+}\, e^{-i\frac{\Delta_\perp^2 q^+}{2\Delta^+}} \\
& \times \bar\psi(y^+,y_\perp)[y^+,+\infty n_1]_y\,\gamma^-[+\infty n_1, x^+]_x\psi(x^+,x_\perp){\rm sign}(\Delta^+) + O(\lambda^{-2})
\end{aligned}
$$

Non-commutativity: small-x_B vs. phase-space integration

Correct: complete phase-space integration *first*, then take high-energy limit
After integrating over $k_\perp$ at fixed x_B, the transverse kernel becomes:

$$\frac{iq^+}{2\pi\Delta^+}\exp\left\{-i\frac{\Delta_\perp^2 q^+}{2\Delta^+}\right\}\xrightarrow{q^+\to\infty}\delta^{(2)}(\Delta_\perp)$$

Operator becomes local in $\perp$, nonlocal in x^+ $\Rightarrow$ light-ray quark distributions at $x_B \neq 0$:

$$\begin{aligned} q_f(x_B) &= \frac{1}{4\pi}\int_0^{+\infty} dx^+ e^{ix_B P^- x^+}\langle N(P)|\hat{\bar{\psi}}_f(0^+,0_\perp)\gamma^-[0^+,x^+]\hat{\psi}_f(x^+,0_\perp)|N(P)\rangle \\ \bar{q}_f(x_B) &= -\frac{1}{4\pi}\int_0^{+\infty} dx^+ e^{-ix_B P^- x^+}\langle N(P)|\hat{\bar{\psi}}_f(x^+,0_\perp)\gamma^-[x^+,0^+]\hat{\psi}_f(0^+,0_\perp)|N(P)\rangle \end{aligned}$$

$\Rightarrow$ The small-x_B limit and the phase-space integration do **not** commute.

G.A.C. (2026)

Helicity distributions

The asymmetry $(\varepsilon^i_+\varepsilon^{*j}_+ - \varepsilon^i_-\varepsilon^{*j}_-)$ gives access to the helicity distribution.
Repeating the analysis with the correct phase-space ordering:

$$\sigma_A^{\gamma^* P} = \frac{\pi}{q^+P^-}\alpha_{\rm em}\sum_f e_f^2\Big(\Delta q_f(x_B) + \Delta\bar{q}_f(x_B)\Big) + O(\lambda^{-2})$$

with helicity distributions at $x_B \neq 0$:

$$\begin{aligned}\Delta q_f(x_B) &= \frac{1}{4\pi}\int_0^{+\infty}dx^+ e^{ix_BP^-x^+}\langle N(P)|\hat{\bar{\psi}}_f(0^+,0_\perp)\gamma^5\gamma^-[0^+,x^+]\hat{\psi}_f(x^+,0_\perp)|N(P)\rangle \\ \Delta\bar{q}_f(x_B) &= -\frac{1}{4\pi}\int_0^{+\infty}dx^+ e^{-ix_BP^-x^+}\langle N(P)|\hat{\bar{\psi}}_f(x^+,0_\perp)\gamma^5\gamma^-[x^+,0^+]\hat{\psi}_f(0^+,0_\perp)|N(P)\rangle\end{aligned}$$

The first sub-eikonal correction already reconstructs, in the inclusive limit, the standard nonlocal quark and helicity distributions at $x_B \neq 0$.

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The two operations do not commute

$$\boxed{\lim_{\text{HE}} \int d\Phi \neq \int d\Phi \lim_{\text{HE}}}$$

Approximate too early

Set the phase to one before the transverse phase-space integration:

$$e^{-i\left(q^- - k_\perp^2/(2q^+)\right)\Delta^+} \to 1.$$

Then one obtains only the naive collinear object at

$$x_B = 0.$$

Correct inclusive limit

Complete the phase-space integration first and keep

$$e^{-iq^-\Delta^+} = e^{ix_B P^- \Delta^+}.$$

Then the finite-x_B light-ray distributions are reconstructed.

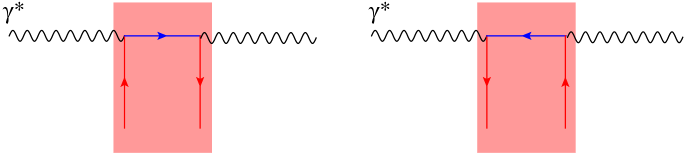

Figure: Diagrams contributing to $T_{\mu\nu}$

Starting point: the non-local OPE propagator in the background gluon field

$$\langle \mathrm{T}\psi_q(x)\bar{\psi}_q(y)\rangle_A = \frac{i(\not x - \not y)}{2\pi^2[(x-y)^2 - i\epsilon]^2}[x,y]_c + O((x-y)^{-2})$$

with *straight* gauge link $[x,y]_c = \mathrm{P}\exp\Big\{ig\int_y^x dz^\mu\, A^c_\mu(z)\Big\}$.

Under a large longitudinal boost, the straight link reduces to a light-cone Wilson line at fixed $x_\perp$.

From non-local OPE to High-energy OPE

$$T_{\mu\nu} = i\int d^4x d^4y\, e^{iq\cdot(x-y)}\frac{i}{2\pi^2[(x-y)^2 - i\epsilon]^2}\sum_f e_f^2$$
$$\times\Big(\big\langle \mathrm{T}\{\bar{\psi}_c(x)\gamma_\mu(\not{x}-\not{y})[x,y]\gamma_\nu\psi_c(y)\}\big\rangle_A - \langle \mathrm{T}\{\bar{\psi}_c(y)\gamma_\nu(\not{x}-\not{y})[y,x]\gamma_\mu\psi_c(x)\}\big\rangle_A\Big)$$

Take high-energy limit and use the boost reduction $[x,y] \to [x^+,y^+]_x$

$$T_{\mu\nu} = i\int d^4x d^4y\, e^{iq\cdot(x-y)}\frac{i}{2\pi^2[(x-y)^2 - i\epsilon]^2}\sum_f e_f^2$$
$$\times\Big(\big\langle \mathrm{T}\{\bar{\psi}_c(x^+,x_\perp)\gamma_\mu(\not{x}-\not{y})[x^+,y^+]_x\gamma_\nu\psi_c(y^+,y_\perp)\}\big\rangle_A - \big\langle \mathrm{T}\{\bar{\psi}_c(y^+,y_\perp)\gamma_\nu(\not{x}-\not{y})[y^+,x^+]_y\gamma_\mu\psi_c(x^+,x_\perp)\}\big\rangle_A\Big)$$

From non-local OPE to high-energy OPE

Consider transverse polarization; and calculate the residue

take $q^- = -\frac{Q^2}{2q^+} = -x_B P^-$, then integrate over Δ^-:

$$\frac{1}{2}\sum_{\lambda=\pm 1} \varepsilon^i_\lambda \varepsilon^{*j}_\lambda T_{ij} = \lim_{\epsilon^+\to 0} \frac{i}{2}\sum_f e_f^2 \int dx^+ dy^+ \theta(x^+ - y^+) d^2x d^2y\, e^{ix_B P^- \Delta^+} \frac{(-i)q^+}{2\pi\Delta^+} e^{i\frac{q^+\Delta_\perp^2}{2\Delta^+}}$$
$$\times \Big[\langle N(P)|\hat{\bar{\psi}}(x^+, x_\perp)\gamma^-[x^+, y^+]_x \hat{\psi}(y^+, y_\perp)|N(P)\rangle + \cdots\Big] + O(\lambda^{-2})$$

High-energy limit: $\frac{-iq^+}{2\pi\Delta^+} \exp\left\{i\frac{q^+\Delta_\perp^2}{2\Delta^+}\right\} \xrightarrow{q^+\to\infty} \delta^{(2)}(\Delta_\perp)$

From non-local OPE to high-energy OPE

$$\frac{1}{2}\sum_{\lambda=\pm 1} \varepsilon^i_\lambda(q)\varepsilon^j_\lambda(q)\mathrm{Im}\, T_{ij} = \frac{\pi}{q^+P^-}\sum_f e_f^2\big(q_f(x_B) + \bar{q}_f(x_B)\big) + O(\lambda^{-2})$$

The optical theorem then gives:

$$\frac{1}{2}\sum_{\lambda=\pm 1} \varepsilon^i_\lambda(q)\varepsilon^j_\lambda(q)W_{ij} = \frac{1}{2\pi}\frac{1}{4\pi\alpha_{\rm em}}\Big[4\pi\alpha_{\rm em}\sum_f e_f^2\big(q_f(x_B) + \bar{q}_f(x_B)\big)\Big]$$

Both methods—shock-wave and non-local OPE—yield the same result. This establishes an explicit *operator-level bridge* between the two formalisms.

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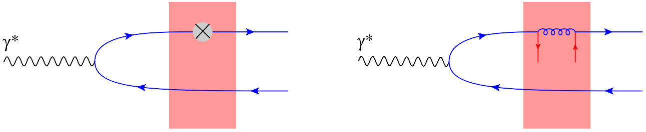
Mixed-space formulation and sub-eikonal operator basis



a) b)

The sub-eikonal quark propagator generates operator insertions localized at $z^+ = 0$:

Gluonic operators on the shock-wave:

- Chromomagnetic: $\mathcal{F}(z_\perp) = ig\frac{s}{4}\int dz^+[\infty n_1, z^+]_z\,\epsilon^{ij}F_{ij}(z^+, z_\perp)[z^+, -\infty n_1]_z$ Parity odd
- Chromoelectric composite: $\mathcal{G}_2(z_\perp) \equiv \mathcal{F}_2(z_\perp) - \mathcal{F}_2'(z_\perp)$ (built from $F_i^{\ -}$) Parity even

Quark operators on the shock-wave:

$$Q_{ij}(x_\perp) = g^2\frac{s}{2}\int dz^+ \int^{z^+} dz'^+\,[\infty n_1, z^+]_x t^a \psi(z^+, x_\perp)[z^+, z'^+]^{ab}\bar{\psi}(z'^+, x_\perp)t^b[z'^+, -\infty n_1]_{x\,ij}$$

All operators vanish as the dipole size $\to 0$ when placed in dipole combinations.

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Sub-eikonal corrections: F_L and F_T

Define dipole-type combinations (all vanish as $x_\perp \to y_\perp$):

Parity even $\qquad \mathcal{Q}^f_{1,xy} \equiv Q^f_{1,x}\mathcal{U}_{xy}, \quad \Psi^f_{1,xy} \equiv \mathrm{Tr}\{\tilde{Q}^f_{1,x}(U^\dagger_x - U^\dagger_y)\}, \quad \mathcal{G}_2(x,y) \equiv \mathrm{Tr}\{(U^\dagger_x - U^\dagger_y)\mathcal{G}_{2,x}\}$

F_L at sub-eikonal order - finite, no UV divergence

$$F_L = \frac{4Q^2 N_c \alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\, z^2\bar{z}^2 \int d^2x d^2y\, \left|K_0(\bar{Q}|x-y|)\right|^2 \left[\mathcal{U}_{xy} + \frac{\sqrt{s/2}}{4\,z\,\bar{z}\,s^2 N_c}\left(N_c\mathcal{Q}^f_{1\,xy} - \frac{1}{N_c}\Psi^f_{1\,xy} + 2\mathcal{G}_{2\,xy} + \mathrm{h.c.}\right)\right]$$

F_T at sub-eikonal order - single-log UV divergence

$$F_T = \frac{Q^2 N_c \alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\, z\bar{z}(z^2+\bar{z}^2) \int d^2x d^2y\, \left|K_1(\bar{Q}|x-y|)\right|^2 \left[\mathcal{U}_{xy} + \frac{\sqrt{s/2}}{4\,z\,\bar{z}\,s^2 N_c}\left(N_c\mathcal{Q}^f_{1xy} - \frac{1}{N_c}\Psi^f_{1\,xy} + 2\mathcal{G}_{2\,xy} + \mathrm{h.c.}\right)\right]$$

Eikonal $\to$ known

Sub-eikonal $\to$ new $\qquad$ G.A.C. (2026)

Operators parity even

$s \simeq 2q^+P^-$

$$\tilde{Q}^f_{1ij}(x_\perp) \equiv g^2\frac{s}{2}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\Big([\infty n_1, z^+]_x \mathrm{tr}\{\psi_f(z^+,x_\perp)\bar{\psi}_f(z'^+,x_\perp)\,\gamma^-\}[z'^+,-\infty n_1]_x\Big)_{ij}$$

$$\mathcal{G}_2(z_\perp) \equiv ig^2 s\int_{-\infty}^{+\infty}d\omega^+\int_{\omega^+}^{+\infty}d\omega'^+(\omega^+ - \omega'^+)[\infty n_1,\omega'^+]_z F^{i-}(\omega'^+,z_\perp)[\omega'^+,\omega^+]_z F_i{}^-(\omega^+,z_\perp)[\omega^+,-\infty n_1]_z$$

Sub-eikonal correction to g_1: Singlet case

Define the helicity dipole-type operators:

Parity odd $\qquad \mathcal{Q}^f_{5\,xy} \equiv \mathcal{Q}^f_{5\,x}\mathcal{U}_{xy}, \quad \Psi^f_{5,xy} \equiv \mathrm{Tr}\{\tilde{\mathcal{Q}}^f_{5\,x}(U^\dagger_x - U^\dagger_y)\}, \quad \mathcal{F}_{xy} \equiv \mathrm{Tr}\{U^\dagger_x \mathcal{F}_y\}$

g_1 at sub-eikonal order - single-log UV divergence

$$g_1 = \frac{Q^2\alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\,(z-\bar{z})^2 \int d^2x d^2y |K_1(\bar{Q}|x-y|)|^2 \frac{\sqrt{s/2}}{s^2}\left[2\mathcal{F}_{xy} + N_c \mathcal{Q}^f_{5\,xy} - \frac{1}{N_c}\Psi^f_{5\,xy} - \mathrm{h.c.}\right]$$

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- The gluon operator $\mathcal{F}(x,y)$ enters in the flavor-singlet sector only.
- The single-log divergences in F_T and g_1 are precisely those generated by the one-loop rapidity evolution of $\mathcal{Q}^f_1$ and $\mathcal{Q}^f_5$ respectively.

Sub-eikonal corrections:

- Quark propagator in the background of quark and gluon fields at sub-eikonal level
- Gluon propagator in the background of quark and gluon fields at sub-eikonal level

G.A.C. (2019), JHEP 01, 118

Operators parity odd

$s \simeq 2q^+P^-$

$$\tilde{Q}^f_{5ij}(x_\perp) \equiv g^2\frac{s}{2}\int_{-\infty}^{\infty}dz^+\int_{-\infty}^{z^+}dz'^+\Big([\infty n_1, z^+]_x\mathrm{tr}\{\psi_f(z^+,x_\perp)\bar{\psi}_f(z'^+,x_\perp)\gamma^5\gamma^-\}[z'^+,-\infty n_1]_x\Big)_{ij}$$

$$\mathcal{F}(z_\perp) = ig\frac{s}{4}\int dz^+[\infty n_1, z^+]_z\,\epsilon^{ij}F_{ij}(z^+,z_\perp)[z^+,-\infty n_1]_z$$

Sub-eikonal correction to g_1: Non-singlet case

Define the helicity dipole-type operators:

$$Q_5^{\rm NS}(x,y) \equiv \sum_f c_f\, Q_5^f(x,y), \qquad \Psi_5^{\rm NS}(x,y) \equiv \sum_f c_f\, \Psi_5^f(x,y)$$

with coefficients c_f satisfying $\sum_f c_f = 0$

g_1 at sub-eikonal order - single-log UV divergence

$$g_1^{\rm NS}(Q^2) = \frac{Q^2\,\alpha_{\rm em}}{\pi^3}\int_0^1 dz\,(z-\bar z)^2 \int d^2x\, d^2y\, \left|K_1(\bar Q|x-y|)\right|^2 \frac{\sqrt{s/2}}{s^2}\left(N_c\, Q^{\rm NS}_{5\,xy} - \frac{1}{N_c}\,\Psi^{\rm NS}_{5\,xy} - \text{h.c.}\right)$$

- In the non-singlet projection $g_1^{\rm NS}$, only $Q_5^{\rm NS}$ and $\Psi_5^{\rm NS}$ remain.

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Diagrams for the evolution of Q_1 and Q_5 operators

$$Q_{1,f}(x_\perp, x_B = 0) \equiv g^2 \frac{s}{2} \int_{-\infty}^{+\infty} dx^+ \int_{-\infty}^{x^+} dy^+ \, \bar{\psi}_f(y^+, x_\perp)[y^+, x^+]_x \, \gamma^- \psi_f(x^+, x_\perp) \, ,$$

$$Q_{5,f}(x_\perp, x_B = 0) \equiv g^2 \frac{s}{2} \int_{-\infty}^{+\infty} dx^+ \int_{-\infty}^{x^+} dy^+ \, \bar{\psi}_f(y^+, x_\perp)[y^+, x^+]_x \, \gamma^5 \gamma^- \psi_f(x^+, x_\perp) \, ,$$

High-energy evolution in dipole form

The $x_B = 0$ operators $\mathcal{Q}^f_1(x_\perp)$, $\mathcal{Q}^S_5(x_\perp)$, $\mathcal{Q}^{\rm NS}_5(x_\perp)$ obey:

$$\begin{aligned}
\frac{d}{d\eta}\mathcal{Q}^f_{1x} &= \frac{\alpha_s}{4\pi^2}\int\frac{d^2z}{(x-z)^2_\perp}\Big[2C_F\mathcal{Q}^f_{1z} - N_c\mathcal{Q}^f_{1,zx} + \frac{1}{N_c}\Psi^f_{1,zx}\Big] \\
\frac{d}{d\eta}\mathcal{Q}^S_{5x} &= \frac{\alpha_s}{4\pi^2}\int\frac{d^2z}{(x-z)^2_\perp}\Big[2C_F\mathcal{Q}^S_{5z} - N_c\mathcal{Q}^S_{5,zx} + \frac{1}{N_c}\Psi^S_{5,zx} + 2N_f\mathcal{F}_{xz}\Big] \\
\frac{d}{d\eta}\mathcal{Q}^{\rm NS}_{5x} &= \frac{\alpha_s}{4\pi^2}\int\frac{d^2z}{(x-z)^2_\perp}\Big[2C_F\mathcal{Q}^{\rm NS}_{5z} - N_c\mathcal{Q}^{\rm NS}_{5,zx} + \frac{1}{N_c}\Psi^{\rm NS}_{5,zx}\Big]
\end{aligned}$$

Key features:

- All bilocal combinations vanish as $x \to y \Rightarrow$ small-dipole behavior manifest.
- Singlet $\mathcal{Q}^S_5$ mixes with the gluon operator $\mathcal{F}$; non-singlet $\mathcal{Q}^{\rm NS}_5$ does not.
- The log divergences of the sub-eikonal dipole cross sections are absorbed by the $2C_F$ terms of these kernels.

G.A.C. (2021), JHEP 06, 096; G.A.C. (2026)

Double-logarithmic approximation

Project onto the local ladder sector ($z_\perp \to x_\perp$; dipole combinations vanish):

$$\frac{d}{d\eta} Q_1^f(x_\perp, \eta) = \frac{\bar{\alpha}_s C_F}{2\pi^2} \int \frac{d^2 z}{(x-z)_\perp^2} Q_1^f(z_\perp, \eta), \quad \bar{\alpha}_s \equiv \frac{\alpha_s C_F}{2\pi}$$

Same equation for $Q_5^{\rm NS}$ (non-singlet).

Independent $\perp$ phase space

Introduce $\rho \equiv \ln(L_\perp^2/r_\perp^2) \quad \eta = \ln(1/x_B)$

$$\frac{\partial^2 Q_1}{\partial\eta\,\partial\rho} = \bar{\alpha}_s Q_1$$

Bessel solution:

$$Q_1 = Q_1^{(0)} I_0(2\sqrt{\bar{\alpha}_s \eta\rho})$$

Resums

$$\left(\alpha_s \ln\frac{1}{x_B} \ln Q_\perp^2\right)^n$$

mixed longitudinal–transverse DLA.

Longitudinally constrained $\perp$ phase space

$$\text{Kinematic constraint:} \quad k_\perp^2 \lesssim \frac{k^+}{q^+} s = 2P^- k^+$$

$$\text{Hence} \quad \rho_{\max}(k^+) \sim \ln\frac{2P^- k^+}{Q^2} = \ln\frac{k^+}{x_B q^+}.$$

The second logarithm is converted into an energy logarithm:

$$\int_{x_B q^+}^{q^+} \frac{dk^+}{k^+} \ln\frac{k^+}{x_B q^+} = \frac{1}{2}\ln^2\frac{1}{x_B}$$

Evaluate the same mixed-DLA solution on the symmetric trajectory:

$$\rho \simeq \eta \quad \Rightarrow \quad Q_1(\eta,\eta) \simeq Q_1^{(0)} \frac{e^{2\sqrt{\bar{\alpha}_s}\,\eta}}{\sqrt{4\pi\sqrt{\bar{\alpha}_s}\,\eta}}.$$

Resums $\left(\alpha_s \ln^2\frac{1}{x_B}\right)^n$ genuine double log of energy

Fixed-coupling Kirschner–Lipatov exponent with full finite-N_c color factor:

$$\Delta = 2\sqrt{\bar{\alpha}_s} = \sqrt{2\alpha_s C_F/\pi}$$

G.A.C. (2026)

Conclusions

- Operator bridge: first sub-eikonal correction reconstructs, in the inclusive limit, the standard nonlocal quark and helicity distributions $q_f(x_B)$, $\Delta q_f(x_B)$ at $x_B \neq 0$. Established from two independent routes: shock-wave formalism and high-energy limit of the non-local OPE.
- Non-commutativity: small-x_B limit and full phase-space integration do *not* commute. Taking the limit prematurely yields only the naive $x_B = 0$ operator.
- Sub-eikonal dipole structure functions: F_L, F_T, g_1 expressed in terms of dipole-type operators that vanish as the dipole size goes to zero. F_L is UV finite; F_T and g_1 carry single-log divergences absorbed by the one-loop operator evolution.
- Double-logarithmic evolution: in the local ladder sector, Q_1^f and $Q_5^{\rm NS}$ obey a mixed DLA equation (Bessel solution). When transverse phase space is tied to longitudinal ordering, the second logarithm converts into an energy logarithm, recovering the fixed-coupling Kirschner–Lipatov exponent $\Delta = \sqrt{2\alpha_s C_F/\pi}$ with full C_F.

Next steps: closed evolution system for the full dipole operator basis; matching to Bartels–Ermolaev–Ryskin beyond the ladder; extension to finite-x_B evolution and EIC phenomenology.

Back-up

Gluon operator even

$$\mathcal{G}_2(z_\perp) \equiv ig^2 s \int_{-\infty}^{+\infty} d\omega^+ \int_{\omega^+}^{+\infty} d\omega'^+ (\omega^+ - \omega'^+)[\infty n_1, \omega'^+]_z F^{i-}(\omega'^+, z_\perp)[\omega'^+, \omega^+]_z F_i{}^-(\omega^+, z_\perp)[\omega^+, -\infty n_1]_z$$

$$G(z_\perp) \equiv \mathrm{Tr}\{\mathcal{G}_{2z} U_z^\dagger\} = \frac{i\, s\, g^2}{2} \int_{-\infty}^{+\infty} d\omega^+ \int_{\omega^+}^{+\infty} d\omega'^+ (\omega'^+ - \omega^+) F_i^{a,-}(\omega'^+, z_\perp)[\omega'^+, \omega^+]_z^{ab} F_i^{b,-}(\omega^+, z_\perp)$$

Quark propagator at sub-eikonal level (background of gluon field)

G.A.C. (2019), JHEP 01, 118

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_A = \left[\int_0^{+\infty}\frac{\text{đ}p^+}{4(p^+)^2}\theta(x^+-y^+) - \int_{-\infty}^{0}\frac{\text{đ}p^+}{4(p^+)^2}\theta(y^+-x^+)\right] e^{-ip^+(x^- - y^-)}\langle x_\perp| e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}$$
$$\times\left\{\hat{\not{p}}\gamma^+[x^+,y^+]\hat{\not{p}} + \hat{\not{p}}\gamma^+\hat{\mathcal{O}}_1(x^+,y^+;p_\perp)\hat{\not{p}} + \hat{\not{p}}\gamma^+\frac{1}{2}\hat{\mathcal{O}}_2(x^+,y^+;p_\perp) - \frac{1}{2}\hat{\mathcal{O}}_2(x^+,y^+;p_\perp)\not{n}_2\hat{\not{p}}\right\} e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}|y_\perp\rangle$$
$$+O(\lambda^{-2})\,.$$

$$\hat{\mathcal{O}}_1(x^+,y^+;p_\perp) = \frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big([x^+,\omega^+]\frac{1}{2}\sigma^{ij}F_{ij}[\omega^+,y^+] + \{\hat{p}^i,[x^+,\omega^+]\ \omega^+\ F_i{}^-[\omega^+,y^+]\}$$
$$+g\int_{\omega^+}^{x^+}d\omega'^+(\omega^+-\omega'^+)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]\,F_i{}^-[\omega^+,y^+]\Big)\,,$$

$$\hat{\mathcal{O}}_2(x^+,y^+;p_\perp) = \frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big[\{\hat{p}^k,[x^+,\omega^+]iF_{kj}\gamma^j[\omega^+,y^+]\} + [x^+,\omega^+]iF_{kj}\gamma^j(i\mathfrak{D}^k[\omega^+,y^+]) - (i\mathfrak{D}^k[x^+,\omega^+])iF_{kj}\gamma^j[\omega^+,y^+]$$
$$-[x^+,\omega^+]\,i\,F^{-+}(i\not{\mathfrak{D}}_\perp[\omega^+,y^+]) + (i\not{\mathfrak{D}}_\perp[x^+,\omega^+])\,i\,F^{-+}[\omega^+,y^+] + (\hat{p}^+\gamma^- - \hat{\not{p}}_\perp)[x^+,\omega^+]\,i\,F^{-+}[\omega^+,y^+]\Big]\,,$$

Quark propagator in the background of quark fields

G.A.C. (2019), JHEP 01, 118

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_{\psi,\bar{\psi}} = \frac{1}{s}\int d^4z\delta(z^+)\langle x|\frac{i\not{p}}{p^+(p^2+i\epsilon)}|z\rangle$$
$$\times\gamma_\perp^\mu\Big(Q(z_\perp)\theta(x^+)\theta(-y^+) - \tilde{Q}(z_\perp)\theta(-x^+)\theta(y^+)\Big)\gamma_\mu^\perp\langle z|\frac{i\not{p}}{p^2+i\epsilon}|y\rangle + O(\lambda^{-2})$$

$$Q_{ij}^{\alpha\beta}(x_\perp) \equiv g^2\frac{s}{2}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\Big([\infty n_1, z^+]_x t^a\psi^\alpha(z^+,x_\perp)[z^+,z'^+]_x^{ab}\bar{\psi}^\beta(z'^+,x_\perp)t^b[z'^+,-\infty n_1]_x\Big)_{ij}$$

$$\tilde{Q}_{ij}^{\alpha\beta}(x_\perp) \equiv g^2\frac{s}{2}\int_{-\infty}^{+\infty}dz^+\int_{z^+}^{+\infty}dz'^+\Big([-\infty n_1, z^+]_x t^a\psi^\alpha(z^+,x_\perp)[z^+,z'^+]_x^{ab}\bar{\psi}^\beta(z'^+,x_\perp)t^b[z'^+,\infty n_1]_x\Big)_{ij}$$