The $\gamma^* \to \eta\gamma$ and $\gamma^* \to \eta'\gamma$ form factors to NNLO accuracy

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based on: V. Braun, K. Chetyrkin, A. Manashov, 2510.15643

QCD Evolution 2026

Time-like form factors

Babar 2006: $e^+e^- \to \gamma^* \to \eta(\eta')\gamma$

$$i\int d^4x\, e^{-iqx}\langle 0|T\{j^{\rm em}_\mu(x) j^{\rm em}_\nu(0)\}|M(p)\rangle = e^2\varepsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta F_M(Q^2)$$

$$q^2|F_\eta(q^2)| = 0.229 \pm 0.030 \pm 0.008\,, \qquad q^2|F_{\eta'}(q^2)| = 0.251 \pm 0.019 \pm 0.008\,,$$

Collinear factorization

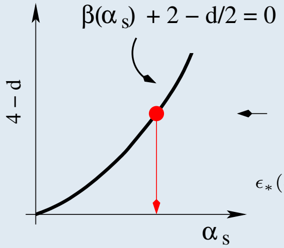
$$Q^2 F_M(Q^2) = \sum_q \int_0^1 du\, \underbrace{\mathbb{T}_q(u,\mu/Q,\alpha_s(\mu))}_{\text{coeff. func.}}\, \underbrace{\Phi^q_M(u,\mu)}_{\text{quark LCDA}} + \int_0^1 du\, \mathbb{T}_g(u,\mu/Q,\alpha_s(\mu))\, \underbrace{\Phi^g_M(u,\mu)}_{\text{gluon LCDA}}$$

Why interesting?

- η, η' at short distances (hard processes)
  — State mixing paradigm? Gluon content?
  — Applications to B-decays etc.
- Highest $q^2 = 112\,\text{GeV}^2$ for all exclusive reactions
- Excellent prospects for high-precision measurements (Belle II)
- High-quality lattice inputs starting to arrive
- Theory input completed at NNLO
  — Two-loop coefficient functions
  Yao Ji, J. Schoenleber, JHEP 01 (2024) 05
  — Three-loop anomalous dimensions matrix
  V.B., K. Chetyrkin, A. Manashov, this work

Conformal QCD

QCD is not a conformal theory, but



$$\epsilon_*(a_s) = -\beta_0 a_s + \ldots, \qquad a_s = \frac{\alpha_s}{4\pi}$$

$$\mathcal{A}_{\mathrm{QCD}} = \mathcal{A}^{\mathrm{conf}}_{\mathrm{QCD}} + \frac{\beta(g)}{g}\Delta\mathcal{Q}$$

"Conformal QCD": QCD in $d-2\epsilon$ at Wilson-Fischer critical point $\beta(\alpha_S)=0$

V.B., A. Manashov, Eur.Phys.J.C 73 (2013) 2544

- No corrections $\sim \beta(\alpha_s)$ for the counterterms
  $\Rightarrow$ exact conformal symmetry of the QCD RG equations in MS-schemes

Mixing with total derivatives

Renormalization of off-forward matrix elements

$$\langle p'|\mathcal{O}_{Nk}|p\rangle = \langle p'|\partial_{\mu_1}\ldots\partial_{\mu_k}\bar{q}(0)\gamma_{\mu_1}(\gamma_5)\overleftrightarrow{D}_{\mu_{k+1}}\ldots\overleftrightarrow{D}_{\mu_N}\, q(0)|p\rangle$$

Mixing matrix with operators containing total derivatives with anomalous dimensions on the diagonal

$$\begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ * & \gamma_2 & 0 & 0 \\ * & * & \gamma_3 & 0 \\ * & * & * & \gamma_4 \end{pmatrix}$$

need off-diagonal elements

[evolution kernels in x-space are functions of two variables]

- In conformal theories, operators with different scaling dimensions are orthogonal

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \frac{\text{const}}{|x_1 - x_2|^{2\Delta_1}}\,\delta_{\Delta_1\Delta_2}$$

- n-loop ADs from n-loop two-particle functions

VB, K.Chetyrkin, A.Manashov, PLB 834 (2022) 137409

- Gauge-invariant two-point functions
- Simple algorithmic implementation

VB, K.Chetyrkin, A.Manashov, PLB 834 (2022) 137409

simple example:

$$O_2(x) = \partial_+^2\,\bar q_1(x)\,C_2^{(3/2)}\left(\frac{\overleftarrow{D}_+ - \overrightarrow{D}_+}{\overleftarrow{D}_+ + \overrightarrow{D}_+}\right)\gamma_+ q_2(x) \qquad O_1(x) = \partial_+^2 \bar q_1(x)\gamma_+ q_2(x)\,,$$

$$\gamma = \begin{pmatrix}\gamma_{11} & 0\\ \gamma_{21} & \gamma_{22}\end{pmatrix} \qquad \begin{aligned}&\gamma_{11} = 0\\ &\gamma_{22} = \frac{100}{9}a_s + \left[\frac{34450}{243} - \frac{830}{81}n_f\right]a_s^2 + \dots\\ &\gamma_{21} = \gamma_{21}^{(2)}a_s^2 + \dots = ?\end{aligned}$$

Let

$$\begin{pmatrix}[\mathbb{O}_1]\\ [\mathbb{O}_2]\end{pmatrix} = \begin{pmatrix}[O_1]\\ [O_2] + \lambda_{21}[O_1]\end{pmatrix}, \qquad \lambda_{21} = \frac{\gamma_{21}}{\gamma_{22} - \gamma_{11}}$$

At the critical point

$$\Big(\mu\partial_\mu + \gamma_{11}\Big)\mathbb{O}_1 = 0\,, \qquad \Big(\mu\partial_\mu + \gamma_{22}\Big)\mathbb{O}_2 = 0\,,$$

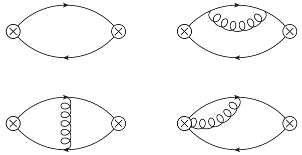
and conformal symmetry requires that to all orders of perturbation theory

$$\langle \mathbb{O}_2(x)\mathbb{O}_1(0)\rangle = \langle [O_2](x)[O_1](0)\rangle + \lambda_{21}\langle [O_1](x)[O_1](0)\rangle = 0$$

— an equation for γ_{21}

NB: this equation holds at $d = 4 - 2\epsilon_*$, but the result for γ_{21} is valid for $d = 4$

to leading order $\mathcal{O}(a_s^2)$



$$\langle \mathsf{T}\{O_1(x)O_1(0)\}\rangle = \mathcal{N}\Big[-105 + \mathcal{O}(a_s,\epsilon)\Big],$$

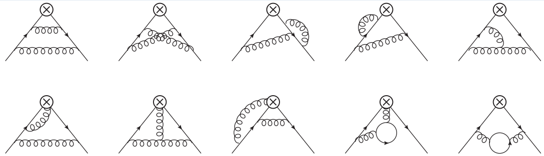
$$\langle \mathsf{T}\{O_2(x)O_1(0)\}\rangle = \mathcal{N}\Big[63\epsilon + 70 a_s + \mathcal{O}(a_s^2, a_s\epsilon, \epsilon^2)\Big],$$

$$\mathcal{N} = \frac{(n\cdot x)^6}{(4\pi)^d}\left(\frac{4}{-x^2+i0}\right)^{d+4}.$$

Using these expressions and the one-loop γ_{22}, expanding to $\mathcal{O}(a_s)$, and replacing $\epsilon \mapsto -\beta_0 a_s$, one obtains

$$\gamma_{21}^{(2)} = \frac{260}{9} - \frac{40}{9} n_F$$

compare:



- Straightforward to generalize to higher orders, more derivatives and flavor-singlets

Three-loop flavor-singlet anomalous dimensions (axial vector)

light-like vectors $n^2 = 0, \quad \bar{n}^2 = 0, \quad (n\bar{n} = 1), \quad p_+ = (pn)$

orthogonal vectors $a^\mu, b^\mu: \quad a, b \perp n, \bar{n}, \quad (ab) = 0$

antisym. product $\Gamma_{\mu\nu\rho} = \frac{1}{6}(\gamma_\mu\gamma_\nu\gamma_\rho \pm \ldots) \qquad \Gamma_+^{ab} = a^\mu b^\nu n^\rho \Gamma_{\mu\nu\rho}$

$$\mathcal{O}^q_{nk} = i\,\partial_+^n \sum_{f=1}^{n_f} \bar{q}^f\, C_k^{(3/2)}\left(\frac{\overleftarrow{D}_+ - \overrightarrow{D}_+}{\overleftarrow{D}_+ + \overrightarrow{D}_+}\right)\Gamma_+^{ab} q^f,$$

$$\mathcal{O}^g_{nk} = 6\,\partial_+^{n-1}\, F^{a+}\, C_{k-1}^{(5/2)}\left(\frac{\overleftarrow{D}_+ - \overrightarrow{D}_+}{\overleftarrow{D}_+ + \overrightarrow{D}_+}\right)F_{b+} - (a \leftrightarrow b).$$

$$\boldsymbol{\gamma}_n(a) = \begin{pmatrix} \gamma_{11} & 0 & \cdots & 0 \\ \gamma_{31} & \gamma_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{pmatrix}.$$

$$\gamma_{nk} = \begin{pmatrix} \gamma^{qq}_{nk} & \gamma^{qg}_{nk} \\ \gamma^{gq}_{nk} & \gamma^{gg}_{nk} \end{pmatrix} = a_s\gamma^{(1)}_{nk} + a_s^2\gamma^{(2)}_{nk} + a_s^3\gamma^{(3)}_{nk} + \ldots$$

- For quarks: a finite renormalization applied to convert to the anticommuting γ_5 scheme

  V.B., A. Manashov, S. Moch, M. Strohmaier, PRD 103 (2021) 094018

- Calculated: Three-loop off-forward anomalous dimensions for $N = 2, 4, 6, 8$

- Advantages of this technique:
  - Gauge-invariant correlation functions
  - Simple algorithmic implementation
  - Straightforward to extend to four loops (MINCER $\to$ FORCER)

- Price to pay:
  - $2N - 2$ open indices, progressing to high N calculationally expensive

Heavy c, b quarks

- Variable flavor-number scheme (VFNS)

$$T_g^{(c)(1)}(u) = T_g^{(\text{light})(1)}(u) + \underbrace{\ln\frac{\mu^2}{m_c^2}\left(\frac{\ln\bar{u}}{u^2} - \frac{\ln u}{\bar{u}^2}\right)}_{\text{c-quark generated through qg-mixing}} + \underbrace{\delta T_g^{(c)(1)}(u)}_{\mathcal{O}(m_c^2/q^2)}$$

- No constant term under log — can put c-quark LCDA to zero at $\mu = m_c$
  — similar to ACOT scheme in DIS

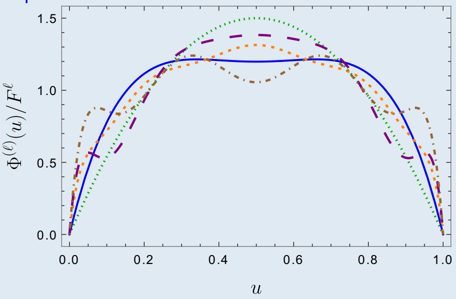
LCDA inputs

- normalizations at $\mu_0^2 = 2\,\mathrm{GeV}^2$ (decay constants) G. Bali *et al.* (RQCD) JHEP 08 (2021) 137

$$F_\eta^{u(d)}(\mu_0) = 72.43 \binom{2.04}{1.99}\,\mathrm{MeV}\,, \qquad F_\eta^{s}(\mu_0) = -109.12 \binom{4.65}{5.02}\,\mathrm{MeV}\,,$$

$$F_{\eta'}^{u(d)}(\mu_0) = 55.48 \binom{3.48}{2.51}\,\mathrm{MeV}\,, \qquad F_{\eta'}^{s}(\mu_0) = 143.22 \binom{4.57}{7.65}\,\mathrm{MeV}\,,$$

- shapes



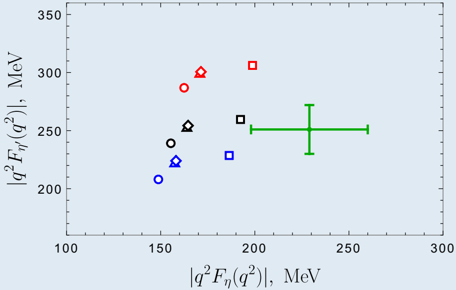
solid: RQCD21 [1903.08038]
short dashes: RQCD21+
long dashes: fit to $\gamma\gamma^*\pi$ [1206.3968]
dash-dotted: LAMET [2407.00206]
dots: asymptotic LCDA

- gluon LCDA — ony the first term $n = 2$ kept at μ_0, free parameter

$$\Phi_M^g(u,\mu) = F_M^0\, 30u^2\bar{u}^2 \sum_{n=2,\dots} b_n^M(\mu) C_{n-1}^{5/2}(2u-1)$$

Results

- The $\gamma^* \to \eta'\gamma$ vs. $\gamma^* \to \eta\gamma$ form factor at $q^2 = 112\,\text{GeV}^2$



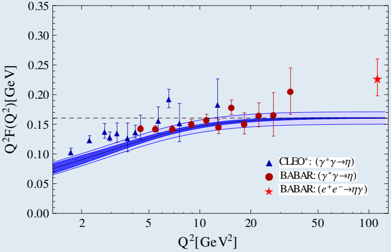
- The three sets of points are obtained for $b_2(\mu_0) = -0.2$ (red), 0 (black) and $+0.2$ (blue)
- circles, triangles, diamonds and squares are calculations with four models of the quark LCDA
- Uncertainty estimated by the residual factorization scale dependence is $\sim 1\%$

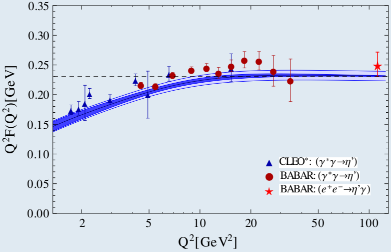
- Sensitivity to inverse moment $\int du\phi(u)/u$
- η': Sensitivity to gluon LCDA and SU(3) breaking

- No hard conclusions at this point, need more data

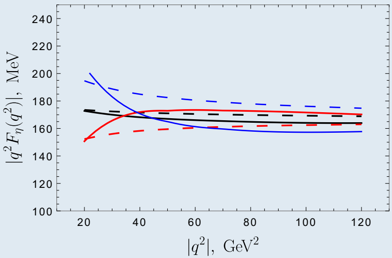
Space-like vs. time-like form factors

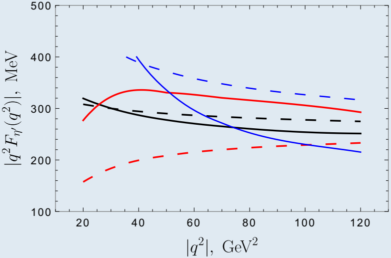
- Space-like FFs with soft corrections from disp. relations and duality (NLO LCSRs) arXiv:1409.431





- NNLO collinear factorization: solid=timelike, dashed=spacelike, black = no gluons



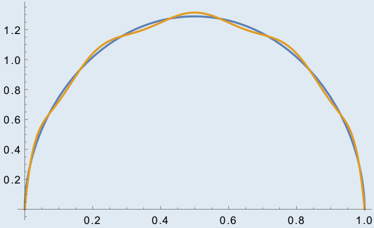


Depending on LCDA model, extreme sensitivity to end-point regions

- Compare two LCDA models

$$\phi_{\mathsf{I}}(u) = B(1+p,1+p)u^p(1-u)^p\,, \qquad p = 0.532$$

$$\phi_{\mathsf{II}}(u) = 6u(1-u)\sum_{n=0}^{8} a_n^{(p)} C_n^{3/2}(2u-1)$$



- The two-loop contribution, nonsinglet

$$2\int_{\frac{1}{2}}^{1} du\, a_s^2 T_{NS}^{(2)}(u)\phi_{\mathsf{I}}(u) = \quad 1.4815 \qquad ?!$$

$$2\int_{\frac{1}{2}}^{1} du\, a_s^2 T_{NS}^{(2)}(u)\phi_{\mathsf{II}}(u) = -0.4998 \qquad ?!$$

Depending on LCDA model, extreme sensitivity to end-point regions

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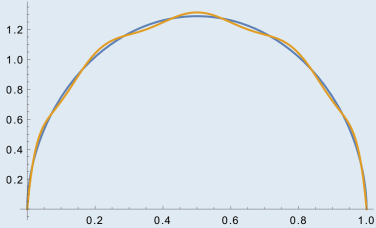

- The two-loop contribution, nonsinglet

$$2\int_{\frac{1}{2}}^{1} du\, a_s^2 T_{NS}^{(2)}(u)\phi_{\text{I}}(u) = \underbrace{-0.967186}_{u<0.99}\ \underbrace{+0.495428}_{0.99<u<0.999}\ \underbrace{+1.95326}_{u>0.999}$$

$$2\int_{\frac{1}{2}}^{1} du\, a_s^2 T_{NS}^{(2)}(u)\phi_{\text{II}}(u) = \underbrace{-0.972059}_{u<0.99}\ \underbrace{+0.27485}_{0.99<u<0.999}\ \underbrace{+0.197422}_{u>0.999}$$

- Threshold resummation or nonperturbative effects (LCSRs)?

Summary

- Three-loop flavor-singlet axial-vector AD matrix calculated for $N = 2, 3, 4, 8$
- Can be extended to four loops with moderate effort

- Application to $\gamma^* \to \eta\gamma$ and $\gamma^* \to \eta'\gamma$

  — Highest Q^2 acessible in exclusive reactions

  — Experimental uncertainies expected to be reduced arXiv:1808.10567
  statistics: factor 8 (luminosity) times 2.5 (trigger efficiency)
  systematics: by at least factor two

  — Lattice QCD approaching the needed precision

  — NNLO corrections $< 2\%$ (NLO $\sim 15\%$); Higher twist corrections estimated $< 1\%$

  — soft corrections ??

- What else can be done:

  — threshold resummation following [2209.09015] (need extension for flavor-singlet)

  — NNLO Light-Cone Sume Rules (need two-loop CFs for two virtual photons, cf. 2411.14985)