Rapidity-only TMD factorization

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QCD Evolution 2026

Outline

1. Part 1: Rapidity factorization and rapidity evolution of TMDs:
   - Rapidity-only cutoff *vs* UV+rapidity regularization
   - Rapidity evolution of TMDs in the Sudakov region.
   - Rapidity-only factorization at one loop.
2. Part 2: Interpolation between Sudakov and BFKL rapidity evolutions for TMD factorization at small x.
   - Interpolation equation between Sudakov and BFKL evolutions.
   - Interpolating terms as corrections to Sudakov and BFKL evolutions.
   - If time permits: Interpolation equation in the Mellin representation
3. Conclusions and outlook

Part 1: Rapidity-only TMD factorization

TMD factorization

TMD factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ ``Y} - \text{terms''}$$

- $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a parton f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$,
- $\mathcal{D}_{f/B}(x, q_\perp - k_\perp)$ is a similar quantity for hadron B,
- $C_i(q, k)$ are determined by the cross section $\sigma(f\!f \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two partons.

Examples: Drell-Yan process with Q being the mass of DY pair and Higgs production by gluon-gluon fusion

TMD approach is relevant when the transverse momentum $q_\perp \ll Q$

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A,k_\perp)\mathcal{D}_{f/B}(x_B,q_\perp - k_\perp)C(q,k_\perp)$$
$$+ \text{ power corrections } + \text{ “Y} - \text{terms}''$$

The quantities $\mathcal{D}_{f/A}(x_A,k_\perp)$, $\mathcal{D}_{f/B}(x_B,q_\perp - k_\perp)$, and $C(q,k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A, x_B: CSS/SCET approach. The TMDs $\mathcal{D}_{f/A}(x_A,k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

It is impossible to extend CSS to small x. (Recently: LO BFKL from SCET)

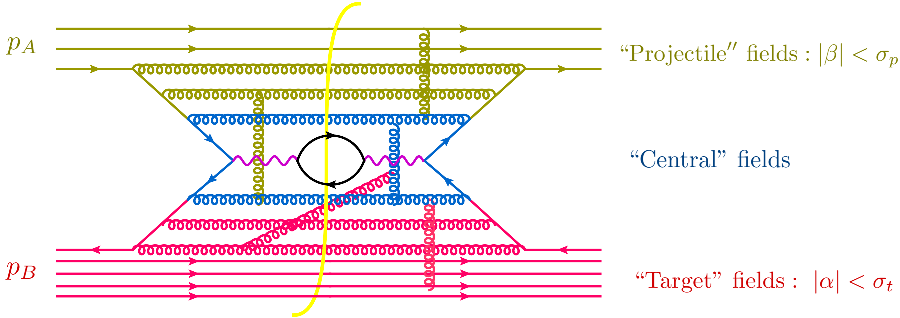
It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

Example: full list of power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor.
They are not obtained (yet?) by CSS or SCET

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables:

$$p \;=\; \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A,\;\; p_2 \simeq p_B,\;\; p_1^2 = p_2^2 = 0$$



The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2} \Rightarrow$ power corrections

Goal: TMD factorization formula

TMD factorization formula structure :

$$\langle p_A', p_B'|J(x_1)J(x_2)|p_A, p_B\rangle = \sum_{\text{TMD operators}} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_i(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t)$$
$$\times \langle p_A'|\hat{\mathcal{O}}_i^{\sigma_p}(z_2^-, x_{2_\perp}; z_1^-, x_{1_\perp})|p_B\rangle \langle p_B'|\hat{\mathcal{O}}_i^{\sigma_t}(z_2^+, x_{2_\perp}; z_1^+, x_{1_\perp})|p_B'\rangle$$

$q_\perp^2 \ll Q^2 \Rightarrow$ no dynamics in the transverse space (to be demonstrated below)

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "projectile" TMD operators with $\beta < \sigma_p$ cutoff, e.g

$$\mathcal{O}(z_{1-}, z_{1_\perp}, z_{2-}, z_{2_\perp}) \equiv \bar{\psi}(z_{1-}, z_{1_\perp})[z_{1-}, -\infty]_{z_{1_\perp}} \Gamma[-\infty, z_{2+}]_{z_2} \psi(z_{2+}, z_{2_\perp})$$

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "target" TMD operators with $\alpha < \sigma_t$ cutoff, e.g

$$\mathcal{O}(z_{1+}, z_{1_\perp}, z_{2+}, z_{2_\perp}) \equiv \bar{\psi}(z_{1+}, z_{1_\perp})[z_{1+}, -\infty]_{z_{1_\perp}} \Gamma[-\infty, z_{2+}]_{z_{2_\perp}} \psi(z_{2+}, z_{2_\perp}).$$

Standard notation for straight-line gauge link

$$[x, y] \equiv \mathrm{P}e^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)} - \text{gauge link}$$

Convenient notations

$$[x_+, y_+]_{z_\perp} \equiv [x_+, 0_-, z_\perp; y_+, 0_-, z_\perp], \quad [x_-, y_-]_{z_\perp} \equiv [x_-, 0_+, z_\perp; y_-, 0_+, z_\perp]$$

Means: “double operator expansion”

Intermediate step: double operator expansion

$$\hat{J}(x_1)\hat{J}(x_2) = \sum_{I,J}\int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t) \times \hat{\mathcal{O}}_I^{\sigma_p}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\hat{\mathcal{O}}_J^{\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})$$

To find relevant operators and coefficients, it is convenient to consider “matrix” elements of the l.h.s. and r.h.s. in suitable background field

Suitable field $\mathbb{A}$: solution of classical YM equations with boundary condition that at the remote past the field is a sum of projectile and target fields

$$\langle\hat{J}(x_1)\hat{J}(x_2)\rangle_{\mathbb{A}} = \sum_{I,J}\int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t) \times \langle\hat{\mathcal{O}}_I^{\sigma_p}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\hat{\mathcal{O}}_J^{\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})\rangle_{\mathbb{A}}$$

Method of solution:

- Start with $\Psi_{\rm trial} = \psi_A + \psi_B$ and $\mathbb{A}_{\rm trial} = \bar{A}_\mu + \bar{B}_\mu$ in the gauge $A^+ = 0$, $A^- = 0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P} + m)(\psi_A + \psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(A + B)$

Classical fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

The solution of such YM equations in general case is yet unsolved problem (goes under the name “glasma” ⇔ scattering of two “color glass condensates”).

Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_\perp^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\Psi, \mathbb{A} \ = \ \text{series in } \frac{q_\perp}{Q}: \qquad \Psi = \psi^{(0)} + \psi^{(1)} + ..., \qquad \mathbb{A} = A^{(0)} + A^{(1)} + ...$$

NB: After the expansion

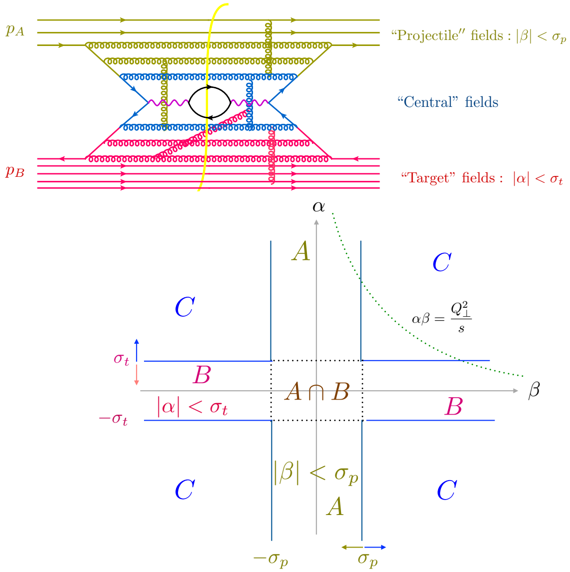
$$\frac{1}{p^2 + i\epsilon p_0} \ = \ \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} \ = \ \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2 + i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2 + i\epsilon p_0} \ + ...$$

the dynamics in transverse space is trivial.

Fields are either at the point $x_\perp$ or at the point $0_\perp \Rightarrow$ TMDs

Part 2: Rapidity factorization and rapidity evolution of TMDs

Rapidity-only cutoffs and matching of logs



Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons
$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons cancel out

Rapidity-only cutoff

Typical diagram in the background
field $\Psi(\beta_B, p_{B_\perp}) = \varrho\int dz^+ dz_\perp\ \Psi(z^+, z_\perp) e^{i\varrho\beta_B z^+ - i(p_B,z)_\perp}$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp)\rangle_\Psi = g^2 c_F \int d\!\!^-\beta_B d\!\!^- p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp})$$

$$\times \int_0^\infty d\!\!^-\alpha \int \frac{d\!\!^- p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,\Delta)}}{\alpha\beta_B s + (p - p_B)_\perp^2 + i\epsilon} \qquad \leftarrow \text{ divergent as } \alpha \to \infty$$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp, -\delta^-)\rangle_\Psi = g^2 c_F \int d\!\!^-\beta_B d\!\!^- p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp}) \qquad \sigma \equiv \frac{1}{\varrho\delta^-}$$

$$\times \int_0^\infty d\!\!^-\alpha \int \frac{d\!\!^- p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,x-y)_\perp}}{\alpha\beta_B s + (p - p_B)_\perp^2 + i\epsilon} e^{-i\frac{\alpha}{\sigma}} \qquad \leftarrow \text{ convergent as } \alpha \to \infty$$

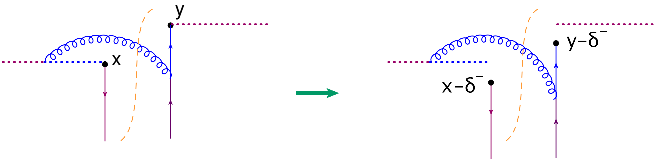


Figure: Point-splitting visualization of “smooth” rapidity-only cutoff.

Rapidity-only cutoff *vs* UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $\text{đ}^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon}\int \text{đ}\alpha\, \text{đ}\beta\, \text{đ}p_\perp \frac{1}{\beta - i\epsilon}\frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \mu^{-2\varepsilon}\int \frac{\text{đ}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\beta_B}\frac{\text{đ}\beta}{\beta_B}\frac{\beta_B - \beta}{\beta - i\epsilon} = -\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\epsilon}$$

Regularization with $A^-(z^+) \to A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\delta} \simeq \frac{1}{8\pi^2}\Big(-\frac{1}{\varepsilon} + \ln\mu^2\frac{x_\perp^2}{4} + \gamma_E\Big)\Big(\ln\frac{\beta_B}{-i\delta} - 1\Big)$$

Rapidity-only cutoff

$$-i\int \text{đ}\alpha\, \text{đ}\beta\, \text{đ}p_\perp \frac{1}{\beta - i\epsilon}\frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \int \frac{\text{đ}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^\infty \text{đ}\alpha \frac{\beta_B s}{\alpha\beta_B s + p_\perp^2}e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2}\ln^2\Big(-i\beta_B\sigma s\frac{x_\perp^2}{4}e^{\gamma_E}\Big)$$

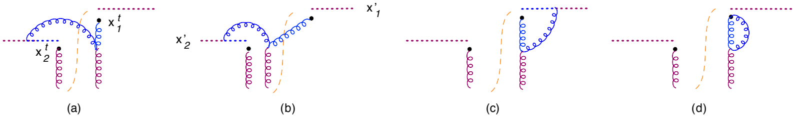
Rapidity evolution of TMDs

Sudakov regime: $Q^2 \gg Q_\perp^2 \Leftrightarrow z_{12+}z_{12-} \ll z_{12_\perp}^2$

Quark TMD operator (for the target)

$$\mathcal{O}(x_{1+}, x_{1_\perp}, x_{2+}, x_{2_\perp}) \equiv \bar\psi(x_{1+}, x_{1_\perp})[x_{1+}, -\infty_+]_{x_1}\Gamma[-\infty_+, x_{2+}]_{x_2}\psi(x_{2+}, x_{2_\perp})$$

Evolution diagrams at the one-loop level



The $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting: positions of $\bar\psi$ and ψ are separated from the beginnings of gauge links. (Violations of gauge invariance are power corrections).

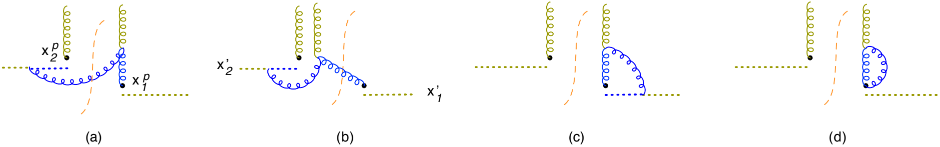
Evolution equation

$$\sigma_t \frac{d}{d\sigma_t}\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{sx_{12_\perp}^2}{4} + \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

For projectile TMDs

"Projectile" TMD matrix elements.

The $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting.



Evolution equation

$$\sigma_t \frac{d}{d\sigma_t}\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{sx^2_{12_\perp}}{4} + \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

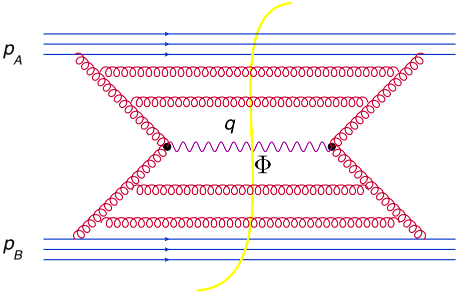
Coefficient function for TMD factorization at one loop

Coefficient function = integral over “central” fields

Calculation of coefficient function: similarly to OPE for DIS

NLO coeff. function =
= {Diagrams at NLO (with quark/gluon tails or in a suitable background field)}
– {α_s correction to matrix elements of the LO operators}

Example which I calculated: NLO coeff. function for Higgs production by gluon fusion

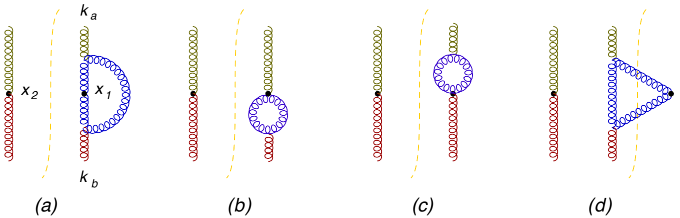


$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

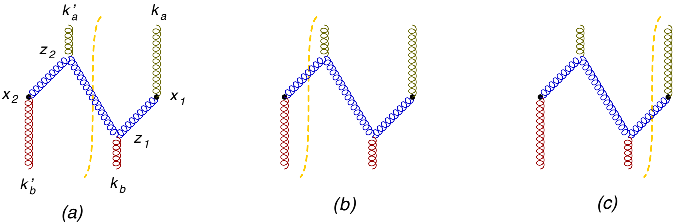
Result for DY is similar up to $\frac{\pi^2}{2}$ in Higgs case $\to \frac{\pi^2}{2} + ?$ in the DY case

Diagrams for $\langle \tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



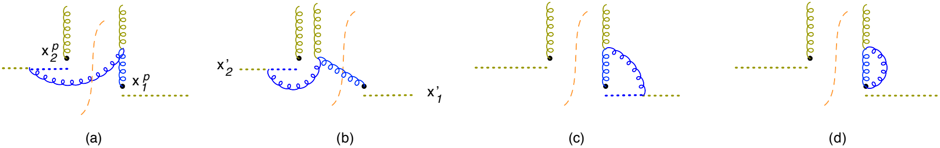
"Real" diagrams



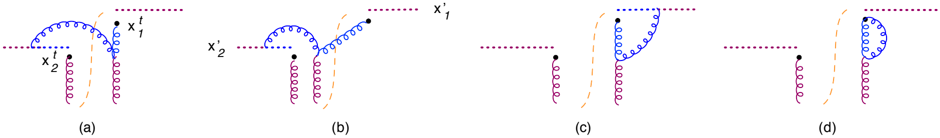
Diagrams for subtracted TMD matrix elements

"Projectile" TMD matrix elements.

The $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting: positions of $F's$ are separated from the beginnings of gauge links. (Violations of gauge invariance are power corrections).



"Target" TMD matrix elements. The $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



Result for the coefficient function

Result of calculations:

$$\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$=\int\mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})$$
$$+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \mathcal{O}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})\ +\ O(\alpha_s^2)\Big]$$

Result

$$\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_{1_\perp},x_{2_\perp};\sigma_p,\sigma_t)$$
$$=\ \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}-\ \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p}-\ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p}+\pi^2$$

Matching of coefficient function and TMDs

TMD evolution equations

$$\sigma_p \frac{d}{d\sigma_p} \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4} + \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$\sigma_t \frac{d}{d\sigma_t} \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4} + \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

Matching of σ_p and σ_t evolutions $\Rightarrow$

$$\sigma_t \frac{d}{d\sigma_t}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4}$$
$$+ \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\mathfrak{C}(x_1, x_2; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$
$$\sigma_p \frac{d}{d\sigma_p}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4}$$
$$+ \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\Big]\ \mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$

Matching of coefficient function and TMDs

The solution of this equations compatible with our first-order result is

$$\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \;=\; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$

$\Rightarrow$ hadronic tensor is

$$W(\alpha'_a, \alpha_a, \beta'_b, \beta_b, x_{1_\perp}, x_{2_\perp}) \;=\; \int d\!\!{}^-\alpha'_a d\!\!{}^-\alpha_a d\!\!{}^-\beta'_b d\!\!{}^-\beta_b \; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$
$$\times \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})|p_B\rangle \;+ \ldots$$

Reminder

$$\mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_{1_\perp}, x_{2_\perp}; \sigma_p, \sigma_t)$$
$$= \ln^2\frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4} - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

Forward case ($\equiv$ particle production by gluon fusion)

Recall $\alpha_q \equiv x_A$, $\beta_q \equiv x_B$.

$$W(p_A,p_B;q) \;=\; \int db_\perp\; e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),$$

$$W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \;=\; \frac{\pi^2}{2} Q^2 \mathcal{G}^{\sigma_p}_{ij}(\alpha_q,b_\perp;p_A)\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)$$

$$\times\;\exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\big)\big(\ln\frac{\beta_q}{\sigma_p}+\gamma\big)+\frac{\pi^2}{2}\Big]\Big\}$$

$$+\;\text{NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections} \qquad (*)$$

where $\mathcal{G}^{\sigma_p}_{ij}$, $\mathcal{G}^{\sigma_t}_{ij}$ are gluon TMDs:

$$\langle p_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z^-,0^-,b_\perp)|p_A\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\mathcal{G}^{\sigma_p}_{ij}(u,b_\perp)\cos u\varrho z^-,$$

$$\langle p_B|\hat{\mathcal{O}}^{\sigma_t}_{ij}(z^-,0^-,b_\perp)|p_B\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\mathcal{G}^{\sigma_t}_{ij}(u,b_\perp)\cos u\varrho z^-.$$

Matching of coefficient function and TMDs

The r.h.s. of this evolution formula $(*)$ does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{x_A s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{x_B s}$. Thus, the result of double-log Sudakov evolution reads

$$W(p_A, p_B; x_A, x_B, b_\perp) \; = \; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(x_A, b_\perp; p_A) \mathcal{G}^{ij;\tilde{\sigma}_t}(x_B, b_\perp; p_B)$$
$$\times \; \exp\Big\{ -\frac{\alpha_s N_c}{2\pi}\Big[\Big(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma\Big)^2 - 2\gamma^2 - \frac{\pi^2}{2}\Big]\Big\} + \; O(\alpha_s^2) \text{ terms } + \text{ power corrections}$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution stops (for projectile) at $\tilde{\sigma}_p \simeq \frac{b_\perp^{-2}}{x_A s}$. After that:

- If $x_A \sim 1$ - DGLAP-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\rm fin} = \frac{m_N^2}{s}$: summation of $\big(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2}\big)^n$
- If $x_A \ll 1$ - BFKL-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\rm fin} = \frac{b_\perp^{-2}}{s}$: summation of $\big(\alpha_s \ln x_B\big)^n$
- Problem: matching of these evolutions

Interpolation between Sudakov and BFKL evolutions

The evolution equation for $\mathcal{G}(\beta_B, b_\perp) \equiv \mathcal{G}_i^i(\beta_B, b_\perp)$ valid at any $b_\perp$ and $\beta \equiv x_B$ is very complicated. It has three correct limits: DGLAP, Sudakov, and BK. (A. Tarasov and I.B.)

$$
\begin{aligned}
&\frac{d}{d\eta}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle^\eta \qquad\qquad (5.5)\\
&= -\alpha_s\langle p|\mathrm{Tr}\Big\{\int d^2k_\perp\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\Big[(x_\perp|\Big(U\frac{1}{\sigma\beta_B s+p_\perp^2}(U^\dagger k_k+p_kU^\dagger)\\
&\quad\times\frac{\sigma\beta_B s g_{\mu i}-2k_\mu^\perp k_i}{\sigma\beta_B s+k_\perp^2}-2k_\mu^\perp g_{ik}U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger-2g_{\mu k}U\frac{p_i}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad\times(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\Big(\frac{\sigma\beta_B s\delta_j^\mu-2k_\perp^\mu k_j}{\sigma\beta_B s+k_\perp^2}(k_lU+Up_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&\quad-2k_\perp^\mu g_{jl}U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger-2\delta_l^\mu U\frac{p_j}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)|y_\perp)\\
&\quad+2(x_\perp|\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\Big(\frac{k_j}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2}(k_lU+Up_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&\quad+2U\frac{g_{jl}}{\sigma\beta_B s+p_\perp^2}U^\dagger-2\frac{k_l}{k_\perp^2}U\frac{p_j}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)|y_\perp)\\
&\quad+2(x_\perp|\Big(U\frac{1}{\sigma\beta_B s+p_\perp^2}(U^\dagger k_k+p_kU^\dagger)\frac{k_i}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2}+2U\frac{g_{ik}}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&\quad-2U\frac{p_i}{\sigma\beta_B s+p_\perp^2}U^\dagger\frac{k_k}{k_\perp^2}\Big)\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|y_\perp)\Big]\\
&\quad+2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta_m^k\delta_j^l-g_{jm}g^{kl})U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger|y_\perp)\\
&\quad+2(x_\perp|U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger(2\delta_i^k\delta_m^l-g_{im}g^{kl})(i\partial_k-U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\\
&\quad-4\int\frac{d^2k_\perp}{k_\perp^2}\Big[\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s},y_\perp\Big)e^{i(k,x-y)_\perp}\\
&\quad-\frac{\sigma\beta_B s}{\sigma\beta_B s+k_\perp^2}\tilde{\mathcal{F}}_i(\beta_B,x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big]\Big\}|p\rangle^\eta+O(\alpha_s^2)
\end{aligned}
$$

Interpolation between Sudakov and BFKL evolutions

However, if we want to describe transition between Sudakov and BFKL evolutions, the equation can be simplified to $\alpha_q \equiv x_A, \ \beta_q \equiv x_B$

$$\sigma\frac{d}{d\sigma}\mathcal{G}_\sigma(\alpha_q, b_\perp) = \frac{\alpha_s N_c}{\pi^2}\int dz_\perp\Big[\frac{e^{-\alpha_q s\sigma\frac{(b-z)_\perp^2}{4}}}{(b-z)_\perp^2}\mathcal{G}_\sigma(\alpha_q, z_\perp) - \frac{(b,z)_\perp}{(b-z)_\perp^2 z_\perp^2}\mathcal{G}_\sigma(\alpha_q, b_\perp)\Big] \qquad (\star)$$

This interpolation equation has two correct limits:

- Sudakov logs at $\sigma \gg \frac{b_\perp^{-2}}{\alpha_q s} \sim \frac{q_\perp^2}{x_A s}$

$$\sigma\frac{d}{d\sigma}\mathcal{G}_\sigma(\alpha_q, b_\perp) = \frac{\alpha_s N_c}{\pi^2}\mathcal{G}_\sigma(\alpha_q, b_\perp)\int dz\Big[\frac{e^{-\alpha_q s\sigma\frac{(b-z)_\perp^2}{4}}}{(b-z)_\perp^2} - \frac{(b,z)_\perp}{(b-z)_\perp^2 z_\perp^2}\Big]$$
$$= -\frac{\alpha_s N_c}{\pi^2}\Big[\ln\frac{\sigma b_\perp^2}{4} + \ln\alpha_q + \gamma_E\Big]\mathcal{G}_\sigma(\alpha_q, x)$$

- BFKL evolution at $\sigma \ll \frac{b_\perp^{-2}}{\alpha_q s} \sim \frac{q_\perp^2}{x_A s}$

$$\sigma\frac{d}{d\sigma}\mathcal{G}_\sigma(\alpha_q, b_\perp) = \frac{\alpha_s N_c}{\pi^2}\int dz_\perp\Big[\frac{1}{(b-z)_\perp^2}\mathcal{G}_\sigma(\alpha_q, z_\perp) - \frac{(b,z)_\perp}{(b-z)_\perp^2 z_\perp^2}\mathcal{G}_\sigma(\alpha_q, b_\perp)\Big]$$

Interpolation between Sudakov and BFKL evolutions

Interpolation equation

$$\sigma\frac{d}{d\sigma}\mathcal{G}_\sigma(\alpha_q,b_\perp) \;=\; \frac{\alpha_s N_c}{\pi^2}\int dz_\perp\Big[\frac{e^{-\alpha_q s\sigma\frac{(b-z)^2_\perp}{4}}}{(b-z)^2_\perp}\mathcal{G}_\sigma(\alpha_q,z_\perp) - \frac{(b,z)_\perp}{(b-z)^2_\perp z^2_\perp}\mathcal{G}_\sigma(\alpha_q,b_\perp)\Big]$$

Hope: use this interpolation equation to obtain complete result for hadronic tensor

For now: one can estimate the size of the corrections from both sides

For estimates, we take simple GBW model

$$\langle p_B|1-\frac{1}{N_c}\mathrm{Tr}\{U_zU_0^\dagger\}|p_B(1+\lambda)\rangle \;=\; 2\pi\delta(\lambda)\boldsymbol{\sigma}_{\mathrm{GBW}}\Big(1-e^{-\frac{1}{4}z^2_\perp Q_s^2}\Big)$$

$$\Rightarrow\;\alpha D(\alpha,\sigma_0,z_\perp) \;=\; CQ_s^2\Big(1-\frac{1}{4}z^2_\perp Q_s^2\Big)e^{-\frac{1}{4}z^2_\perp Q_s^2},$$

$$C \;\equiv\; \frac{N_c}{g^2\pi}Q_s^2\boldsymbol{\sigma}_{\mathrm{GBW}},\qquad \boldsymbol{\sigma}_{\mathrm{GBW}}\sim 30\mathrm{mb}$$

Correction to Sudakov evolution

In the Sudakov region $\sigma_p \gg \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{x_A s}$ one can rewrite the interpolation equation as ($\varsigma \equiv \alpha\sigma s$)

$$\sigma\frac{d}{d\sigma}D_g(\alpha, b_\perp; \sigma) \;=\; -a_s\big[\ln \varsigma b_\perp^2 + \gamma_E\big]D_g(\alpha, b_\perp; \sigma)$$
$$+\;\frac{a_s}{\pi}\int dz_\perp \frac{e^{-\varsigma z_\perp^2}}{z_\perp^2}\big[D_g(\alpha, z_\perp + b_\perp; \sigma) - D_g(\alpha, b_\perp; \sigma)\big]$$
$$\simeq\; -a_s\Big[\ln\frac{\alpha\sigma s b_\perp^2}{4} + \gamma_E\Big]D_g(\alpha, b_\perp; \sigma) + \frac{a_s}{\alpha\sigma s}\partial_\perp^2 D_g(\alpha, b_\perp; \sigma)\Big]$$

The size of the correction in GBW model is

$$\frac{1}{\alpha\sigma s}\frac{\partial_\perp^2 D_g(\alpha, b_\perp; \sigma)}{D_g(\alpha, b_\perp; \sigma)} \simeq \frac{Q_s^2}{\alpha\sigma s} = \frac{\tilde{\sigma}_p}{\sigma}b_\perp^2 Q_s^2$$

so it becomes ~ 1 when $\sigma_* = \frac{\tilde{\sigma} Q_s^2 b_\perp^2}{4} \sim \tilde{\sigma}_p \frac{Q_s^2}{q_\perp^2}$. If $q_\perp \sim 10 \div 30$ GeV $\quad \sigma_* \ll \tilde{\sigma}_p$

Correction to the BFKL evolution

The interpolation equation in the BFKL region $\tilde\sigma_p \geq \frac{Q_s^2}{s}$ can be rewritten as

$$\sigma\frac{d}{d\sigma}D_g(\alpha,x_\perp;\sigma) \;=\; \frac{\alpha_s N_c}{\pi^2}\int dz\Big[\frac{D_g(\alpha,z_\perp;\sigma)}{(x-z)_\perp^2} - \frac{(x,z)}{(x-z)^2z^2}D_g(\alpha,x_\perp;\sigma)\Big]$$

$$+\;\frac{\alpha_s N_c}{\pi^2}\int dz\frac{e^{-\frac{\alpha s\sigma(x-z)_\perp^2}{4}}-1}{(x-z)_\perp^2}D_g(\alpha,z_\perp;\sigma)$$

$$\simeq\;\frac{\alpha_s N_c}{\pi^2}\int dz\Big\{\Big[\frac{D_g(\alpha,z_\perp;\sigma)}{(x-z)_\perp^2} - \frac{(x,z)}{(x-z)^2z^2}D_g(\alpha,x_\perp;\sigma)\Big] + \frac{(\alpha s\sigma)^2}{2}z^2D_g(\alpha,z_\perp;\sigma) \;+\;\ldots$$

The GBW estimate of the last term last term gives

$$\sigma\frac{d}{d\sigma}D_g(\alpha,x_\perp;\sigma) \;\stackrel{\sigma\sim\sigma_0}{\simeq}\; -a_sD_g(\alpha,x_\perp;\sigma)\big[\ln\frac{Q_s^2x^2}{4}+\gamma_E\big] + 8a_sCQ_s^2\Big(\frac{x_A\sigma s}{Q_s^2}\Big)^2$$

The relative weight of the correction is $\left(\frac{x_A\sigma s}{Q_s^2}\right)^2 \sim \left(\frac{\sigma}{\tilde\sigma}\frac{b_\perp^{-2}}{Q_s^2}\right)^2$. Again, it becomes ~ 1 when $\sigma_* = \tilde\sigma Q_s^2 b_\perp^2 \sim \tilde\sigma Q_s^2/q_\perp^2$.

In both cases it looks like that the switch between Sudakov and BFKL evolution occurs at σ_* which is lower than $\tilde\sigma_p$ when $q_\perp \gg Q_s$

With σ_*: BFKL summation of $\left(\alpha_s\ln\frac{1}{x_A}\right)^n$

With $\tilde\sigma_p$: BFKL summation of $\left(\alpha_s\ln\frac{1}{x_A}\frac{q_\perp^2}{Q_s^2}\right)^n$

Reminder: hadronic tensor for Higgs production

$$W(p_A,p_B;q) \ = \ \int db_\perp \ e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),$$

$$W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \ = \ \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)$$
$$\times \ \exp\Big\{\frac{a_s}{2}\Big[-\Big(\ln\frac{Q^2b_\perp^2}{4}e^{2\gamma_E}\Big)^2 + \Big(\ln\frac{\alpha_q\sigma_p s b_\perp^2}{4}e^{\gamma_E}\Big)^2 + \Big(\ln\frac{\sigma_t\beta_q s b_\perp^2}{4}e^{\gamma_E}\Big)^2 + \frac{\pi^2}{2}\Big]\Big\}$$

Result with $\tilde{\sigma}$'s

$$W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \ = \ \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(\alpha_q,b_\perp;p_A)\mathcal{G}^{ij;\tilde{\sigma}_t}(\beta_q,b_\perp;p_B)$$
$$\times \ \exp\Big\{\frac{a_s}{2}\Big[-\Big(\ln\frac{Q^2b_\perp^2}{4}e^{2\gamma_E}\Big)^2 + 2\gamma_E^2 + \frac{\pi^2}{2}\Big]\Big\}$$

Result with σ_*'s

$$W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \ = \ \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)$$
$$\times \ \exp\Big\{\frac{a_s}{2}\Big[-\Big(\ln\frac{Q^2b_\perp^2}{4}e^{2\gamma_E}\Big)^2 + 2\Big(\ln\frac{Q_s^2b_\perp^2}{4}e^{\gamma_E}\Big)^2 + \frac{\pi^2}{2}\Big]\Big\}$$

The difference is essential, especially for Fourier transformation to $W(q_\perp)$

Conclusions and Outlook

1. Conclusions:
   - Rapidity-only factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B
   - Proposed evolution equation interpolates between Sudakov and BFKL evolutions.
   - Study of the interpolation equation suggests that point of transition from Sudakov to BFKL depends on $\frac{q_\perp}{Q_s}$
2. Outlook: (Numerical?) solution of the interpolation equation

Conclusions and Outlook

1. Conclusions:
   - Rapidity-only factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B
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2. Outlook: (Numerical?) solution of the interpolation equation

Thank you for attention!

BACKUP SLIDES

Interpolation equation in the Mellin representation

Mellin transformation

$$D_g(\alpha,\nu;\sigma) \;=\; \int dz_\perp^2 (z_\perp^2)^{-\frac{1}{2}-i\nu} D_g(\alpha,z_\perp;\sigma)$$
$$D_g(\alpha,z_\perp;\sigma) \;=\; \int \frac{d\nu}{2\pi} (z_\perp^2)^{-\frac{1}{2}+i\nu} D_g(\alpha,\nu;\sigma)$$

From the interpolation equation $(\star)$ we obtain

$$\sigma\frac{d}{d\sigma}D_g(\alpha,\gamma;\sigma) \;=\; \frac{\alpha_s N_c}{\pi}\lim_{\varepsilon\to 0}\Big[\frac{\Gamma(1-\gamma-\varepsilon)}{\Gamma(\gamma+\varepsilon)}$$
$$\times\int_{\gamma+\frac{\varepsilon}{2}-i\infty}^{\gamma+\frac{\varepsilon}{2}+i\infty}\frac{d\xi}{2\pi i}(\alpha s\sigma)^{\gamma-\xi}\frac{\Gamma(\xi-\gamma)\Gamma(\gamma-\xi+\varepsilon)\Gamma(\xi)}{\Gamma(1+\xi-\gamma-\varepsilon)\Gamma(1-\xi)}D_g(\alpha,\xi;\sigma) - \frac{1}{\varepsilon}D_g(\alpha,\gamma;\sigma)\Big]$$

where we introduced the notations $\gamma = \frac{1}{2}+i\nu$ and $\xi = \frac{1}{2}+i\nu'$

Interpolation between Sudakov and BFKL evolutions

To get the interpolation equation in the form of BFKL equation + correction term, we move contour to the left, taking residue at $\xi = \gamma$. After that one can set $\varepsilon = 0$ and get

$$\sigma \frac{d}{d\sigma} D_g(\alpha, \gamma; \sigma) = \frac{\alpha_s N_c}{\pi} \times \Big[\chi(\gamma) D_g(\alpha, \gamma; \sigma) - \frac{\Gamma(1-\gamma)}{\Gamma(\gamma)} \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{d\xi}{2\pi i} (\alpha s \sigma)^{\gamma-\xi} \frac{\Gamma(\gamma-\xi)\Gamma(\xi)}{(\gamma-\xi)\Gamma(1-\xi)} D_g(\alpha, \xi; \sigma)\Big]$$

If we move contour to the right and take residue at $\xi = \gamma + \varepsilon$, we get the equation in the form "Sudakov+correction"

$$\varsigma \frac{d}{d\sigma} D_g(\alpha, \gamma; \sigma) = \frac{\alpha_s N_c}{\pi} \Big[-(\ln \alpha s \sigma + \gamma_E) D_g(\alpha, \gamma; \sigma) + \partial_\gamma D_g(\alpha, \gamma; \sigma) - \frac{\Gamma(1-\gamma)}{\Gamma(\gamma)} \int_{\frac{3}{4}-i\infty}^{\frac{3}{4}+i\infty} \frac{d\xi}{2\pi i} (\alpha s \sigma)^{\gamma-\xi} \frac{\Gamma(\gamma-\xi)\Gamma(\xi)}{(\gamma-\xi)\Gamma(1-\xi)} D_g(\alpha, \xi; \sigma)\Big]$$

Easy to see that the two equations are identical.