# Hadronic form factors in QCD and the incompleteness problem in the time-like region

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# Motivation

- Hadrons are not elementary
- Finite Size depends on the process
- Form factors characterize these properties as matrix elements

$$\langle p'|J(0)|p\rangle = F((p'-p)^2)$$

- QCD implies sum rules from analyticity high momenta behaviour
- Conclusions

# PION VECTOR FORM FACTOR

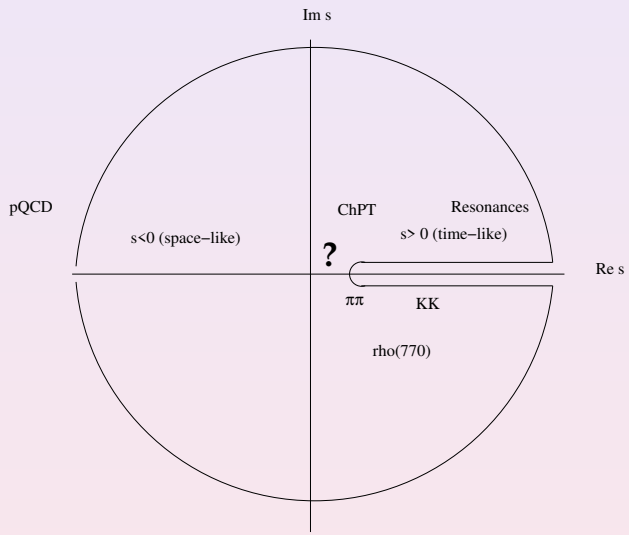
# Dispersion relations and sum rules

Example: pion vector form factor

$$e^-\pi^+ \to e^-\pi^+\,, \qquad \langle \pi^+(p')|J_3^\mu(0)|\pi^+(p)\rangle = F_\pi((p'-p)^2)(p'^\mu + p^\mu)\,, \qquad (p'-p)^2 < 0 \qquad \text{space-like}$$

$$e^+e^- \to \pi^+\pi^-\,, \qquad \langle \pi^+(-p')\pi^-(p)|J_3^\mu(0)|0\rangle = F_\pi((p'+p)^2)(-p'^\mu + p^\mu)\,, \qquad (p'+p)^2 > 4m_\pi^2 \qquad \text{time-like}$$

Analyticity: the two processes correspond to the same function in different domains of the complex plane




$$F(q^2) = F(q^2)^*\,, \quad q^2 < 0 \implies F(z^*) = F(z)^* \implies \text{Disc}F(q^2) = 2i\text{Im}F(q^2+i\epsilon)\,, \qquad q^2 > 4m_\pi^2$$

Unitarity cuts: line $q^2 > 4m_\pi^2 \implies$ Two Riemann sheets $F_{\rm I}(s)$ and $F_{\rm II}(s)$
Resonances:

$$F_{\rm II}(s) = S_{\rm II}(s)F_{\rm I}(s) \implies F_{\rm II}(s) \to \frac{Z_R}{s - m_R^2 + im_R\Gamma_R} + \ldots$$

# Large momentum behaviour (pQCD)

- Asymptotic space-like behaviour

$$F(-Q^2) \quad \to \quad \frac{16\pi F_\pi^2 \alpha_s(-Q^2)}{Q^2} = \frac{16\pi F_\pi^2}{Q^2} \frac{4\pi\beta_0}{\log(Q^2/\Lambda^2_{\rm QCD})}$$

  where $\beta_0 = (11N_c - 2N_f)/3$ and $\Lambda_{\rm QCD}$ is the QCD scale parameter

- Analytic continuation to time-like : $Q^2 \to e^{-i\pi} s \implies \log Q^2 \to -i\pi + \log s$

$$F(s) = \frac{64\pi^2 F_\pi^2/\beta_0}{s(\log(s/\Lambda^2_{\rm QCD}) - i\pi)} \implies {\rm Im}F(s) = -\frac{64\pi^2 F_\pi^2/\beta_0}{s} \frac{\pi}{(\log s/\Lambda^2_{\rm QCD})^2 + \pi^2)} < 0$$

- Unsubtracted Dispersion relations

$$F(-Q^2) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{{\rm Im}F(s)}{s + Q^2}$$

- Normalization

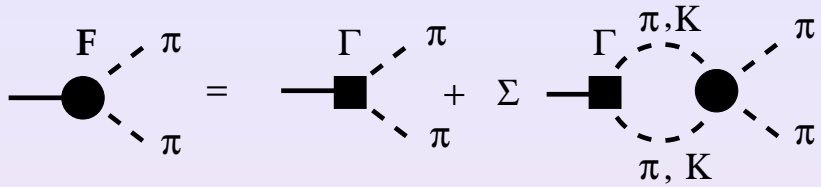
$$F(0) = 1 = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{{\rm Im}F(s)}{s}$$

- Superconvergent sum rule (Donoghue:1996bt)

$$Q^2 F(-Q^2) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds {\rm Im}F(s) + \mathcal{O}\left(\frac{1}{\log Q^2}\right) \implies \int_{4m_\pi^2}^{\infty} ds {\rm Im}F(s) = 0 \implies {\rm Im}F(s) \quad \text{changes sign}$$

# t-channel unitarity

- Bethe-Salpeter (coupled channel)




- Watson's theorem: $\pi\pi$ scattering in $J = I = 1$ channel

$$F(s) = |F(s)|e^{i\delta_{11}(s)} \implies \mathrm{Im}F(s) = |F(s)| \sin\delta_{11}(s) > 0\,, \qquad 4m_\pi^2 < s < 4m_K^2$$

- Threshold behaviour

$$\delta_{11}(s) \sim a_{11}(s/4 - m_\pi^2)^{\frac{3}{2}} \implies \mathrm{Im}F(s) \sim |F(4m_\pi^2)|a_{11}(s/4 - m_\pi^2)^{\frac{3}{2}}$$

- Omnes-Mushkelisvili solution in the spacelike region

$$F(-Q^2) = \exp\left[-\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{Q^2}{s}\frac{\delta_{11}(s)}{s+Q^2}\right] \sim \underbrace{\rightarrow}_{\Gamma_\rho \to 0} = \frac{m_\rho^2}{m_\rho^2 + Q^2}$$

- Question of modeling/using the spectral density

$$\rho(s) = \frac{1}{\pi}\mathrm{Im}F(s) = \begin{cases} \rho_{\mathrm{ChPT}}(s) & 4m_\pi^2 \le s \le 16m_\pi^2\,, \quad \text{threshold region} \\ \rho_{\mathrm{R}}(s) & 16m_\pi^2 \le s \le \Lambda^2_{\mathrm{pQCD}}\,, \quad \text{resonance region} \\ \rho_{\mathrm{pQCD}}(s) & \Lambda^2_{\mathrm{pQCD}} \le s \le \infty\,, \quad \text{pQCD region} \end{cases}$$

- What would be a reasonable $\Lambda_{\mathrm{pQCD}}$ ?

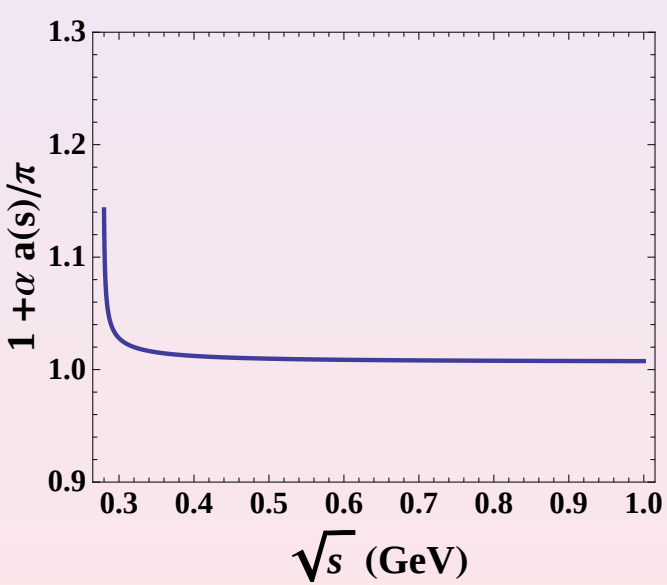
# The phase problem

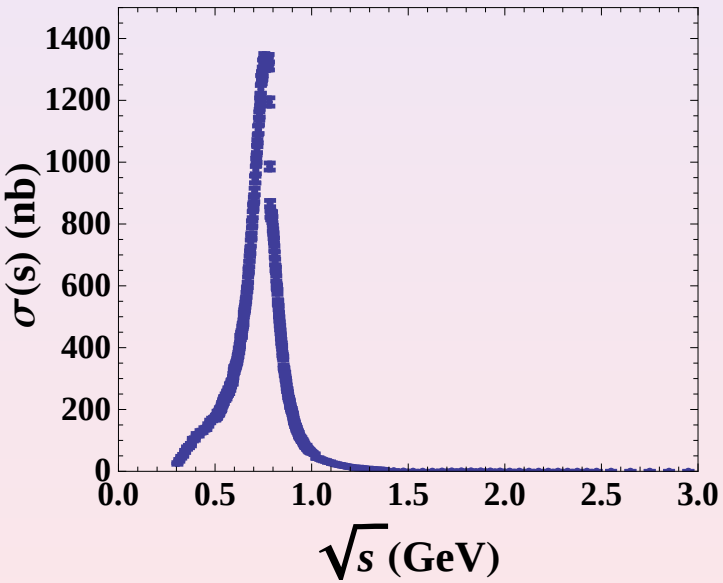
- Theoretically we have the cross sections

$$\sigma_{e^+e^-\to\pi^+\pi^-}(s) = \frac{\pi\alpha^2}{3s^{5/2}}\left(s-4m^2\right)\left(1+\frac{a(s)\alpha}{\pi}\right)|F(s)|^2$$

- Experimentally we have correlated $N = 300$ BABAR data (avoid d'Agostinni bias)

$$\sigma_i = \sigma_{e^+e^-\to\pi^+\pi^-}(s_i)\,, \quad \mathcal{C}(\sigma_i,\sigma_j)\,, \qquad 0.3\text{GeV} \le \sqrt{s} \le 3\text{GeV}$$




- We obtain the MODULUS

$$|F(s_i)| \pm \Delta|F(s_i)|\,, \qquad \mathcal{C}(|F(s_i)|,|F(s_j)|)$$

# Reconstruction of phase

- Modulus-phase representation

$$F(s) = |F(s)| e^{i\Delta(s)}$$

- Watson's theorem

$$\Delta(s) = \delta_{11}(s)\,, \qquad 2m_\pi \leq \sqrt{s} \leq 2M_K$$

- Inverse Omnes functions

$$\Delta(s) = -\frac{s\sqrt{s-s_{th}}}{\pi}\,\mathrm{PV}\int_{s_{th}}^{\infty} dz\,\frac{\ln|F_Q^\pi(z)|}{z\sqrt{z-s_{th}}(z-s)}\,, \tag{1}$$

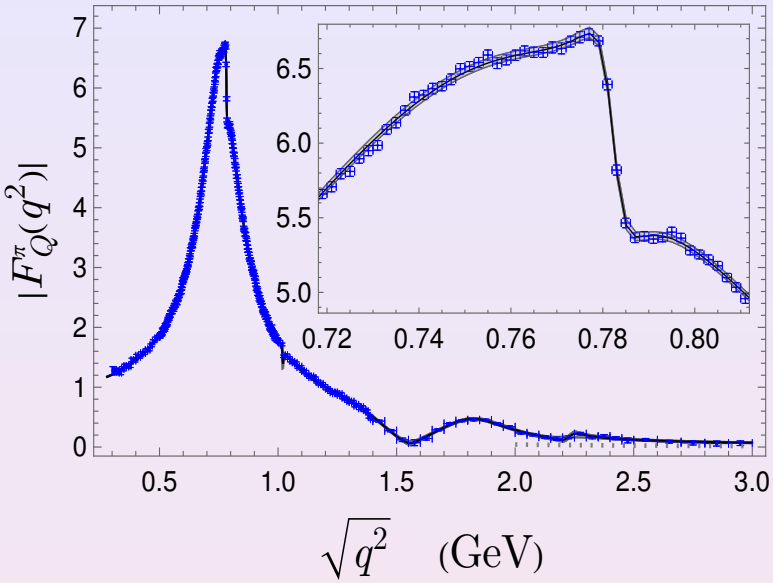
$$\Delta(s) = -\ln F_Q^\pi(s_{th})\left(\frac{s-s_{th}}{s_{th}}\right)^{3/2} - \frac{s(s-s_{th})^{3/2}}{\pi}\,\mathrm{PV}\int_{s_{th}}^{\infty} dz\,\frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}(z-s)}\,, \tag{2}$$

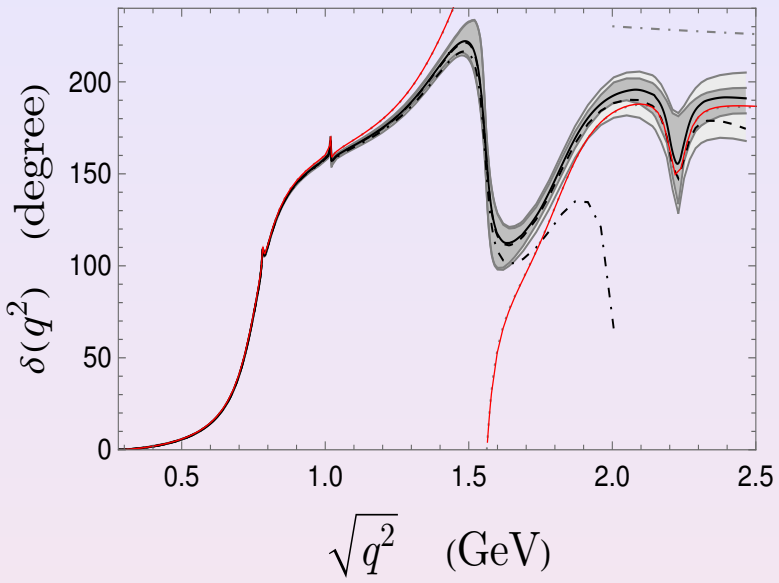
- Sum rule

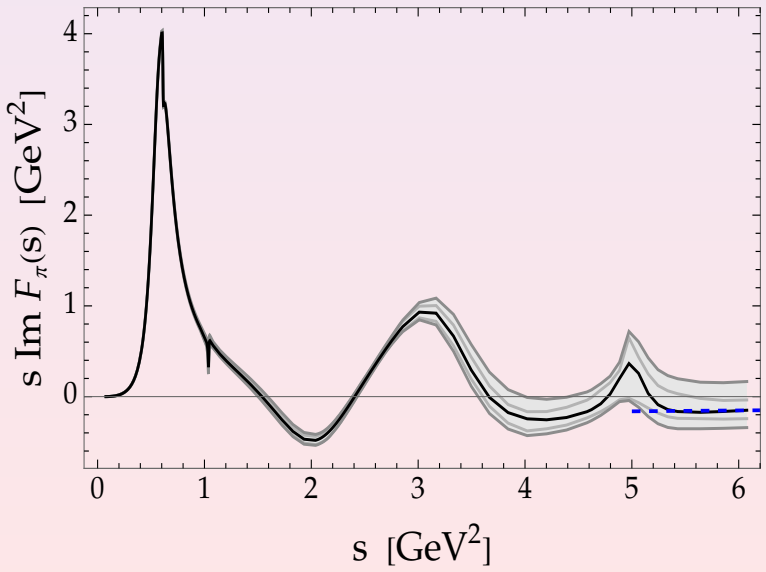
$$\frac{s_{th}^{3/2}}{\pi \ln F_Q^\pi(s_{th})}\int_{s_{th}}^{\infty} dz\,\frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}} = 1\,, \tag{3}$$

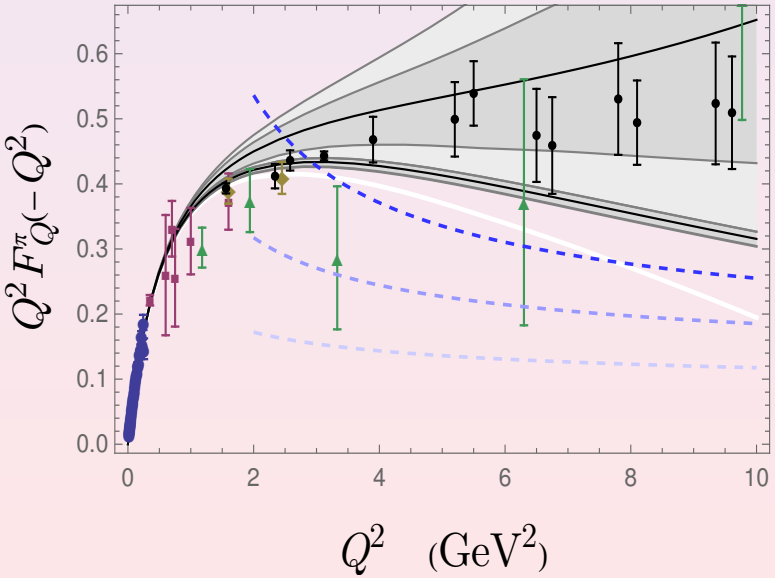
# Analysis of Babar

ERA, Pablo Sanchez-Puertas (RuizArriola:2024gwb) for $\sim 3m_\pi \leq \sqrt{s} \leq 3\text{GeV}$, DR: $|F(s)| \to \arg F(s)$



# Radial Regge trajectories from pion form factor

- Cross section $e^+e^- \to \pi^+\pi^-$

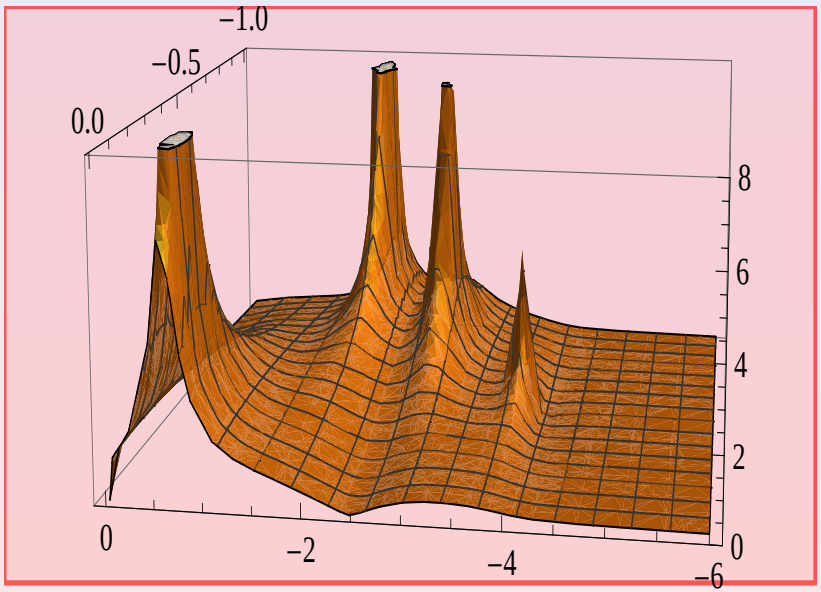
$$|F(s)|^2 = F(s+i\epsilon)F(s-i\epsilon) \equiv F_I(s+i\epsilon)F_{II}(s+i\epsilon)$$

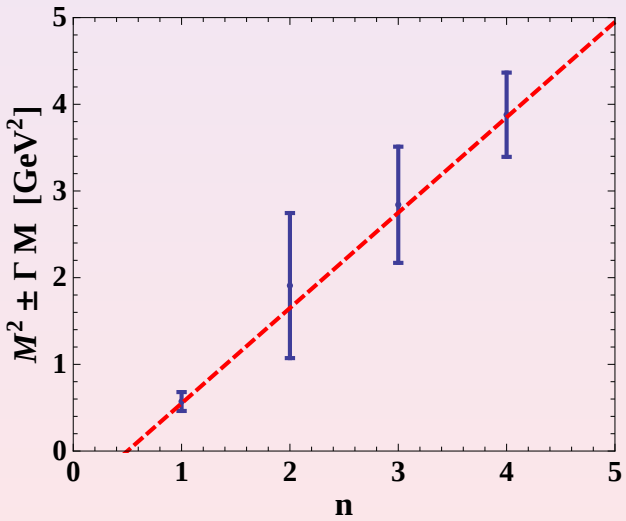
- Form factor in the second Riemann sheet (from Watson's theorem)

$$F_{II}(s) = S_{II}(s)F_I(s)$$

- Poles of form factor and S-matrix in the second Riemann sheet coincide

$$s_n = m_n^2 - i\Gamma_n m_n$$




- Linear Regge trajectories

$$M_{V,n}^2 = m_\rho^2(2n-1)$$

# The incompleteness problem I

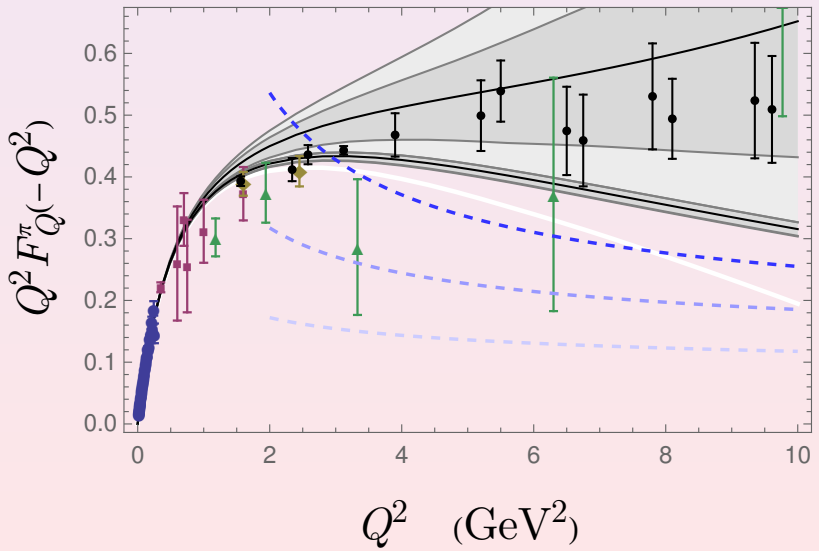
- The maximum Babar $s_{\rm max} = 9{\rm GeV}^2$

$$\frac{1}{\pi}\int_{s_0}^{s_{\rm max}} ds \frac{{\rm Im}\, F(s)}{s}\Big|_{\rm Data} = 1.01(1)_{\rm st}(^{+2}_{-1})_{\rm syst},$$

$$\frac{1}{\pi}\int_{s_0}^{s_{\rm max}} ds\, {\rm Im}\, F(s)\Big|_{\rm Data} = 0.63(2)_{\rm st}(^{+7}_{-4})_{\rm syst}\ {\rm GeV}^2. \qquad m_\rho^2 = 0.6{\rm GeV}^2$$

- The pQCD space-like contribution

$$F_Q^\pi(-t) \to \frac{16\pi F_\pi^2 \alpha_s}{t}\left(1 + 6.58\frac{\alpha_s}{\pi}\left[1 + 11.75\frac{\alpha_s}{\pi}\right]\right), \tag{4}$$



# The incompleteness problem II

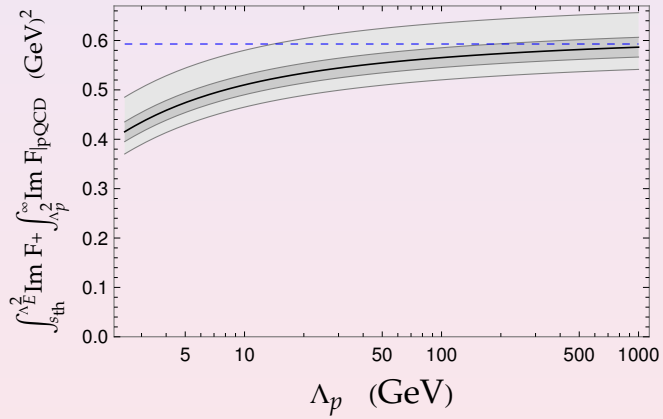
- The pQCD extrapolated to the time-like region

$$\mathrm{Im}F_Q^{\pi}(s) \to -\frac{16\pi F_\pi^2}{s}\mathrm{Im}\left\{\alpha_s(1++6.58\frac{\alpha_s}{\pi}\left[1+11.75\frac{\alpha_s}{\pi}\right]\right), \tag{5}$$

- pQCD contribution to Sum rules

$$\frac{1}{\pi}\int_{s_{\max}}^{\infty} ds \frac{\mathrm{Im}\,F(s)}{s}\Big|_{\mathrm{pQCD}} = -\underbrace{0.0025}_{\mathrm{LO}} - \underbrace{0.0011}_{\mathrm{NLO}} - \underbrace{0.0006}_{\mathrm{NNLO}},$$

$$\frac{1}{\pi}\int_{s_{\max}}^{\infty} ds\,\mathrm{Im}\,F(s)\Big|_{\mathrm{pQCD}} = -\underbrace{0.114}_{\mathrm{LO}} - \underbrace{0.030}_{\mathrm{NLO}} - \underbrace{0.013}_{\mathrm{NNLO}}\ \mathrm{GeV}^2.$$




- Flagrant violation of superconvergent sum rule UNLESS we fill the gap

$$0 = \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{\infty} ds\,\mathrm{Im}\,F(s) = \underbrace{\frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_{\max}} ds\,\mathrm{Im}\,F(s)}_{m_\rho^2} + \underbrace{\frac{1}{\pi}\int_{s_{\max}}^{s_{\mathrm{pQCD}}} ds\,\mathrm{Im}\,F(s)}_{-m_\rho^2+\epsilon} + \underbrace{\frac{1}{\pi}\int_{s_{\mathrm{pQCD}}}^{\infty} ds\,\mathrm{Im}\,F(s)}_{-\epsilon}$$

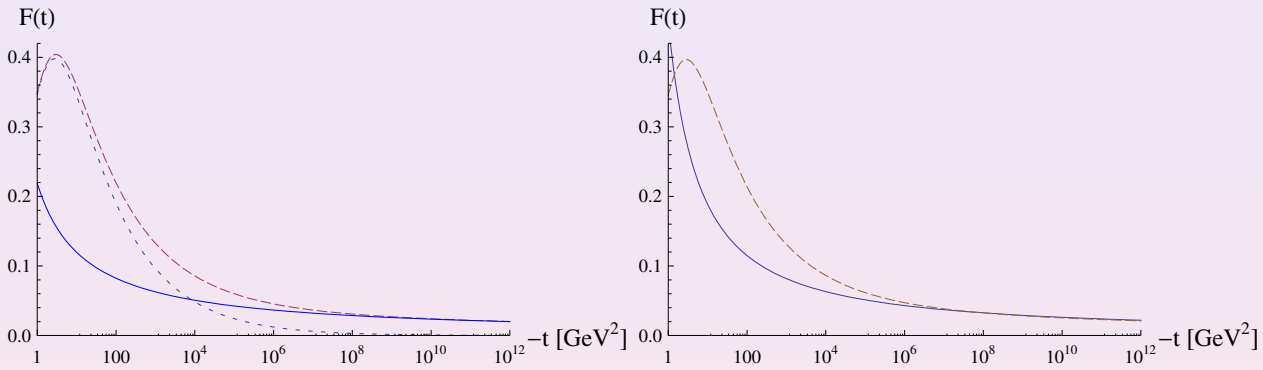
# The delayed onset of pQCD

- When does pQCD become valid ? (Early days $\Lambda_{\rm pQCD} \sim 2$GeV).
- We need Models from the measured region to asymptotics.

$$s_{\rm max} \leq s \leq s_{\rm pQCD} \equiv \Lambda^2_{\rm pQCD}$$

- Broniowski+ERA Phys.Rev.D 78 (2008) 034031 (Cosmological scales)

$$\Lambda_{\rm pQCD} \sim 10^4 {\rm GeV}$$




- Ananthanarayan+Caprini+Imsong ; Phys.Rev.D 85 (2012) 096006 $\Lambda^2_{\rm pQCD} > 7{\rm GeV}^2$
- ERA+Sanchez-Puertas+Weiss Phys.Lett.B 866 (2025) 139585

$$\mathrm{Im}F_R(s) = \mathrm{Im}F(s_R)\left(\frac{s_R}{s}\right)^{\epsilon} \Longrightarrow \mathrm{Im}F_R(\Lambda^2_{\rm pQCD}) = \mathrm{Im}F_{\rm pQCD}(\Lambda^2_{\rm pQCD}) \Longrightarrow \Lambda^2_{\rm pQCD} \sim 10^{10}{\rm GeV}^2$$

- Superconvergence is a theorem but pQCD is FAR AWAY
- Solution: subtractions (but need constants independently)

$$F(-Q^2) = 1 - Q^2 F'(0) + \frac{1}{\pi}\left[\int_{4m_\pi^2}^{s_{\rm max}} + \int_{s_{\rm max}}^{\infty}\right] ds \frac{Q^4}{s^2}\frac{\mathrm{Im}F(s)}{s+Q^2}\,, \qquad \text{Last term} \quad \mathcal{O}(Q^4/s^2_{\rm max})$$

# Extended meson dominance

- Generalized Current field identity

$$J_V^\mu = \sum_{\rho,\rho',\ldots} f_V M_V^2 V^\mu \implies F_V(t) = \sum_V c_V \frac{M_V^2}{M_V^2 - t}\,, \qquad c_V = f_V g_{V\pi\pi}$$

- Short distance constraints

$$F_V(t) \sim \frac{\sum_T c_T m_V^2}{Q^2} + \ldots$$

- Normalization

$$F_V(0) = 1 = \sum_T c_T$$

- Minimal hadronic ansatz

$$F_V(t) = \frac{m_\rho^2}{m_\rho^2 - t}$$

- Improved hadronic ansatz

$$F_V(t) = (1 + at)\frac{m_\rho^2}{m_\rho^2 - t}\frac{m_{\rho'}^2}{m_{\rho'}^2 - t}$$

- Infinitely many mesons (Dominguez 1980)

$$F_V(t) = \frac{B(\epsilon, \frac{M^2 - t}{a})}{B(\epsilon, \frac{M^2}{a})} \to \frac{\Gamma\left(\frac{M^2}{a} + \epsilon\right)}{\Gamma\left(\frac{M^2}{a}\right)}\left(\frac{a}{-t}\right)^\epsilon .$$

- Neither of these simple models fulfill the pQCD behaviour

# PION GRAVITATIONAL FORM FACTOR

# Pion GFF

- Matrix element of the stress-energy-momentum tensor

$$\langle \pi^a(p')|\Theta^{\mu\nu}(0)|\pi^b(p)\rangle = \delta_{ab}\left[2P^\mu P^\nu A(t) + \frac{1}{2}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)D(t)\right]$$

$a, b$ - isospin, $P = \frac{1}{2}(p' + p)$, $q = p' + p$, $t = q^2 = -Q^2$

- Trace form factor

$$\Theta^\mu_\mu \equiv \Theta(q^2) = 2\left(m_\pi^2 - \frac{q^2}{4}\right)A(q^2) - \frac{3}{2}q^2 D(q^2). \tag{6}$$

- Raman decomposition (Raman:1971jg) conserved irreducible tensors corresponding to well-defined total angular momentum, $J^{PC} = 0^{++}$ (scalar) and $2^{++}$ (tensor)

$$\Theta^{\mu\nu} = \Theta_S^{\mu\nu} + \Theta_T^{\mu\nu}\,, \quad \begin{cases} \Theta_S^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Theta \\ \Theta_T^{\mu\nu} = 2\left[P^\mu P^\nu - \frac{P^2}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\right]A \end{cases} \underbrace{\Longrightarrow}_{q^2\to 0} \begin{cases} A(0) = 1 \\ \Theta(0) = 2m_\pi^2 \end{cases} \tag{7}$$

- Since $\Theta$ and $A$ carry the information on good $J^{PC}$ channels, they should be regarded as the primary objects, whereas the $D$ -term form factor mixes the quantum numbers, with the explicit formula

$$D = -\frac{2}{3t}\left[\Theta - \left(2m_\pi^2 - \tfrac{1}{2}\,t\right)A\right]\,, \qquad D_\pi(0) = -1 + \mathcal{O}(m_\pi^2)\,, \qquad \text{(chiral theorem)}$$

# Dispersion relations

- pQCD

$$A(t) = -3D(t)\,(1+\mathcal{O}(\alpha)) = -\frac{48\pi\alpha(t)f_\pi^2}{t}\,(1+\mathcal{O}(\alpha)) \tag{8}$$

$$\Theta(t) = -4\beta_0\alpha(t)^2 f_\pi^2 + O(\alpha^3). \tag{9}$$

- Dispersion relations

$$\Theta(t) = \int_{4m_\pi^2}^{\infty} ds\,\frac{\mathrm{Im}\,\Theta(s)}{s-t}\,, \qquad A(t) = \int_{4m_\pi^2}^{\infty} ds\,\frac{\mathrm{Im}\,A(s)}{s-t}\,, \tag{10}$$

- Sum rules obtained from the vanishing at large $t$

$$2m_\pi^2 = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds\frac{\mathrm{Im}\,\Theta(s)}{s}\,, \qquad 1 = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds\frac{1}{s}\mathrm{Im}\,A(s),\,, \qquad 0 = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds\mathrm{Im}\,A(s)\,, \tag{11}$$

- $\mathrm{Im}\,A(s)$ and $\mathrm{Im}\Theta(s)$ must change sign.
- Watson's theorem in the elastic region

$$\mathrm{Im}\Theta(s) = |\Theta(s)|\,\sin\delta_{00}(s) > 0\,, \qquad \mathrm{Im}A(s) = |A(s)|\,\sin\delta_{02}(s) > 0\,, \tag{12}$$

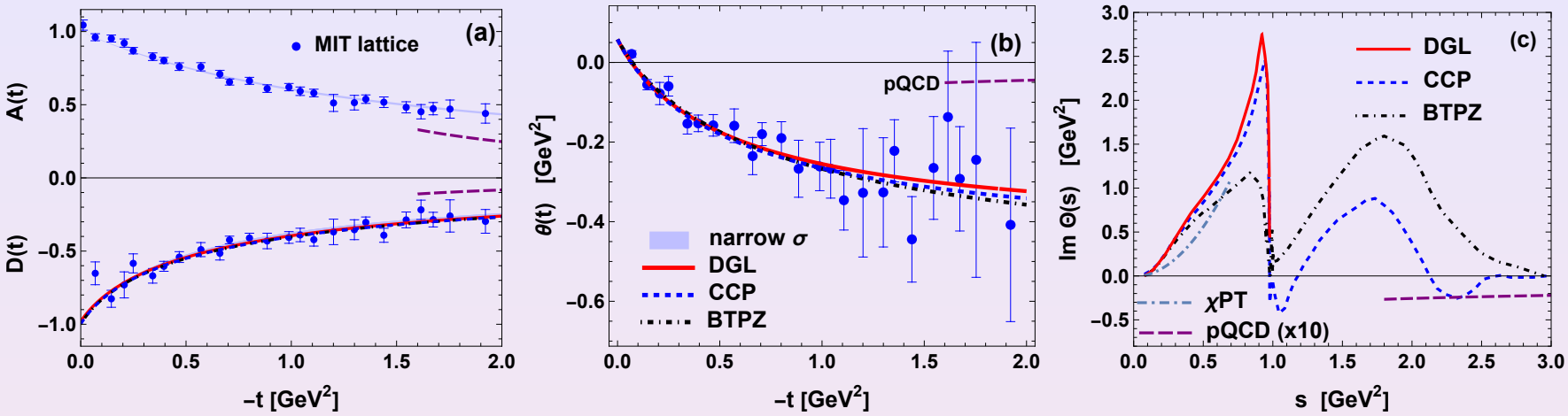
- pQCD after the analytic continuation $Q^2 \to e^{-i\pi}s$,

$$\frac{1}{\pi}\mathrm{Im}\,\Theta(s) = -\left(\frac{4\pi}{\beta_0}\right)^2\frac{4\beta_0 L f_\pi^2}{(L^2+\pi^2)^2} + \mathcal{O}(\alpha^3)\,, \qquad \frac{1}{\pi}\mathrm{Im}\,A(s) = -\left(\frac{4\pi}{\beta_0}\right)\frac{48\pi f_\pi^2}{s(L^2+\pi^2)} + \mathcal{O}(\alpha^2), \tag{13}$$

with $L = \log s/\Lambda^2_{\mathrm{QCD}}$, which is manifestly negative.

# Incompleteness for pion and kaon GFF's

- Lattice MIT data




- Meson dominance (after exptrapolation to physical point)

$$A(-Q^2) = \frac{m_{f_2}^2}{m_{f_2}^2+Q^2}\,, \qquad \Theta(-Q^2) = 2m_\pi^2 - \frac{m_\sigma^2\, Q^2}{m_\sigma^2+Q^2} \Longrightarrow \begin{cases} m_\sigma = 630(60)\text{MeV} \\ m_{f_2} = 1270(40)\text{MeV} \end{cases}$$

- Roy solutions and sum rules for Pion and Kaon e-Print: 2507.05375 (F.K. Guo et al)

$$\frac{1}{\pi}\int_{s_\pi}^{\infty} ds\ \text{Im}\, A^{\pi,K}(s) = 0 \Longrightarrow \begin{cases} \frac{1}{\pi}\int_{s_\pi}^{4m_N^2} ds\ \text{Im}\, A^{\pi}(s) = 1.86\text{GeV}^2 \\ \frac{1}{\pi}\int_{s_\pi}^{4m_N^2} ds\ \text{Im}\, A^{K}(s) = 0.77\text{GeV}^2 \end{cases}, \qquad (14)$$

- Strong violations of superconvergence sum rules

# NUCLEON FORM FACTORS

# The incompleteness problem for nucleon form factors

- Roy-Steiner analyses contain all (important) coupled channels, $\pi\pi$, $K\bar{K}$ below $N\bar{N}$ threshold and implement unitarity and crossing BELOW a certain scale which acts as a high energy cut-off.
- Asymptotics pQCD for nucleons, dispersion relations, sum rules

$$F(-Q^2) \to f_N^2 \left[\frac{\alpha(Q^2)}{Q^2}\right]^2 \Longrightarrow F(t) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{\mathrm{Im}F(s)}{s-t} ds \Longrightarrow \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{Im}F(s) s^{n-1} ds = c_n \delta_{n,0}$$

  $c_n = 1$ for conserved currents
- Isovector-vector form factor (n=0) Eur.Phys.J.A 52 (2016) 11, 331 (Hoferichter et al.)

| | $\Lambda = 1$ | $\Lambda = 2$ | exact |
|---|---|---|---|
| $G_E^v(0)$ | 0.76(8) | 0.68(11) | 0.5 |
| $G_M^v(0)$ | 3.34(25) | 3.21(30) | 2.35 |

Table: Sum rules for form-factor normalizations and radii, evaluated with integral cutoff $\Lambda$. No superconvergence sume rules included

- Gravitational form factors e-Print: 2507.05375 (Guo et al.)

Table:

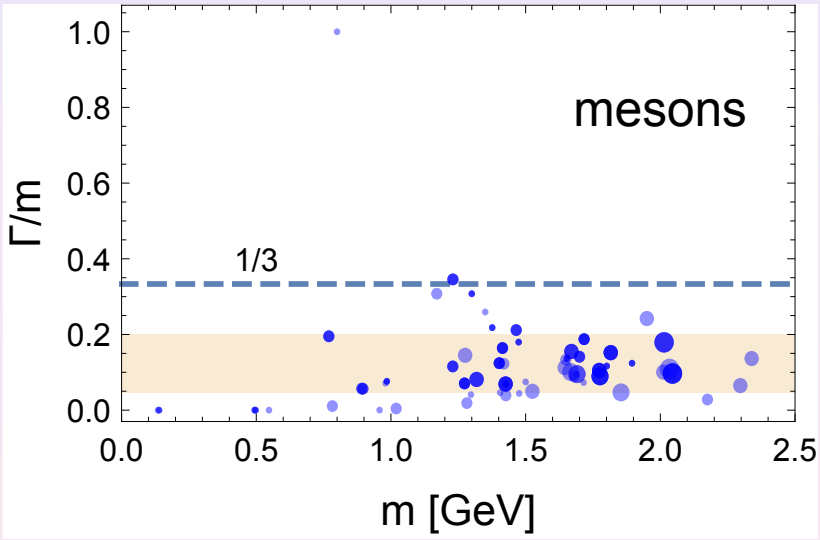
| | $n=0$ | | | $n=1$ | | | $n=2$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $\Lambda$ | 1.3 GeV | $2m_N$ | exact | 1.3 GeV | $2m_N$ | exact | 1.3 GeV | $2m_N$ | exact |
| $A$ | 1.35 | 1.77 | 1 | 1.79 | 2.58 | 0 | 2.49 | 4.04 | 0 |
| $2J$ | 1.19 | 1.42 | 1 | 1.43 | 1.88 | 0 | 1.91 | 2.76 | 0 |
| $\Theta/m_N$ | 1.32 | 2.60 | 1 | 0.67 | 3.71 | 0 | 0.59 | 8.09 | 0 |

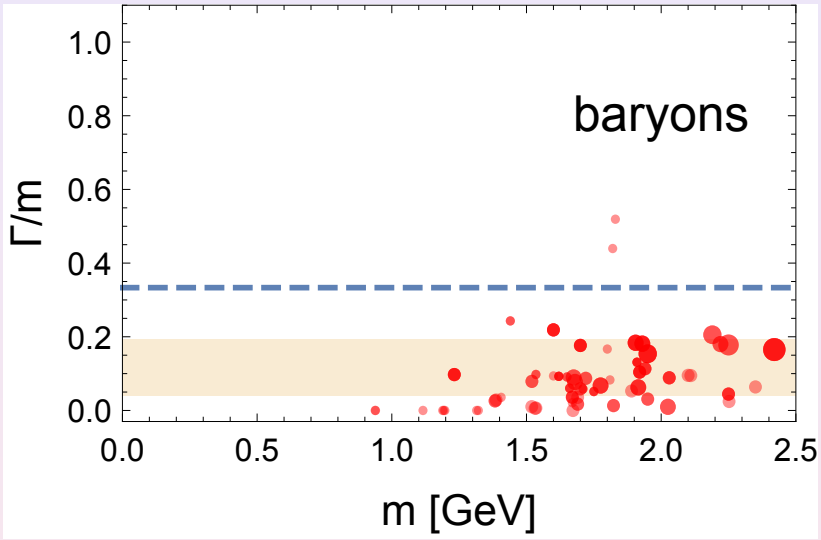
# LARGE $N_c$ REGGE MODELS

# The Suranyi Width/Mass ratio

In the large $N_c$ limit

$$\begin{cases} m = \mathcal{O}(N_c^0) & \Gamma = \mathcal{O}(N_c^{-1}) \\ m = \mathcal{O}(N_c) & \Gamma = \mathcal{O}(N_c^0) \end{cases} \quad \Longrightarrow \quad \Gamma/m = 1/N_c$$

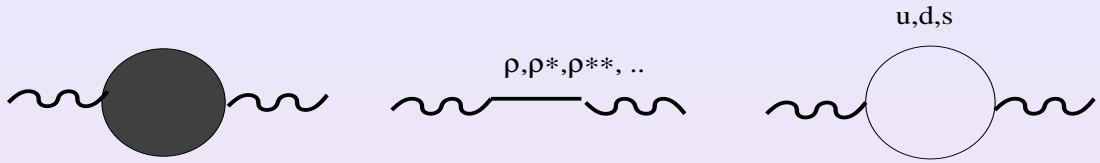



$$\frac{\Gamma}{m} \equiv \sum_{J,\alpha} (2J+1) \frac{\Gamma_{J,\alpha}}{m_{J,\alpha}} = 0.12(8) \tag{15}$$

In average resonances are even narrower than the naive large $N_c$ limit suggests.

# Radial Regge spectrum from Quark-hadron duality




- Vector-Vector and Axial-Axial correlators

$$\Pi_V^{\mu a,\nu b}(q) = i\int d^4x e^{-iq\cdot x}\langle 0|T\left\{J_V^{\mu a}(x)J_V^{\nu b}(0)\right\}|0\rangle = \Pi_V^T(q^2)\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\delta^{ab},$$

$$\Pi_A^{\mu a,\nu b}(q) = i\int d^4x e^{-iq\cdot x}\langle 0|T\left\{J_A^{\mu a}(x)J_A^{\nu b}(0)\right\}|0\rangle = \Pi_A^T(q^2)\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\delta^{ab} + \Pi_A^L(q^2)\,q^\mu q^\nu$$

with $J_{V,A}^{\mu a} = \bar\psi i\gamma^\mu\{1,\gamma_5\}\frac{\tau^a}{2}\psi$ QCD currents. At large $Q$

$$\Pi_{V+A}^T(Q^2) = \frac{1}{4\pi^2}\left\{-\frac{N_c}{3}\log\frac{Q^2}{\mu^2} + \frac{\pi}{3}\frac{\langle\alpha_S G^2\rangle}{Q^4} + \frac{256\pi^3}{81}\frac{\alpha_S\langle\bar qq\rangle^2}{Q^6}\right\} + \dots,$$

$$\Pi_{V-A}^T(Q^2) = -\frac{32\pi}{9}\frac{\alpha_S\langle\bar qq\rangle^2}{Q^6} + \dots, \qquad (16)$$

# Narrow Meson resonances

- Saturating with narrow resonances

$$\Pi_V^T(Q^2) = \sum_{n=0}^{\infty} \frac{F_{V,n}^2}{M_{V,n}^2 + Q^2} + c.t.,$$

$$\Pi_A^T(Q^2) = \frac{f_\pi^2}{Q^2} + \sum_{n=0}^{\infty} \frac{F_{A,n}^2}{M_{A,n}^2 + Q^2} + c.t.,$$

- Spectral function

$$\mathrm{Im}\Pi_V^T(s) \to \sum_{n=0}^{\infty} F_{V,n}^2 \delta(M_{V,n}^2 - s) \sim \lim_{n\to\infty} \frac{F_{V,n}^2}{dn/dM_{V,n}^2}$$

$$\mathrm{Im}\Pi_V^T(s) \to \frac{N_c}{24\pi^2}\,\mathrm{Im}\log(-s) = \frac{N_c}{24\pi}$$

# The radial Regge spectrum for light quarks

- Two scalar quarks in the CM frame

$$M = 2\sqrt{p^2 + m^2} + \sigma r \to 2m + \frac{p^2}{m} + \sigma r + \dots \tag{17}$$

- Bohr-Sommerfeld quantization $L = 0$ and $m = 0$

$$2\int_0^{M/\sigma} p_r dr = 2\pi(n + \alpha) \tag{18}$$

- Radial Regge spectrum for large $n$ Anisovich, Anisovich, '02

$$M_n^2 = \mu^2 n + M_0^2 \qquad \mu^2 = 1.25(15)\mathrm{GeV}^2 \tag{19}$$

# Regge phase-space

- For large meson masses, the level density is given by

$$\begin{aligned}\rho(M^2) &= \sum_n \delta(M^2 - M_n^2) \to \int dn \delta(M^2 - M_n^2) \\ &= \frac{1}{dM_n^2/dn}\Big|_{M_n^2=M^2} = \frac{dn}{dM^2} = \frac{1}{2\pi\sigma}\end{aligned} \tag{20}$$

- So, using the WKB approximation we get

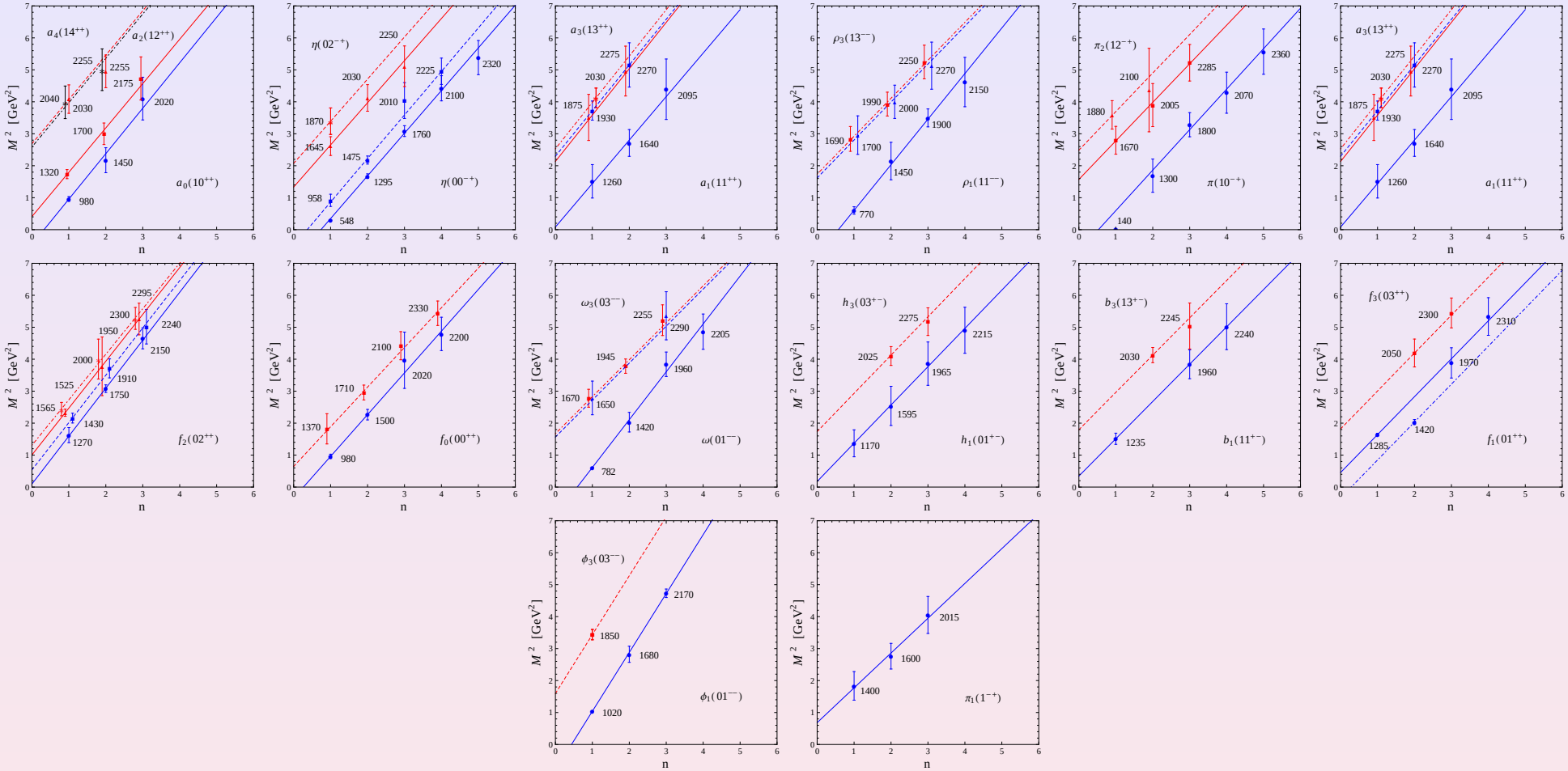
$$\frac{dn}{dM^2} = \frac{1}{\pi}\int_0^a \frac{dr}{p_r} = \frac{1}{2\pi\sigma} \tag{21}$$

- Finite quark mass corrections

$$\frac{dn}{dM^2} = \frac{1}{2\pi\sigma}\sqrt{1 - \frac{4m^2}{M^2}}\,\theta(M^2 - 4m^2) \tag{22}$$

- Two body phase space factor = Absortive part of two point correlators for FREE PARTICLES.

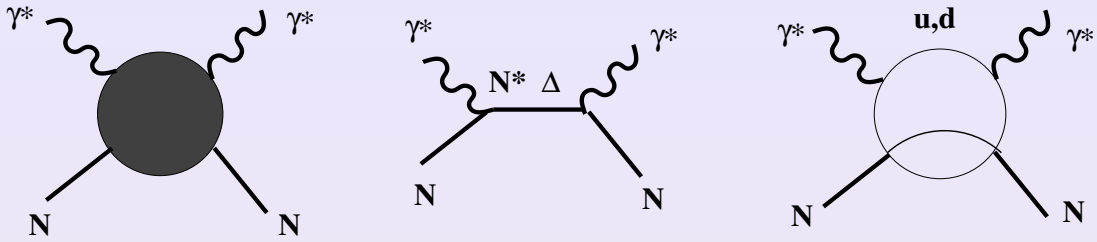
# Empirical Radial Regge Spectrum for Mesons




- The **half-width rule**: $\Delta M_R = \Gamma_R/2$ or $\Delta M_R^2 = M_R \Gamma_R$

$$M^2 = 1.38(4)n + 1.12(4)J - 1.25(4)\,.$$

# Deep inelastic scattering




- Forward Compton scattering

$$W^{ab}_{\mu\nu}(p,q;s) = \frac{1}{4\pi}\int d^4x e^{iq\cdot\xi}\left\langle p,s\left|[J^a_\mu(\xi),J^{b\dagger}_\nu(0)]\right|p,s\right\rangle \tag{23}$$

$$= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)M_N W_1(\nu,Q^2)+\left(p_\mu - q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu\frac{p\cdot q}{q^2}\right)\frac{1}{M_N}W_2(\nu,Q^2)$$

$$= \sum_R \langle N|J_\mu|R\rangle\langle R|J_\nu|N\rangle\delta((p+q)^2-M_R^2) \tag{24}$$

- Bjorken limit:

$$Q^2\to\infty \qquad \text{with} \qquad x = Q^2/2p\cdot q \quad \text{fixed}\,.$$

- Scaling and Spin 1/2 (parton model)

$$M_N W_1(x,Q^2)\ \to, F_1(x) \quad \text{and} \quad \frac{p\cdot q}{M_N}W_2(x,Q^2)\ \to F_2(x) \qquad F_2(x) = 2xF_1(x)$$

# Narrow resonance approximation

- Invariant mass

$$W^2 = (p+q)^2 = M_N^2 + Q^2\left(\frac{1}{x} - 1\right)$$

- Bjorken limit

$$W_2 = \sum_R \left[G_{N\to R}(q^2/M_R^2)\right]^2 \delta(W^2 - M_R^2) \to \int d\mu^2 \sum_i \left[G_{N\to i}(-Q^2/\mu^2)\right]^2 \frac{dn}{d\mu^2} \to \quad (25)$$

$$F_2(x) = \sum_i \left[G_{N\to i}(x/(1-x))\right]^2 \frac{dn}{2\mu^2} \quad (26)$$

- Scaling limit

$$M_R^2 = \mu^2 n + M^2 \qquad G_{N\to R}(q) \equiv F(q^2/M_R^2)$$

- Drell-Yang relation

$$G_{N\to R}(-Q^2) \to \frac{1}{Q^n} \qquad F_2(x) \to (1-x)^{n-1}$$

# The radial Regge spectrum for quark-diquark states

- Quark-diquark in the CM frame

$$M = \sqrt{p^2 + m_D^2} + p + \sigma r \tag{27}$$

- Bohr-Sommerfeld quantization $L = 0$

$$2\int_0^{M/\sigma} p_r dr = 2\pi(n+\alpha) \rightarrow \frac{dn}{dM^2} = \frac{1}{2\pi\sigma}(1 - \frac{m_D^2}{M^2}) \tag{28}$$

- Radial Regge spectrum for large $n$

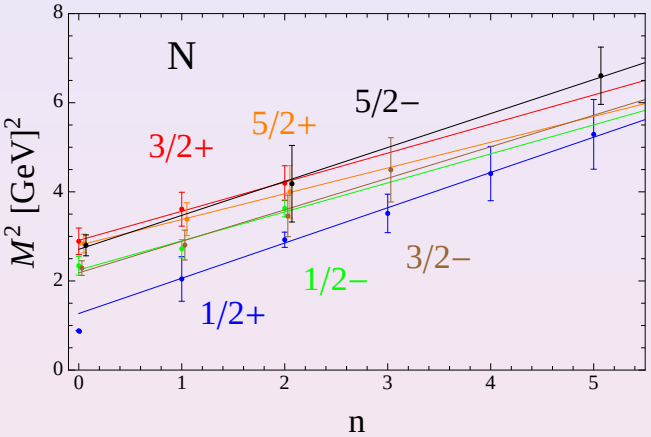
$$M_n^2 = \mu^2 n + M_0^2 \tag{29}$$

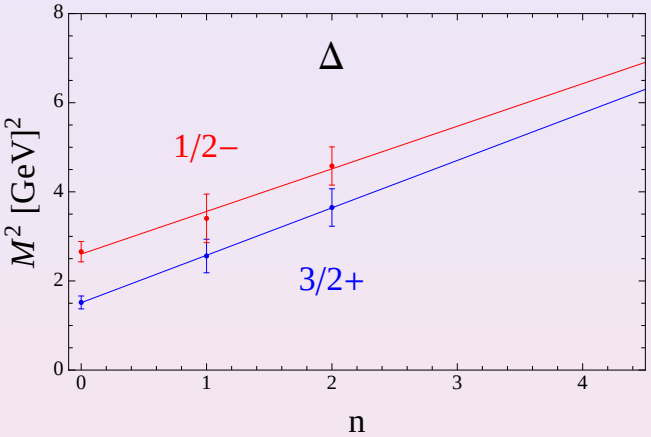
- Quark-Hadron duality in DIS requires Quark-Diquark dynamics !
- Problem of missing resonances $\rightarrow$Quark model with qqq states predicts more states than observed

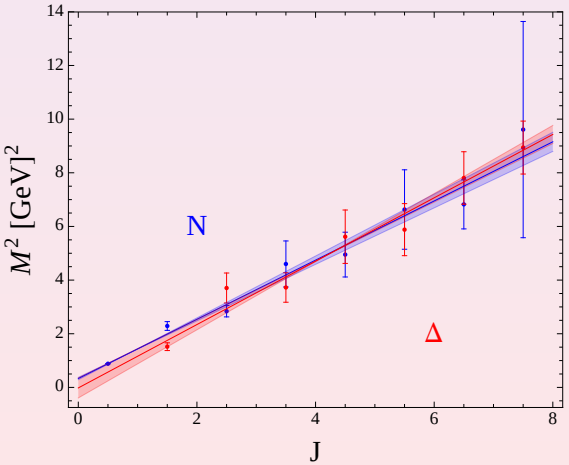
# Regge Fits

Radial Regge trajectory parameterized as:

$$M^2_{n,J} = a + \mu^2 n + \beta J \tag{30}$$

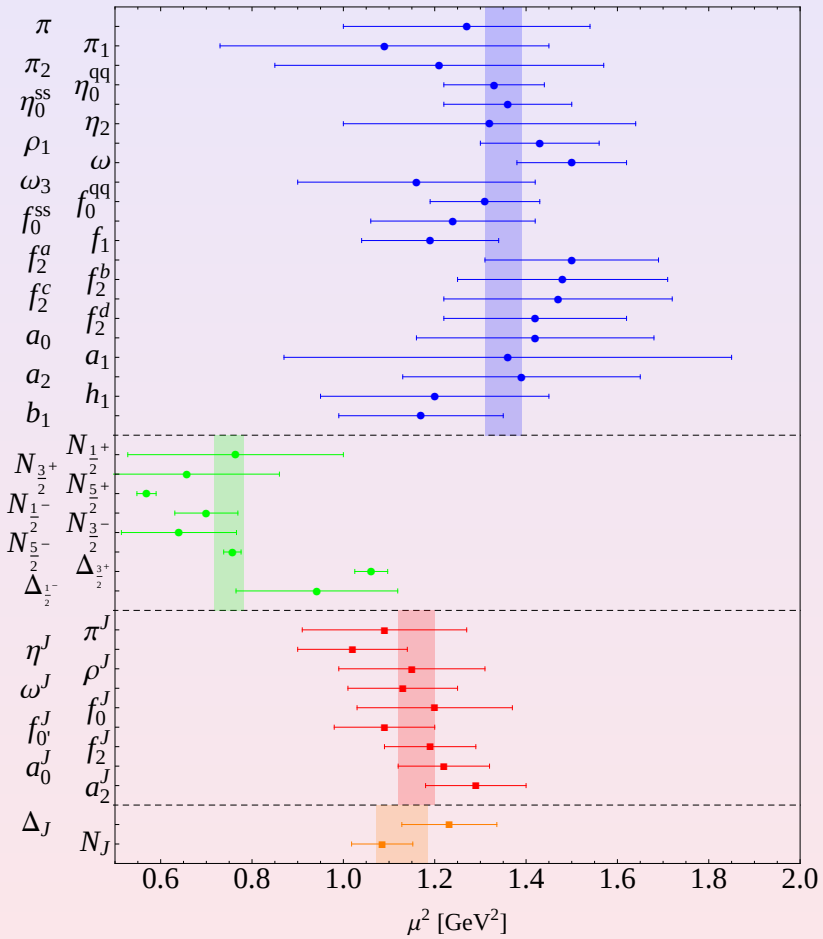


# Summary of Regge trajectories from PDG

$$M^2_{n,J} = a + \mu^2 n + \beta J \tag{31}$$



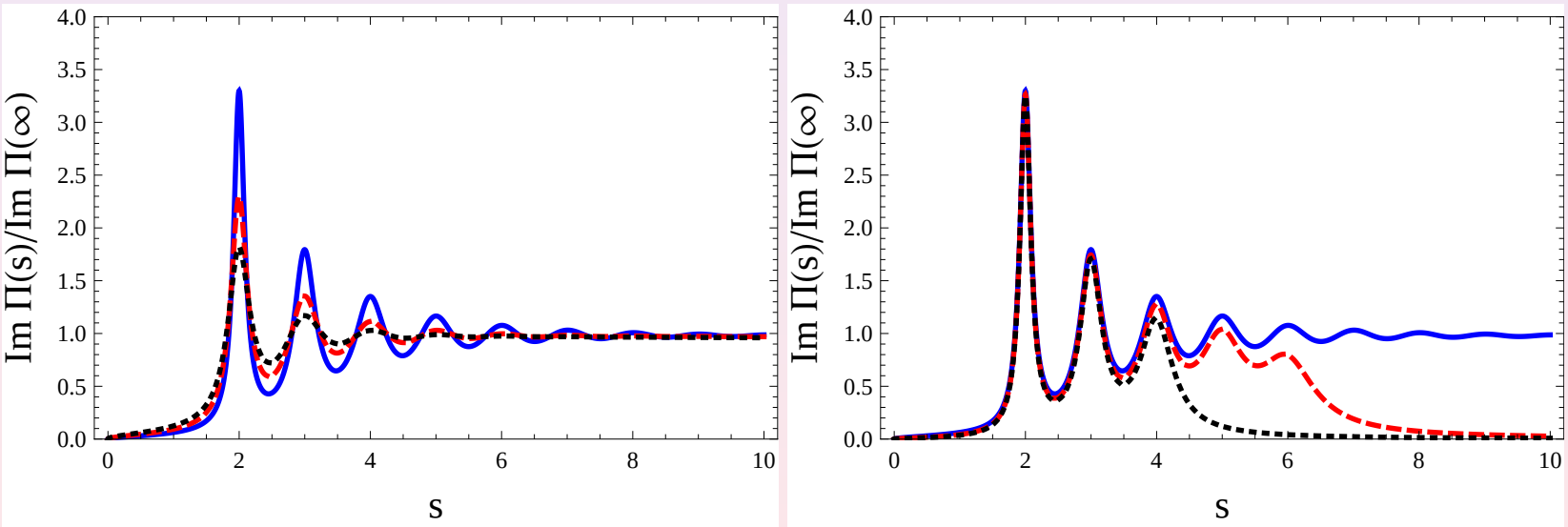
# Width effects

- Radial Regge spectrum + Suranyi ratio

$$M_n^2 = n + 1\,, \qquad \Gamma_n = \gamma M_n\,, \qquad \gamma = 0.11, 0.12, 0.13$$

- Correlation function

$$\mathrm{Im}\Pi(s) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{M_n\Gamma_n}{(M_n^2-s)^2+\Gamma_n^2M_n^2} \underbrace{\rightarrow}_{s\gg 1} \frac{1}{\pi}\int_0^\infty \frac{\gamma n}{(n-s)^2+\gamma^2 n^2} \underbrace{\rightarrow}_{n=sx} \frac{1}{\pi}\int_0^\infty \frac{\gamma}{x^2+\gamma^2} \rightarrow 1+\mathcal{O}(\gamma)$$




- Transition between separate and overlapping resonances

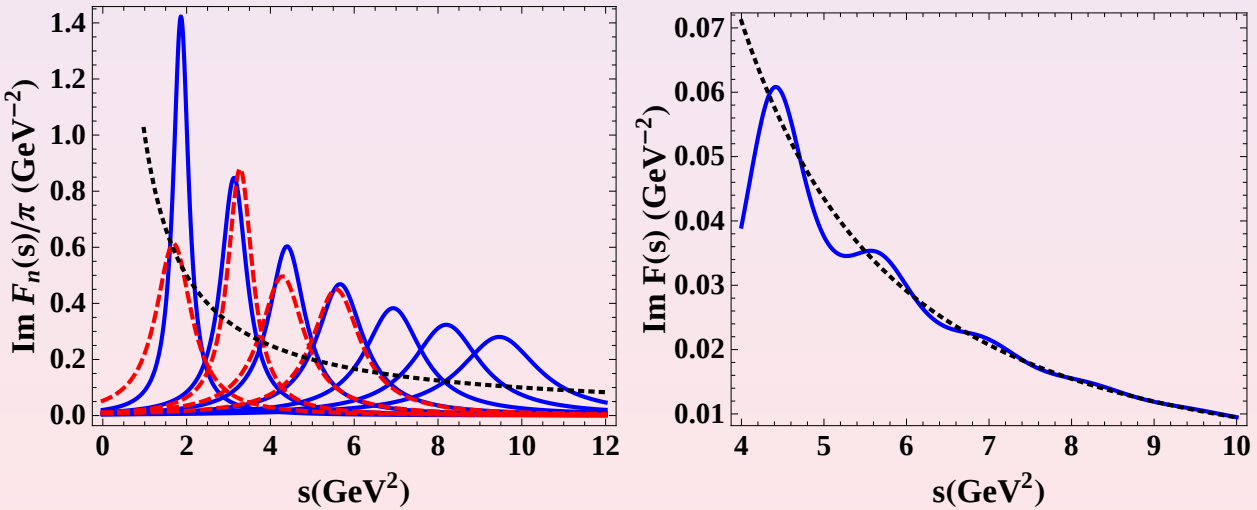
# Form factors smoothing

- Breit-Wigner profiles

$$\text{Im}\, F(s) = \sum_n c_n \,\text{Im}\, F_n(s) = \sum_n c_n \frac{\gamma M_n^2}{(s - M_n^2)^2 + \gamma^2 M_n^4} \,. \tag{32}$$

- For large $s$, we may replace the sum in $n$ by an integral, so that

$$\begin{aligned} \text{Im}\, F(s) \quad &\to \quad \int dn c_n \frac{\gamma M_n^2}{(s - M_n^2)^2 + \gamma^2 M_n^4} = \int \frac{dM^2}{a} c_{M^2/a} \frac{\gamma M^2}{(s - M^2)^2 + \gamma^2 M^4} \\ &= \quad \int dx \frac{1}{a} c_{s/ax} \frac{\gamma x}{(1 - x)^2 + \gamma^2 x^2} \,. \end{aligned} \tag{33}$$




- The oscillations dissapear after the first resonances !

# The Regge-pQCD transition

- Pion

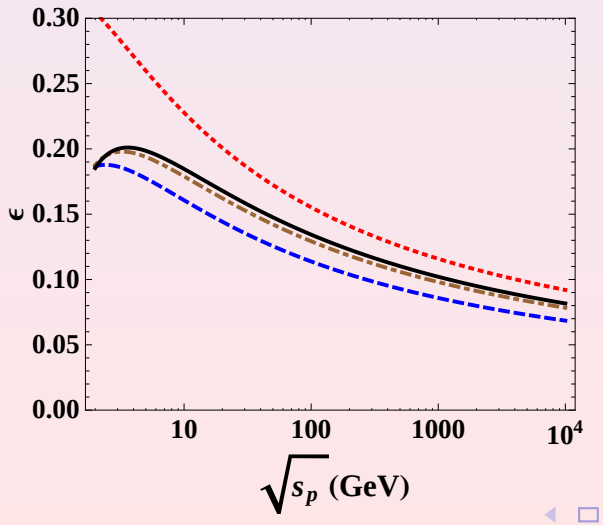
$$\text{Im}\, F(s) = \begin{cases} \text{Im}\, F(\Lambda_R^2) \left(\frac{\Lambda_R^2}{s}\right)^{1+\epsilon}, & \Lambda_R^2 \le s \le \Lambda_{\text{pQCD}}^2 \\ \frac{64\pi^2 F_\pi^2}{\beta_0 s} \text{Im}\alpha_s(s), & \Lambda_{\text{pQCD}}^2 \le s \end{cases} \quad \rightarrow \frac{d}{ds} \log \text{Im}\alpha_s(s) = -1 - \epsilon$$

- Nucleon

$$\text{Im}\, F(s) = \begin{cases} \text{Im}\, F(\Lambda_R^2) \left(\frac{\Lambda_R^2}{s}\right)^{2+2\epsilon}, & \Lambda_R^2 \le s \le \Lambda_{\text{pQCD}}^2 \\ \frac{f_N^2}{s^2} \text{Im}\alpha_s(s)^{2+\gamma}, & \Lambda_{\text{pQCD}}^2 \le s \end{cases} \quad \rightarrow \frac{d}{ds} \log \text{Im}\alpha_s(s)^2 = -2 - 2\epsilon$$

- Solutions

$$\epsilon_\pi = \frac{2 \log(\Lambda_{\text{pQCD}}^2/\Lambda_{\text{QCD}}^2)}{\log^2(\Lambda_{\text{pQCD}}^2/\Lambda_{\text{QCD}}^2) + \pi^2}, \qquad \epsilon_N = \epsilon_\pi - \frac{1}{2 \log(\Lambda_{\text{pQCD}}^2/\Lambda_{\text{QCD}}^2)}. \tag{34}$$



# Nucleon amplitudes

Why logarithmic matching ?
Nucleon form factors

$$F(t) \to \left[\frac{f_N \alpha}{t}\right]^2 , \qquad f_N = \int dxdy \Psi[x]\Psi[y] , \qquad f_N(\mu) = f_N(\mu_0)\left(\frac{\alpha_S(\mu)}{\alpha_S(\mu_0)}\right)^{2/3\beta_0} .$$

Table: The pQCD prediction (in GeV$^4$ units and modulo $\alpha_s^2/Q^4$ overall factor $f_N$) for different DAs in the literature

| | Latt | KS | CZ | COZ | SB | BK | BLW | ABO1/2 |
|---|---|---|---|---|---|---|---|---|
| $P/S$ | $-0.04(^{+0.40}_{-0.09})$ | 21 | 14 | 15 | 24 | $-0.04$ | 0.08 | 1(0.4)/0.07(30) |
| $P/S_u$ | $-0.15(^{+0.11}_{-0.16})$ | 18 | 13.4 | 14 | 11 | $-0.18$ | 0.05 | 0.6(5)/ $-$ 0.08(30) |
| $P/S_d$ | $-0.11(^{+0.00}_{-0.40})$ | $-3$ | $-0.5$ | $-0.7$ | $-13$ | $-0.14$ | $-0.03$ | $-0.4(3)/-0.16(20)$ |
| $A_u$ | $-0.06(^{+0.38}_{-0.00})$ | 21.1 | 15.2 | 15.6 | 22 | 0.13 | 0.30 | 1.1(5)/0.22(22) |
| $A_d$ | $-0.04(^{+0.19}_{-0.04})$ | $-0.4$ | $-0.5$ | $-0.7$ | $-7$ | $-0.5$ | $-0.253$ | $-0.15(10)/-0.28(9)$ |
| $V_u$ | $0.01(^{+0.40}_{-0.01})$ | 22 | 15.5 | 16.2 | 23 | 0.5 | 0.5 | 1.3(5)/0.48(25) |
| $V_d$ | $0.10(^{+0.30}_{-0.05})$ | 2 | 0.3 | 0.6 | 10 | 0.3 | 0.12 | 0.4(2)/0.24(10) |

The numerical value of the amplitude is VERY model dependent

# SUMMARY, APPLICATIONS AND CONCLUSIONS

# General idea

- Dispersion relation with a cut in $s \in \mathbb{R}^+$

$$F(t) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\mathrm{Im} F(s)}{s - t}$$

- Reflection principle

$$F(t) = F^*(t)\,, \qquad t \in \mathbb{R}^- \implies F(s^*) = F(s)^* \qquad s \in \mathbb{C} - \mathbb{R}^-$$

- pQCD behaviour

$$F(t) \to \frac{f\alpha(t)}{t}\,, \qquad F(t) \to \left(\frac{f_N \alpha(t)}{t}\right)^2$$

- Normalization (conserved current)

$$F(0) = \frac{1}{\pi} \int_{s_0}^{\infty} ds\, \frac{\mathrm{Im} F(s)}{s} = 1$$

- Superconvergent Sum rules

$$\int_{s_0}^{\infty} ds \mathrm{Im} F(s) = 0\,, \qquad \int_{s_0}^{\infty} ds s \mathrm{Im} F(s) = 0$$

- Regge interpolation

$$\mathrm{Im}\, F(s) = \mathrm{Im}\, F(\Lambda_R^2) \left(\frac{\Lambda_R^2}{s}\right)^{n+n\epsilon}\,, \quad \Lambda_R^2 \leq s \leq \Lambda^2_{\mathrm{pQCD}}\,. \tag{35}$$

- pQCD becomes tiny but $\epsilon \sim 0.1 - 0.2$ (systematic uncertainty)

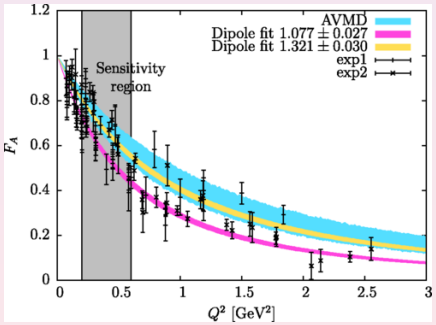
# Axial nucleon form factor

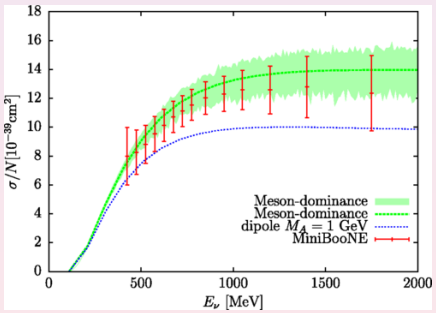
- Controversy Lattice QCD vs MINERVA analysis
- Axial-vector dominance (minimal ansatz)

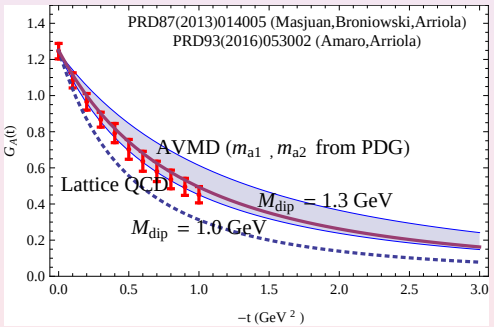
$$G_A(t) = g_A \frac{m_{a_1}^2}{m_{a_1}^2 - t} \frac{m_{a_2}^2}{m_{a_2}^2 - t}$$

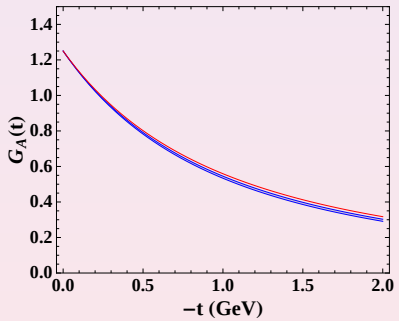
- Adding Regge + 3 sum rules

$$G_A(t) = g_A \left\{ Z_1 \frac{m_{a_1}^2}{m_{a_1}^2 - t} + Z_2 \frac{m_{a_1}^2}{m_{a_1}^2 - t} \right\} + \frac{1}{\pi} \int_{s_R}^{\infty} \mathrm{Im} G_A(s_R) \left[ \frac{s_R}{s} \right]^{2+2\epsilon} \frac{ds}{s - t}$$



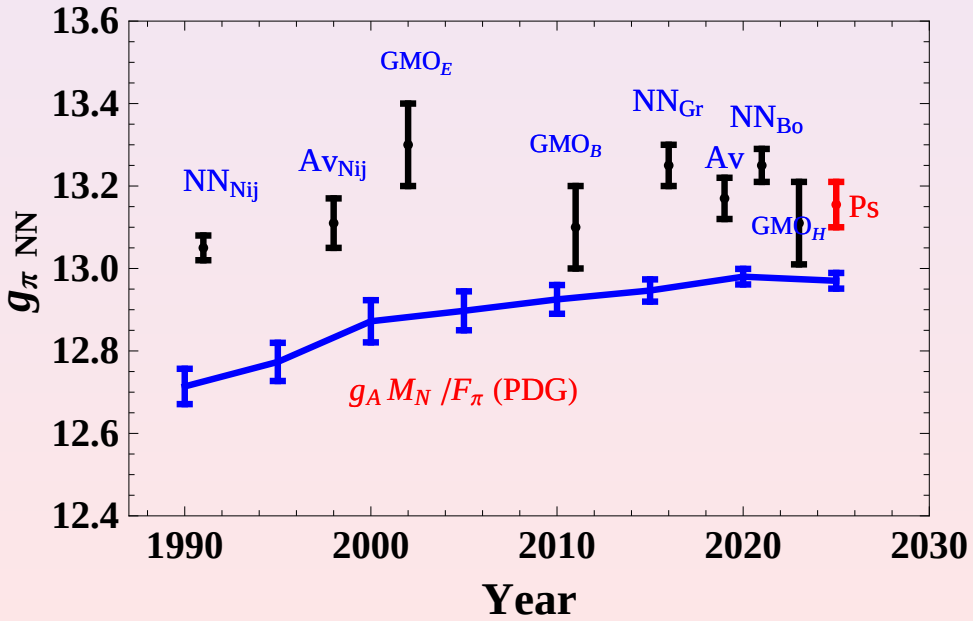

- Our results are compatible with lattice QCD
- Regge effects are tiny

# Goldberger-Treiman discrepancy from PS dominance

- Goldberger–Treiman discrepancy $\Delta_{\rm GT} = 1 - \frac{m_N g_A}{F_\pi g_{\pi NN}}$,
- Results for the charged channel (with isospin breaking estimate)

$$g_{\pi^+ pn} = 13.14(^{+6}_{-4})7_{\rm IB}, \quad \Delta_{\rm GT} = 1.26(^{+51}_{-34})(50)_{\rm IB}\%,$$

- To be compared with
  1. The most precise determinations, $g_{\pi^+ np} = 13.25(5)$ (and hence $\Delta_{\rm GT} = 2.1(4)\%$), from *np*, *pp* scattering analysis of the Granada-2013 database and
  2. $g_{\pi^+ pn} = 13.11(10)$, $\Delta_{\rm GT} = 1.0(7)\%$ from the GMO sum rule.



# Conclusions

- Dispersive methods have a reputation of being rigorous until it comes to practical applications where a high energy cut-off becomes explicit and influential.
- Most analyses ignore QCD superconvergence sum rules which are flagantry violated.
- The infinite radial Regge tower of states provides a rationale where the smallness of the high energy tail becomes natural and simple.
- This result is highly independent on the onset of pQCD and provides a systematic uncertainty which is rather competitive in the studied applications.
- Typical decomposition of spectral function

$$\rho(s) = \frac{1}{\pi}\mathrm{Im}F(s) = \begin{cases} \rho_{\mathrm{ChPT}}(s) & 4m_\pi^2 \leq s \leq 16m_\pi^2\,, \quad \text{threshold region} \\ \rho_{\mathrm{R}}(s) & 16m_\pi^2 \leq s \leq \Lambda_R^2\,, \quad \text{separate resonance region} \\ \rho_{\mathrm{Reg}}(s) & \Lambda_R^2 \leq s \leq \Lambda_{\mathrm{pQCD}}^2\,, \quad \text{Regge region} \\ \rho_{\mathrm{pQCD}}(s) & \Lambda_{\mathrm{pQCD}}^2 \leq s \leq \infty\,, \quad \text{pQCD region} \end{cases}$$

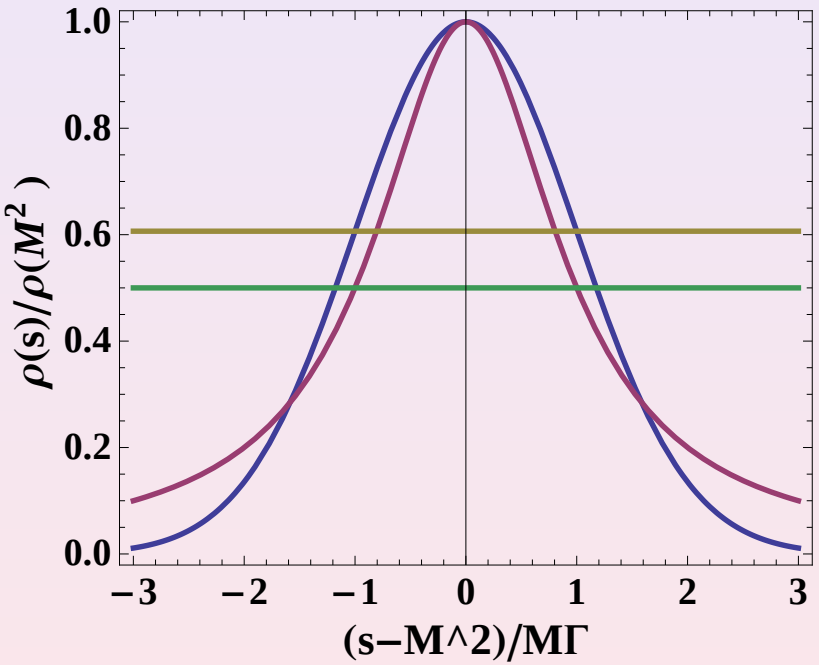
# The half-width rule

- Amplitude = Background + Resonance
- A BW resonance profile

$$\rho_R(s) = \frac{1}{\pi}\frac{M\Gamma}{(M^2 - s)^2 + \Gamma^2 M^2}\,, \qquad \int \rho(s) = 1$$

- Realistically

$$\rho(s) = \rho_B(s) + Z_R \rho_R(s)$$




- Probabilistic interpretation

$$\frac{\rho(M_R^2 \pm \Gamma_R M_R)}{\rho(M_R^2)} = \frac{1}{2}$$

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