Effective parton splitting functions in dense QCD media

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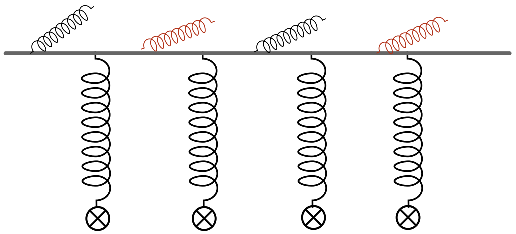
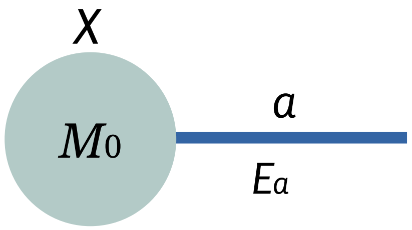
INTRODUCTION TO THE FORMALISM



Splitting functions

Parton splitting functions constitute the basic building block for describing the branching of a parton a to partons $b + c$, with E_b = **z** E_a and E_c = (1-**z**) E_a



$$\mathrm{d}\sigma_{X\to a} = |\mathcal{M}_0|\mathrm{d}\Phi_a$$

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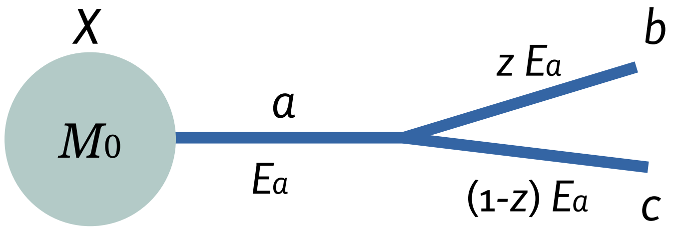

$$\mathrm{d}\sigma_{X\to a\to bc} = \underbrace{|\mathcal{M}_0|\mathrm{d}\Phi_a}_{\mathrm{d}\sigma_{X\to a}} \times \underbrace{\frac{\alpha_S}{2\pi}\frac{\mathrm{d}k_T^2}{k_T^2}\frac{\mathrm{d}z}{z}[zP_{b\leftarrow a}(z)]}_{\mathrm{d}N_{a\to bc}}$$

(LO, collinear limit)

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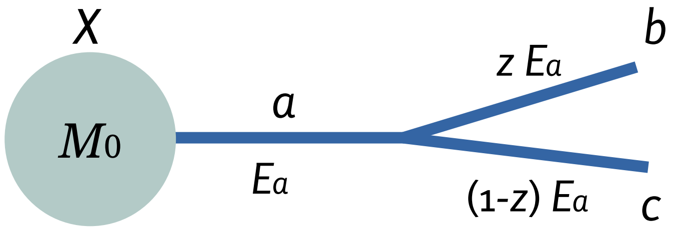

DGLAP splitting functions

$$\mathrm{d}\sigma_{X\to a\to bc} = \underbrace{|\mathcal{M}_0|\mathrm{d}\Phi_a}_{\mathrm{d}\sigma_{X\to a}} \times \underbrace{\frac{\alpha_S}{2\pi}\frac{\mathrm{d}k_T^2}{k_T^2}\frac{\mathrm{d}z}{z}[zP_{b\leftarrow a}(z)]}_{\mathrm{d}N_{a\to bc}}$$

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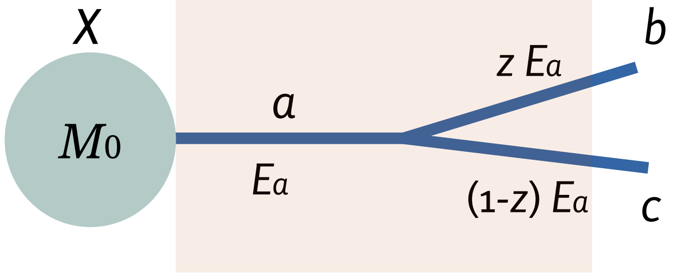


Colored medium (e.g. QGP)

In-medium splitting functions

$$\mathrm{d}\sigma_{X\to a\to bc} = \underbrace{|\mathcal{M}_0|\mathrm{d}\Phi_a}_{\mathrm{d}\sigma_{X\to a}} \times \underbrace{\frac{\alpha_S}{2\pi}\frac{\mathrm{d}k_T^2}{k_T^2}\frac{\mathrm{d}z}{z}\left[z P^{(\mathrm{med})}_{b\leftarrow a}(z,k_T)\right]}_{\mathrm{d}N^{(\mathrm{med})}_{a\to bc}}$$

(LO, collinear limit)

In this factorized picture, how to find effective in-medium parton splitting functions?

Formalism (BDMPS-Z)

__Assumptions__

i) near-eikonal parton in longitudinal direction: $p^- \ll |\boldsymbol{p}_\perp| \ll p^+ \quad x^+ \simeq t \simeq z$

→ Relevant dynamics for quenching on **transverse** plane

ii) medium = **gluon background field:** short-range scattering interactions, gaussian statistics

$$\langle A^{-a}(\boldsymbol{r},t)A^{-b}(\boldsymbol{r}',t')\rangle = n(t)\delta^{ab}\delta(t-t')\gamma(\boldsymbol{r}-\boldsymbol{r}')$$

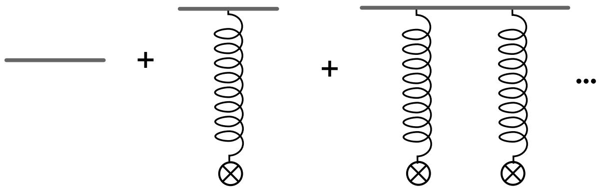
$$\boxed{\gamma(\boldsymbol{x}) = \int_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}^2\boldsymbol{q}}}$$

elastic scattering rate

iii) medium with finite size *L* in the longitudinal (x^+) direction vacuum

= $G(p) \sim \dfrac{1}{\not{p} - m}$ regular QCD Feynman rules

medium

=  $= \mathcal{G}_{ab}(t, t_0|\boldsymbol{x}, \boldsymbol{x}_0)$

$$\mathcal{G}_{ab}(t, t_0|\boldsymbol{x}, \boldsymbol{x}_0) = \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \mathcal{D}\bar{\boldsymbol{x}} \exp\left(i \int \mathrm{d}s \frac{p^+}{2} \dot{\bar{\boldsymbol{x}}}(s)^2\right) U_{R,ab}(L, t_0|\bar{\boldsymbol{x}})$$

Transverse space broadening

Wilson line

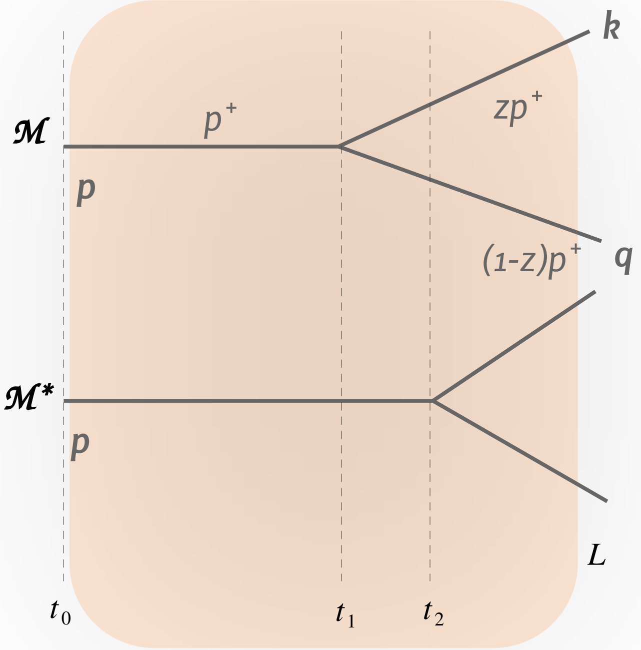
color rotations

$$U_R = \exp\left(ig \int_{t_0}^{t} \mathrm{d}s A^c(s) t_R^c\right)$$

→ In the medium, the propagators written in a mixed (t+*x* or t+*p*) representation

This is crucial so as to keep track the evolution of the parton over the distance traveled in the medium

In-medium cross-section



$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_k \mathrm{d}\Omega_q} \sim S^{(2)}(t_0,t_1)S^{(3)}(t_1,t_2)S^{(4)}(t_1,L)$$

(n)-point correlators of $\mathcal{G}_{ab}$

$$\int_{\boldsymbol{p}'} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\boldsymbol{q}} \mathrm{d}\Omega_{\boldsymbol{k}}} = \frac{\mathrm{d}\sigma_0}{\mathrm{d}p^+} \frac{\mathrm{d}N}{\mathrm{d}z\,\mathrm{d}^2\boldsymbol{p}},$$

$$\omega = z(1-z)p^+$$

$$\frac{\mathrm{d}N}{\mathrm{d}z\,\mathrm{d}^2\boldsymbol{p}} = \frac{\alpha_s}{4\pi^2}\frac{P_{ba}(z)}{\omega^2} \mathrm{Re}\int_{\boldsymbol{u}\bar{\boldsymbol{u}}} e^{-i\boldsymbol{p}\cdot(\boldsymbol{u}-\bar{\boldsymbol{u}})}\mathcal{F}(L,\boldsymbol{u},\bar{\boldsymbol{u}}),$$

Double time integration of path integrals of Wilson line correlators

To facilitate the computation, we can write $\dfrac{\mathrm{d}N}{\mathrm{d}z\mathrm{d}k_T^2} = \dfrac{\mathrm{d}N_{\mathrm{vac}}}{\mathrm{d}z\mathrm{d}k_T^2}(1 + r_{\mathrm{in-in}} + r_{\mathrm{in-out}})$

equivalently: $P_{ba}^{(\mathrm{med})}(z,\boldsymbol{k}_T;L) = r(k_T,z)\,P_{ba}^{(\mathrm{vac})}(z)$

$$\frac{\mathrm{d}N}{\mathrm{d}z\mathrm{d}k_T^2} = \frac{\mathrm{d}N_{\mathrm{vac}}}{\mathrm{d}z\mathrm{d}k_T^2}(1 + r_{\mathrm{in-in}} + r_{\mathrm{in-out}})$$

Isaksen & Tywoniuk ([1], 2023):

[1] JHEP 09 (2023) 049

The problem of finding the ratio components with **exact transverse kinematics** (indirectly computing the Wilson line correlators) can be formulated through evolution equations

$$i\partial_t \mathcal{K}(\boldsymbol{u}, \boldsymbol{u}_1; L, t_0) + \left(\frac{\nabla_{\boldsymbol{u}}^2}{2\omega} - iv_{ab}(\boldsymbol{u})\right)\mathcal{K}(\boldsymbol{u}, \boldsymbol{u}_1; L, t_0) = i\delta^{(2)}(\boldsymbol{u} - \boldsymbol{u}_1)\delta(L - t_0)$$

$$i\partial_L \mathcal{F}_\sigma(L|\boldsymbol{u}, \bar{\boldsymbol{u}}) = \left(-\frac{\nabla_{\boldsymbol{u}}^2 - \nabla_{\bar{\boldsymbol{u}}}^2}{2\omega}\delta_\sigma^{\sigma'} + i\mathbb{M}_\sigma^{\sigma'}(\boldsymbol{u}, \bar{\boldsymbol{u}})\right)\mathcal{F}_{\sigma'}(\boldsymbol{u}, \bar{\boldsymbol{u}}, L) + \mathcal{S}_\sigma(\boldsymbol{u}, \bar{\boldsymbol{u}}, L)$$

(4+1)d PDE

$$\mathcal{S}_\sigma = -i\,\nabla_{\bar{\boldsymbol{u}}}\delta(\bar{\boldsymbol{u}}) \cdot \nabla_{\boldsymbol{u}_1}\int_0^L \mathrm{d}t_0\, \mathcal{K}(\boldsymbol{u}, \boldsymbol{u}_1; L, t_0)|_{\boldsymbol{u}_1=0}$$

$$\tilde{\mathcal{F}}_\sigma(\boldsymbol{p}, t) = \int_{\boldsymbol{u}, \bar{\boldsymbol{u}}} \mathcal{F}_\sigma(\boldsymbol{u}, \bar{\boldsymbol{u}}, t)e^{-i\boldsymbol{p}\cdot(\boldsymbol{u}-\bar{\boldsymbol{u}})}$$

$$r_{\textbf{in-out}} = \mathrm{Re}\left[\int_0^L \mathrm{d}s \int_{\boldsymbol{u}} e^{-i\boldsymbol{u}\cdot\boldsymbol{p}}\boldsymbol{p}\cdot\nabla_{\boldsymbol{u}_1}\mathcal{K}(\boldsymbol{u}, \boldsymbol{u}_1; L, s)|_{\boldsymbol{u}_1=0}\right]$$

$$r_{\textbf{in-in}} = \frac{\boldsymbol{p}^2}{2\omega}\mathrm{Re}\,\tilde{\mathcal{F}}_{\sigma=1}(\boldsymbol{p}; L).$$

State of the art

→ Results available for q → qg at small **z** (BMDPS-Z limit) and large N_c

→ Available finite *z* are computed in a strong **straight-line assumption** of transverse kinematics (provides simple analytical formulas)

→ True finite z results were extracted from previous equation for very limiting cases: low energies, small media, HO potentials (w/ analytical Kernel) with RK4 methods (!)

Should we care?

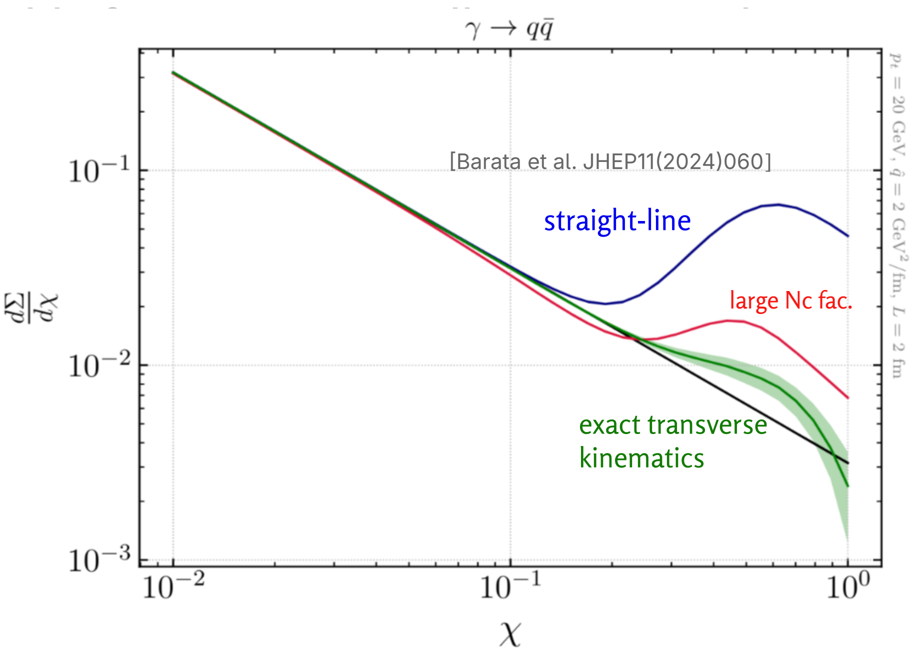
State of the art

→ Results avail ... arge Nc

→ Available fin ... mption of transverse kinematics

→ True finite z ... or very limiting cases: low energies, small ... RK4 methods (!)

Should we



YES!

straight-line approximation is not satisfactory

This work

→ Provide a stable, precise and efficient algorithm to extract in-medium contributions for a wide range of p^+, medium length, parameters, ...

→ advanced time-integration kernel, work on evolution equations

→ Results for leading vertices $q \to qg$ and $g \to gg$

→ Analyse limits of large N_c

→ Analyse non-straight-line corrections in transverse kinematics

→ Go beyond the HO (on-going)

Extraction of parton splitting functions

Methodology and preliminary results

Equations: transformed

$$i\partial_L \mathcal{F}_\sigma(L|\boldsymbol{u},\bar{\boldsymbol{u}}) = \left(-\frac{\nabla^2_{\boldsymbol{u}} - \nabla^2_{\bar{\boldsymbol{u}}}}{2\omega}\delta^{\sigma'}_\sigma + i\mathbb{M}^{\sigma'}_\sigma(\boldsymbol{u},\bar{\boldsymbol{u}})\right)\mathcal{F}_{\sigma'}(\boldsymbol{u},\bar{\boldsymbol{u}},L) + \mathcal{S}_\sigma(\boldsymbol{u},\bar{\boldsymbol{u}},L)$$

$$\tilde{\mathcal{F}}_\sigma(\boldsymbol{p},t) = \int_{\boldsymbol{u},\bar{\boldsymbol{u}}} \mathcal{F}_\sigma(\boldsymbol{u},\bar{\boldsymbol{u}},t)e^{-i\boldsymbol{p}\cdot(\boldsymbol{u}-\bar{\boldsymbol{u}})}$$

$$r_{\textbf{in-in}} = \frac{\boldsymbol{p}^2}{2\omega}\mathrm{Re}\,\tilde{\mathcal{F}}_{\sigma=1}(\boldsymbol{p};L).$$

This formulation has serious issues, the main one being the necessity to extract high momentum Fourier modes out of a 4D function! **Proposed solution:** momentum space + reworking coordinates

$$\int_0^L \mathrm{d}t_0 \int_{\boldsymbol{u}} e^{-i\boldsymbol{u}\cdot\boldsymbol{p}}\,\nabla_{\boldsymbol{u}_1}\mathcal{K}(\boldsymbol{u},\boldsymbol{u}_1;L,t_0)|_{\boldsymbol{u}_2=0} \equiv \boldsymbol{J}(L,\boldsymbol{p})$$

$$i\partial_t \boldsymbol{J}(t,\,\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2\omega}\boldsymbol{J}(t,\boldsymbol{p}) + i(\tilde{v}_{ba} * \boldsymbol{J})(t,\boldsymbol{p}) - \boldsymbol{p},$$

$$r_{\textbf{in-out}} = \frac{1}{\omega}\mathrm{Re}\left[\boldsymbol{p}\cdot\boldsymbol{J}(\boldsymbol{p})\right],$$

$$i\partial_t \mathcal{F}_\sigma(t,\boldsymbol{k},\boldsymbol{l}) = \left[\frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\omega}\delta^{\sigma'}_\sigma + i\left(\tilde{\mathbb{M}}^{\sigma'}_\sigma * \cdot\right)\right]\mathcal{F}_{\sigma'}(t,\boldsymbol{k},\boldsymbol{l}) + (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{J}(\boldsymbol{k}+\boldsymbol{l})\;1_\sigma.$$

$$r_{\text{in-in}} = \frac{\boldsymbol{k}_T^2}{2\omega}\mathrm{Re}\,\hat{\mathcal{F}}_{\sigma=1}(\boldsymbol{k}_T,\boldsymbol{l}=0)$$

Numerical time evolution

$$i\partial_t g_a(t,\boldsymbol{K}) = [\hat{\mathcal{H}}(t,\boldsymbol{K})]_{ab}\, g_b(t,\boldsymbol{K}) + s_a(t,\boldsymbol{K})$$

Recursive solution in array form:

$$\mathbf{g}^{(t+\delta t)} = \mathbb{U}^{(t+\delta t,t)}\,\mathbf{g}^{(t)} + \int_t^{t+\delta t} \mathbb{U}^{(s',t)}\,\mathbf{s}^{(s')}\,\mathrm{d}s, \qquad \int_t^{t+\delta t} \mathbb{U}^{(s',t)}\,\mathbf{s}^{(t)}\,\mathrm{d}s = \sum_{n=0}^{N-1} w_n \mathbb{U}^{(t_i,t)}\,\mathbf{s}^{(t_i)}\epsilon + \mathcal{O}(\epsilon^\alpha),$$

Need to compute terms of the form: $e^{-i\mathbb{H}\delta t}\mathbf{f}.$

→ **Faber polynomial expansion:** $e^{-i\mathbb{H}_{\rm sc}\,\delta\tau}\mathbf{f} = \sum_n c_n(\delta\tau)\,\Psi_n(\mathbb{H}_{\rm sc})\mathbf{f}.$

Recursion relations of Faber Polynomials

Spectral properties of H

★ Provides a very stable time evolution!

Harmonic Oscillator

Potentials v_{ba} and elements of **M** are all combinations of $\sigma(\boldsymbol{u}) = g^2(\gamma(0) - \gamma(\boldsymbol{u}))$

Given the GW model, we take small dipole separation $\boldsymbol{u}$ and slow varying logarithms

$$n(t)\sigma(\boldsymbol{r}) \simeq \frac{1}{4}\tilde{q}\,\boldsymbol{r}^2,$$

(good for multiple soft scattering, but fails UV tail)

quenching coefficient

This results in an analytical solution for $\boldsymbol{J}$: $\boldsymbol{J}(t,\,\boldsymbol{p}) = 2\omega\frac{\boldsymbol{p}}{\boldsymbol{p}^2}\left(1 - e^{-\frac{i}{2\omega\Omega}\tan(\Omega t)\,\boldsymbol{p}^2}\right)$, $\Omega = \frac{1-i}{2}\sqrt{\frac{\hat{q}_{ab}}{\omega}}$.

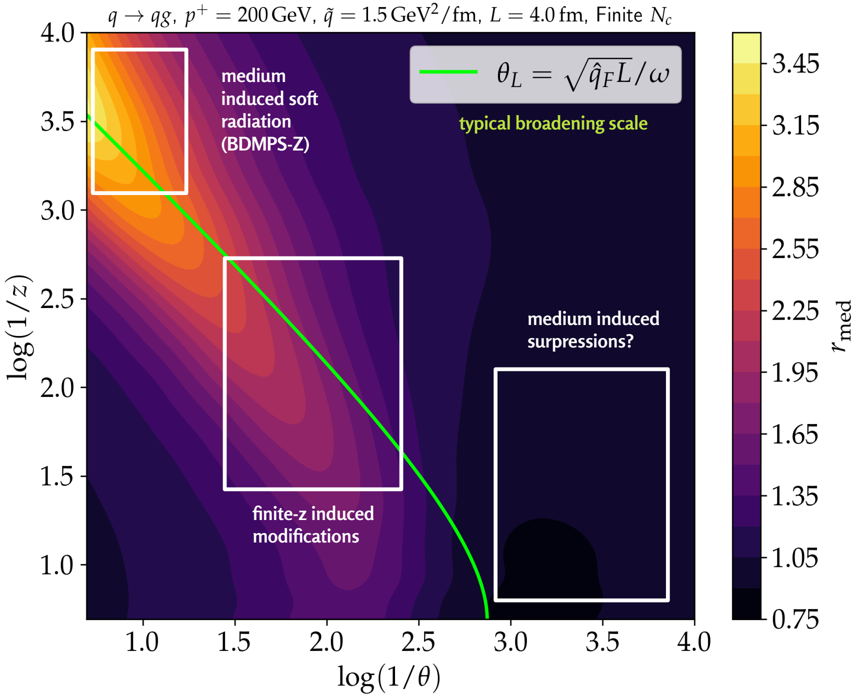
Given isotropy of the medium, equation for $\boldsymbol{F}$ can be reduced to 3D equation of the form:

$$i\partial_t\bar{\mathcal{F}}_\sigma(t,k,l,\psi) = \left[\frac{kl\cos(\psi)}{\omega}\delta_\sigma^{\sigma'} + i\mathbb{A}_\sigma^{\sigma'}\left(\partial_k^2 + \frac{1}{k}\partial_k\right) + i\mathbb{B}_\sigma^{\sigma'}\left(\partial_l^2 + \frac{1}{l}\partial_l + \frac{1}{l^2}\partial_\psi^2\right)\right]\bar{\mathcal{F}}_{\sigma'}(k,l,\psi,t)$$
$$+\bar{S}_\sigma(k,l,\psi,t),$$

Preliminary results: full solution

$q \to gq$

We are now able to compute modifications for all relevant phase space, high energy and large media!



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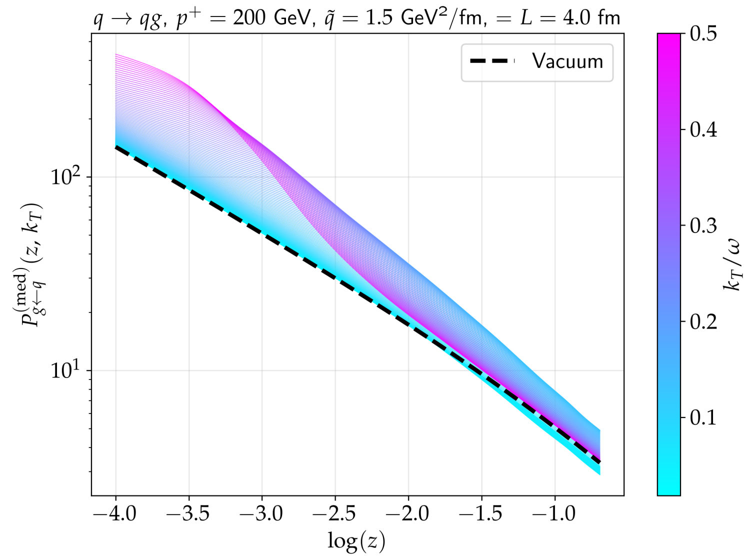
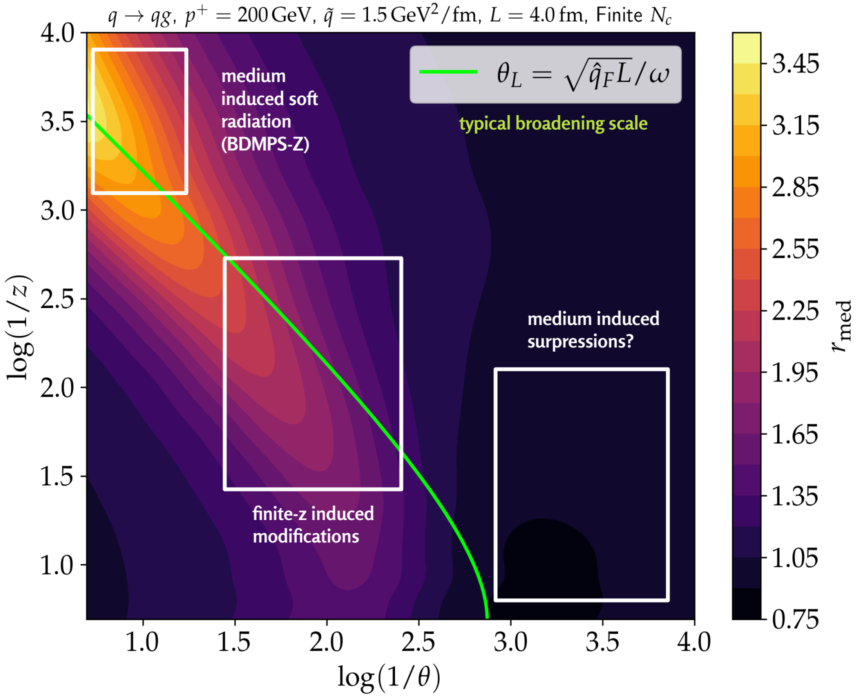
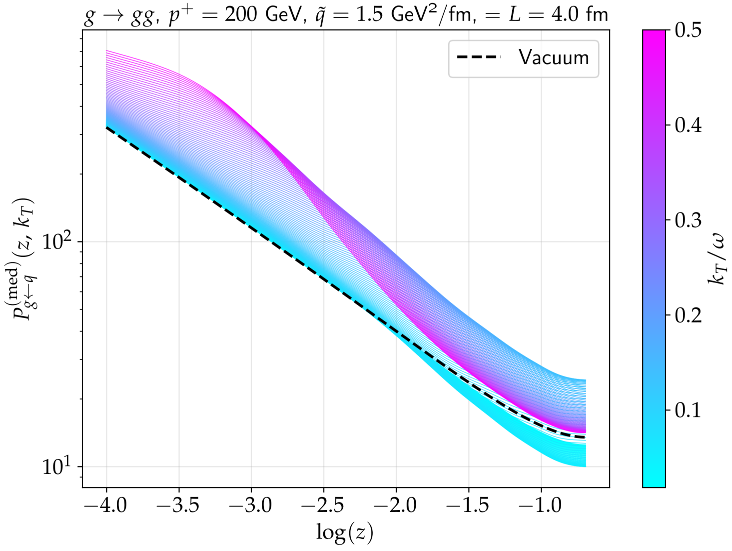


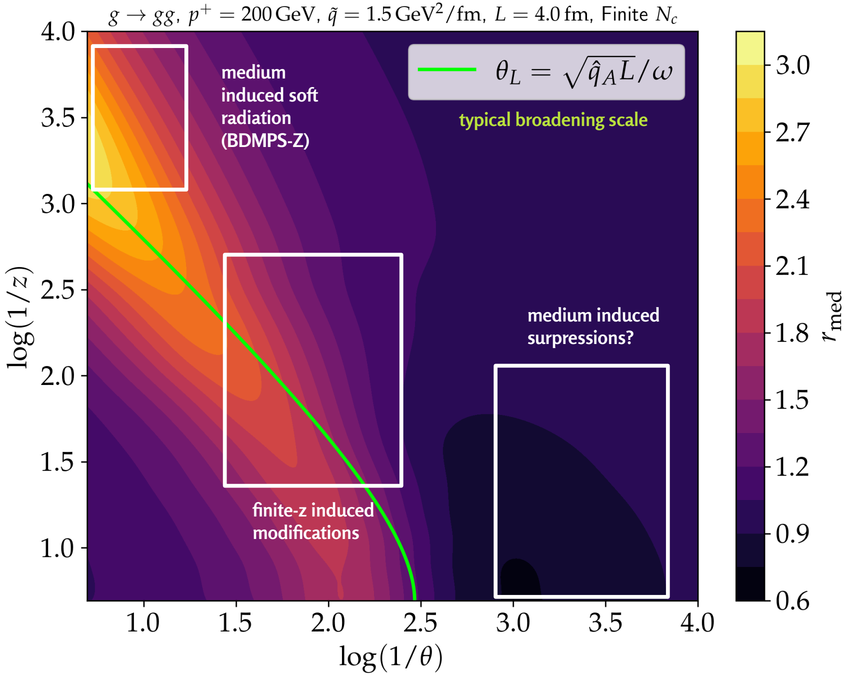


Preliminary results: full solution

$g \to gg$

Qualitatively similar to $q \to gq$, slightly smaller maximum enhancement, some stronger surpressions

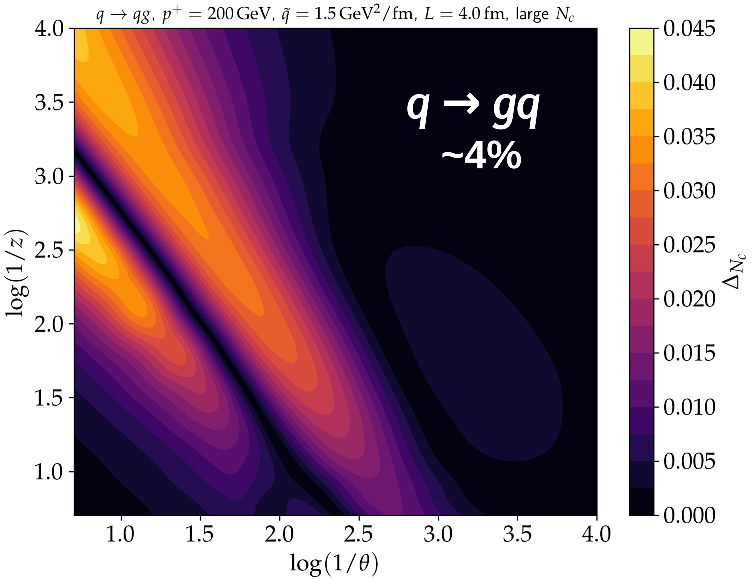


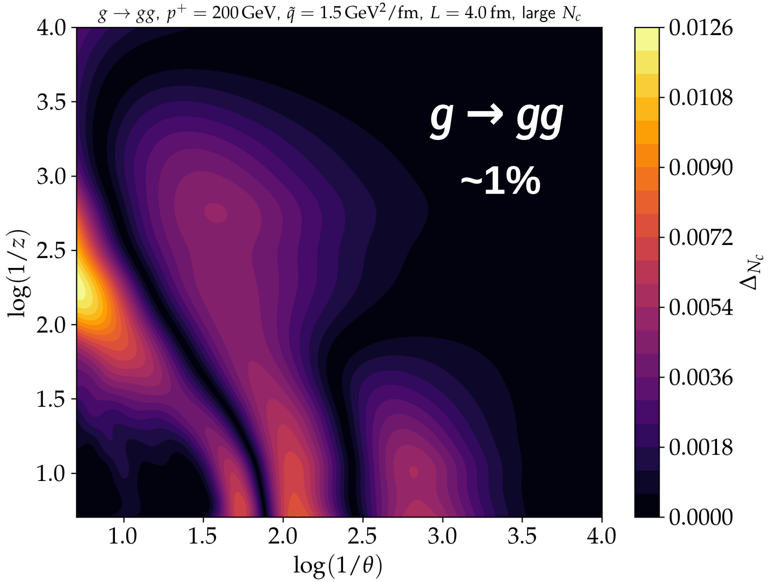


Preliminary results: N_c corrections

Large N_c limit helps reducing equation complexity: <u>3x3</u> for q→ qg and <u>8x8</u> for g→ gg reduce both to **<u>2x2</u>**!

How relevant are finite N_c corrections?





quite small...

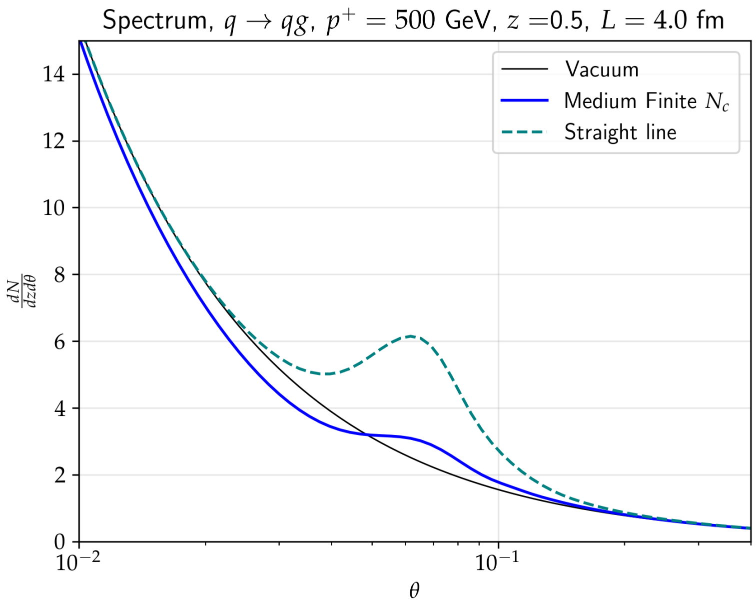
Preliminary results: straight line error

→ Straight line transverse dynamics allow for simple computation of the spectrum

→ Assumed to be the classical path and should work at very high energies ($p^+ \to \infty$) and **z** ~ 0.5 (hard splitting)

→ Even at a very high energy, it clearly shows great discrepancy to the real result

→ Does this assumption make sense? For $p^+ \to \infty$, we naturally have straight-line trajectory, but is this meaningful for modifications? (**on-going debate**)



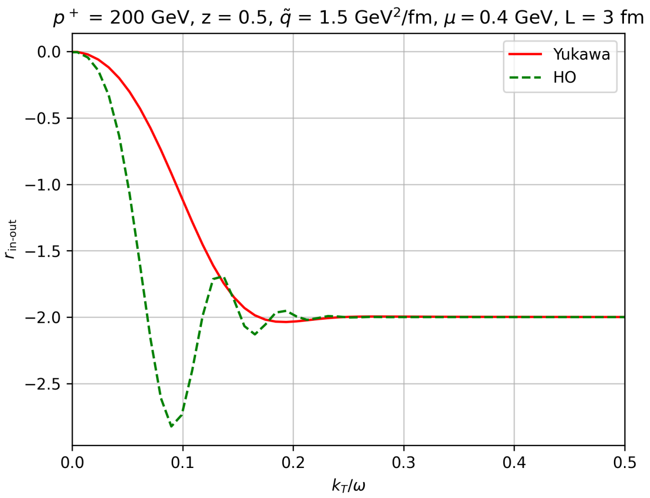
Beyond the HO

→ General potentials (GW or HTL) require evaluating PDE with convolutions

→ $V(q) \sim 1/q^4$, so we don't need many neighbors per point

→ Have to develop an efficient strategy to work these convolutions (have some preliminary ideas)

in-out



Conclusions

- We have computed, for the first time, the **effective in-medium parton splitting functions in exact transverse kinematics** for phenomenologically relevant parameters and leading processes $\boldsymbol{q \to gq}$ and $\boldsymbol{g \to gg}$

- Large N_c corrections were found to be the order of ~4% for the q → gq and ~1% for the g → gg.

- We have shown that straight-line approximation for transverse dynamics, commonly used in literature, produces very large discripancies

- We have started to work on beyond HO, which will provide a more acurate UV tail

¡MUCHAS GRACIAS!

Acknowledgements