January 8-14, 2026 - University of Granada

Chiral Symmetry Breaking in a Magnetic Field & Pion Weak Decay

Perspectives from Chiral Perturbation Theory

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Acknowledgements. *U.S. Dept. of Energy, Office of Science, Office of Nuclear Physics and Quantum Horizons Program*

KITP Scholars Program, Kavli Institute for Theoretical Physics, University of California, Santa Barbara

Context & Motivation

- Pioneering Lattice QCD Study of Pion Decay Width in a Magnetic Field *[Bali, Brandt, Endrodi and Gläßle, PRL **121**, 072001 (2018)]*
  - Identified novel decay constant $F_{\pi Q}^{(V)}$ absent in absence of magnetic field $\pi^+ \to \mu^+ \nu_\mu$
  - Constructed decay width within the lowest Landau level (LLL) approximation for the final state muon
- Nambu-Jona-Lasinio (NJL) Model Study of Charged Pion Decay Width in Magnetic Field *[Coppola, Dumm, Noguero and Scoccola, PRD **99**, 054031 (2019); PRD **101**, 034003 (2020), JHEP 2020, **1** (2020)]*
  - Constructed decay width without restriction to LLL $\pi^+ \to \ell^+ \nu_\ell$
- Two-flavor Chiral Perturbation Theory (ChPT) *[Adhikari and Tiburzi, arXiv:2408.09472]*
  - Motivation i) Results are model-independent — complements NJL & lattice QCD $\pi^+ \to \ell^+ \nu_\ell$ $\pi^0 \to \gamma\gamma$
  - Motivation ii) Finite volume corrections in a magnetic field are significant *[PRD **107**, 094504 (2023)]*

Finite Volume Local Quark Condensate

- The quark condensate is inhomogeneous
  (Green's function breaks translation)

$$\mathscr{L}_{\text{mass}} = \frac{F^2}{4}\text{Tr}[\chi^\dagger\Sigma + \chi\Sigma^\dagger] = = 2B_0 m_q \left(F^2 - \frac{1}{2}\pi^0\pi^0 - \pi^-\pi^+ + \cdots\right)$$

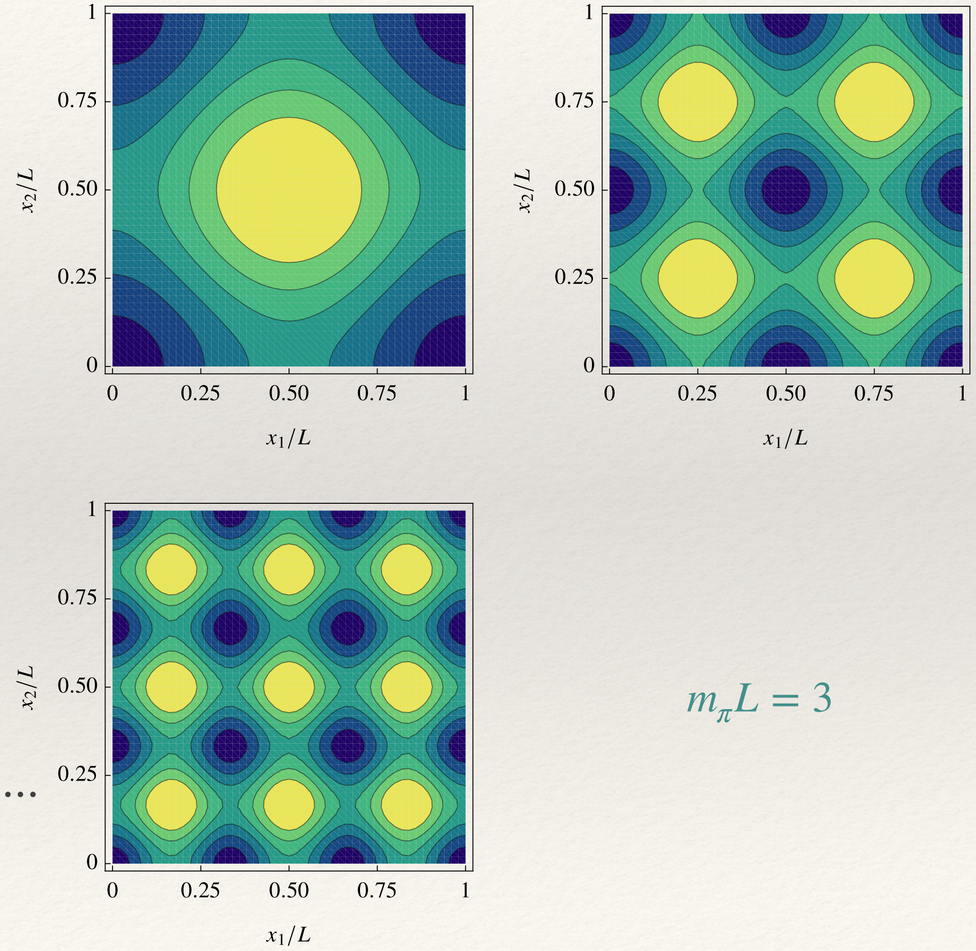
$$\langle\bar{\psi}(x)\psi(x)\rangle = -2B_0[F^2 - \tfrac{1}{2}G_0(0,0) - G_+(x,x)]$$

$$\langle\bar{\psi}\psi\rangle_0^\infty = -2B_0F^2$$

- For asymptotically large boxes

$$R(x_\perp) = \frac{\langle\bar{\psi}(x)\psi(x)\rangle - \langle\bar{\psi}\psi\rangle^\infty}{\langle\bar{\psi}\psi\rangle_0^\infty}$$

$$= -\left[\frac{5}{2} + \cos\left(\frac{2\pi N_\Phi x_1}{L}\right) + \cos\left(\frac{2\pi N_\Phi x_2}{L}\right)\right] \times \frac{m_\pi^2}{F_\pi^2}\frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}} + \cdots$$



$m_\pi L = 3$

*Adhikari, Tiburzi, PRD **107**, 094504 (2023)*

Finite Volume Average Quark Condensate

- The quark condensate is therefore spatially variant

$$\mathscr{L}_{\text{mass}} = \frac{F^2}{4}\text{Tr}[\chi^\dagger\Sigma + \chi\Sigma^\dagger] = = 2B_0 m_q \left(F^2 - \frac{1}{2}\pi^0\pi^0 - \pi^-\pi^+ + \cdots\right)$$

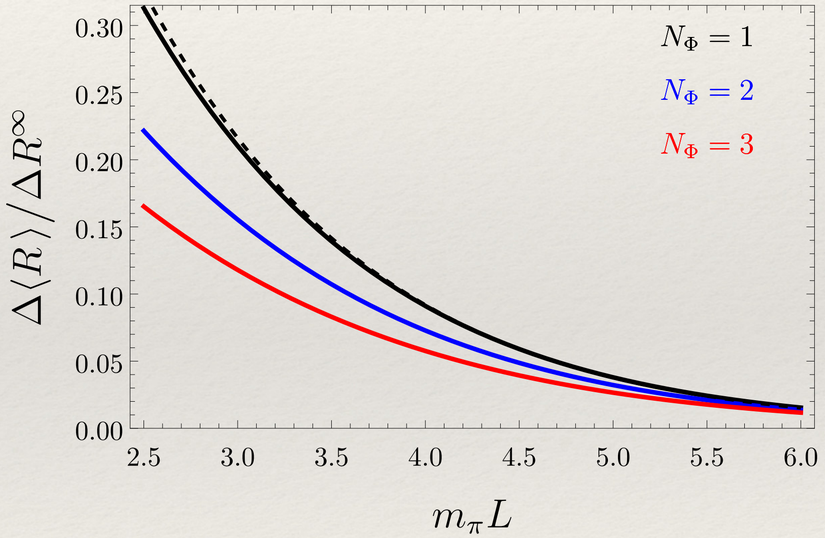
$$\langle\bar{\psi}(x)\psi(x)\rangle = -2B_0[F^2 - \frac{1}{2}G_0(0,0) - G_+(x,x)]$$

$$\langle\bar{\psi}\psi\rangle_0^\infty = -2B_0F^2$$

- Finite volume correction of the spatially averaged condensate is characterized by

$$\frac{\Delta\langle R\rangle}{\Delta R^\infty} = \frac{\langle R\rangle - \langle R\rangle_{B=0}}{R^\infty - R^\infty_{B=0}}$$

$$= e^{-m_\pi L}\sqrt{2\pi m_\pi L}\left[1 + \mathcal{O}\left(\frac{1}{m_\pi L}\right) + \mathcal{O}(e^{-m_\pi L}) + \mathcal{O}\left(\frac{N_\Phi^2}{(m_\pi L)^2}\right) + \cdots\right]$$



*Adhikari, Tiburzi, PRD **107**, 094504 (2023)*

Outline

- Chiral Perturbation Theory
  - Introduction & Discussion of Relevant Ingredients
  - Effective Lagrangian for Pion Weak Decay
    - Renormalized Pion Masses
    - Heuristic Description of Pion Decay Constants
- Model-Independent Predictions
  - Neutral Pion Decay Width $\Gamma_{\pi^0 \to \gamma\gamma}$
  - Charged Pion Decay Width $\Gamma_{\pi^+ \to \ell^+ \nu_\ell}$
    - Branching Ratio for the μ^+ and e^+ channels
- Concluding Remarks

Chiral Perturbation Theory - Ingredients

- Chiral Perturbation Theory is the low-energy *effective theory* of QCD
- In *massless QCD,* left-and-right-handed quarks do not mix and therefore can be rotated independently $\psi_L \to L\,\psi_L$ and $\psi_R \to R\,\psi_R$. The symmetry of the QCD partition function is
  - $U(1)_V \times SU(2)_V \times SU(2)_A \to U(1)_V \times SU(2)_V$ due to the formation of the chiral condensate, $\langle \bar{\psi}\psi \rangle \neq 0$
    - The $U(1)_A : \psi \to e^{i\theta_5\gamma_5}\,\psi$ subgroup of $U(2)_L \times U(2)_R$ is broken by the partition function.
- The building blocks of the effective theory are
  - *Goldstone Manifold.* $\Sigma \in SU(2)_L \times SU(2)_R / SU(2)_V$
  - *Scalar-Pseudoscalar Source.* $\chi = s + i\,p$
  - *Left-and-Right Current.* $\nabla_\mu \Sigma = \partial_\mu \Sigma - i\,R_\mu\,\Sigma + i\,\Sigma\,L_\mu \qquad L_\mu = R_\mu = -\,eQA_\mu \qquad A_\mu = -\,B\,y\,\delta_{\mu 1}$
- The results are *model-independent* for $p^2/(4\pi F_\pi)^2 \ll 1$ with $p = -\,i\partial\,, \sqrt{eB}\,, m_\pi$

Gasser, Leutwyler, Ann. Phys. ***158*** *(1984) 142*

Chiral Perturbation Theory - LO (Gasser-Leutwyler)

- Goldstone manifold, $\Sigma \in SU(2)_L \times SU(2)_R / SU(2)_V$
- The fluctuations are axial since the chiral condensate $\langle \bar{\psi}\psi \rangle$ is only invariant if $L = R \equiv V$.

$$1 \to \Sigma = 1 + \cdots$$

$$\Sigma \equiv U1U \in SU(2)_L \times SU(2)_R / SU(2)_V$$

$$U = \exp\left(\frac{i\Phi}{2F}\right) \qquad \Phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \pi^0 \end{pmatrix}$$

- The construction of the Lagrangian involves building a chirally invariant theory under $\Sigma \to L\Sigma R^\dagger$.

$$\mathscr{L}_2 = \frac{1}{4}F^2 \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle + \frac{1}{4}F^2 \langle \chi \Sigma^\dagger + \chi^\dagger \Sigma \rangle$$

- The mass term of QCD $\Delta\mathscr{L} = m_q \bar{\psi}\psi$ also explicitly breaks chiral symmetry $SU(2)_L \times SU(2)_R \to SU(2)_V$. Chiral symmetry can be preserved through the introduction of spurion sources in the QCD Lagrangian that transform accordingly.
  - The leading order mass term is by introducing sources in the QCD Lagrangian that preserves chiral symmetry $\Delta\mathscr{L} = \bar{\psi}_L \chi \psi_R + \bar{\psi}_R \chi^\dagger \psi_L$ with $\chi \to L\chi R^\dagger$.

Gasser, Leutwyler, Ann. Phys. ***158*** *(1984) 142*

Chiral Perturbation Theory - NLO (Kaiser)

- The non-anomalous NLO Lagrangian required for mass renormalization is

$$\mathscr{L}_4 \supset \frac{1}{16} l_3 \langle \chi \Sigma^\dagger + \chi^\dagger \Sigma \rangle^2 + \frac{1}{4} l_4 \langle \nabla_\mu \Sigma^\dagger \nabla^\mu \chi + \nabla_\mu \chi^\dagger \nabla^\mu \Sigma \rangle$$
$$+ l_5 \langle \Sigma^\dagger \hat{R}_{\mu\nu} \Sigma \hat{L}^{\mu\nu} \rangle + \frac{i}{2} l_6 \langle \hat{R}_{\mu\nu} \nabla^\mu \Sigma \nabla^\nu \Sigma^\dagger + \hat{L}_{\mu\nu} \nabla^\mu \Sigma^\dagger \nabla^\nu \Sigma \rangle$$

$$L^{\mu\nu} = \partial^\mu L^\nu - \partial^\nu L^\mu - i [L^\mu, L^\nu]$$
$$R^{\mu\nu} = \partial^\mu R^\nu - \partial^\nu R^\mu - i [R^\mu, R^\nu]$$
$$\hat{O} = O - \frac{1}{2} \langle O \rangle \mathbf{1}$$

  - l_3 (l_4) term renormalizes the pion mass (decay constant) at zero field

$$\nabla_\mu \sim \mathcal{O}(p)$$
$$\chi \sim \mathcal{O}(p^2)$$
$$L, R \sim \mathcal{O}(p^2)$$

- The anomalous (WZW) Lagrangian in a background field has the structure

$$\mathscr{L}_4^{WZW} \supset \frac{N_c}{64\pi^2} \epsilon^{\mu\nu\alpha\beta} \left\langle i \left(\Sigma^\dagger \partial_\mu \Sigma \hat{L}_\nu - \Sigma \partial_\mu \Sigma^\dagger \hat{R}_\nu \right) + \Sigma^\dagger [\hat{R}_\mu, \Sigma] \hat{L}_\nu \right\rangle \langle L_{\alpha\beta} + R_{\alpha\beta} \rangle$$

  - The contribution is non-vanishing due to the singlet contribution to $Q = \frac{1}{6} 1 + \frac{1}{2} \tau_3$
  - Conveniently, inclusion of the singlet anomaly does not modify pion mass and decay constant renormalization

Kaiser, Phys. Rev. D ***63****, 076010*

Effective Lagrangian For Weak Decay

- Charged pion decay width depends on pion `good' quantum number
- The S-matrix depends on $\Gamma(B, P_z) = \sum_{\ell=\mu\, e} \oint d\rho \frac{1}{T} |\langle \ell^+(\widetilde{p})\, \nu_\ell(q) | \hat{S} | \pi^+(\widetilde{P}) \rangle|^2$

$$\widetilde{P}^\mu = (E_{N,P_z}, P_x, 0, P_z)$$
$$E_{N,P_z} = \sqrt{P_z^2 + m_\pi^2(B) + (2N+1)eB}$$
$$\widetilde{p}^\mu = (E_{n,p_z}, P_x, 0, p_z)$$
$$E_{n,p_z} = \sqrt{p_z^2 + m_\ell^2 + 2neB}$$

$$\langle \ell^+(\widetilde{p})\, \nu_\ell(q) | \hat{S} | \pi^+(\widetilde{P}) \rangle = \int d^4x \langle \ell^+(\widetilde{p})\, \nu_\ell(q) | \hat{J}_W(x) | 0 \rangle \langle 0 | \hat{J}_{L\mu}(x) | \pi^+(\widetilde{P}) \rangle$$

- The effective Lagrangian comprises of contributions from the strong and weak sectors

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_\pi + \mathscr{L}_{\ell\nu} + \mathscr{L}_{\text{int}}$$

$$\mathscr{L}_\pi = D_\mu \pi^+ D^\mu \pi^- - \underline{m_{\pi^\pm}^2(B)} \pi^+\pi^- + \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 - \frac{1}{2}\underline{m_{\pi^0}^2(B)} \pi^0\pi^0 + \cdots$$

$$\mathscr{L}_{\ell\nu} = \bar{\ell}\left(i\gamma^\mu D_\mu - m_\ell\right)\ell + \bar{\nu}_\ell\left(i\gamma^\mu \partial_\mu\right)\nu_\ell$$

$$\mathscr{L}_{\text{int}} = J_W^\mu \underline{J_{L\mu}} \qquad J_W^\mu = 2G_F V_{ud}\left[\bar{\nu}_\ell \gamma^\mu (1-\gamma_5)\ell + \bar{\ell}\gamma^\mu(1-\gamma_5)\nu_\ell\right] \qquad eB \ll (4\pi F_\pi)^2 \lll M_W^2$$

Renormalized Pion Mass

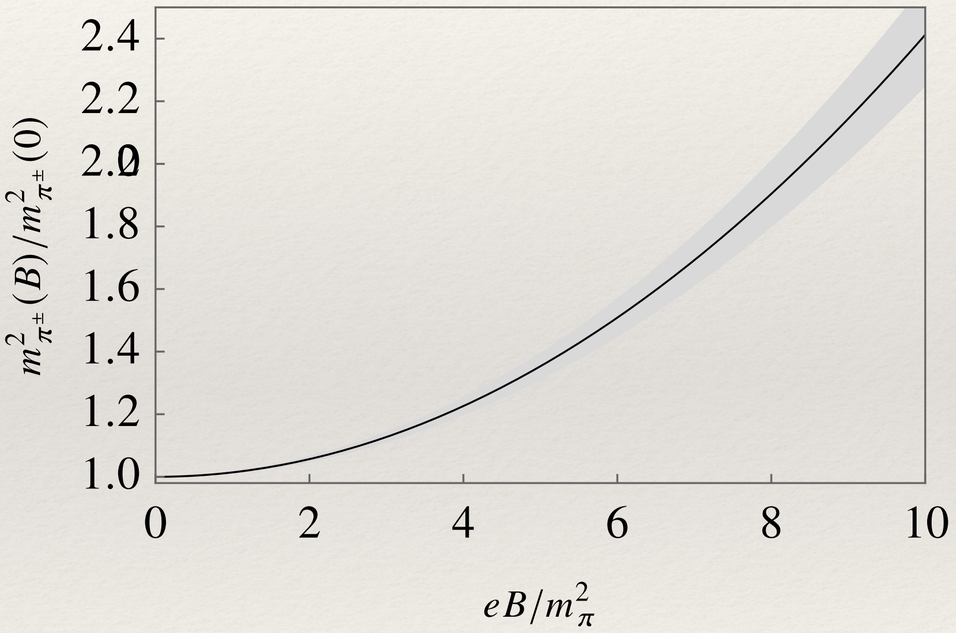
- Magnetic pion masses at $\mathcal{O}(p^4)$ are

$$m^2_{\pi^\pm}(B) = m^2_\pi \left[1 + \frac{eB}{m^2_\pi}\frac{eB}{(4\pi F_\pi)^2}\bar{l}\right] \qquad \bar{l} = \tfrac{1}{3}(\bar{l}_6 - \bar{l}_5) = 1.0(1)$$

$$m^2_{\pi^0}(B) = m^2_\pi \left[1 + \frac{eB}{(4\pi F_\pi)^2}\mathcal{J}(\frac{m^2_\pi}{eB})\right]$$

- The Schwinger integral $\mathcal{J}$ arises through tadpole contributions

$$\frac{eB}{(4\pi)^2}\mathcal{J}(\frac{m^2_\pi}{eB}) \equiv \int_0^\infty \frac{ds}{(4\pi s)^2} e^{-m^2_\pi s}\left(\frac{eBs}{\sinh eBs} - 1\right)$$



Tiburzi, Nucl. Phys. A ***814****, 74 (2008); Andersen, Phys. Rev. D* ***86*** *025020 (2012)*

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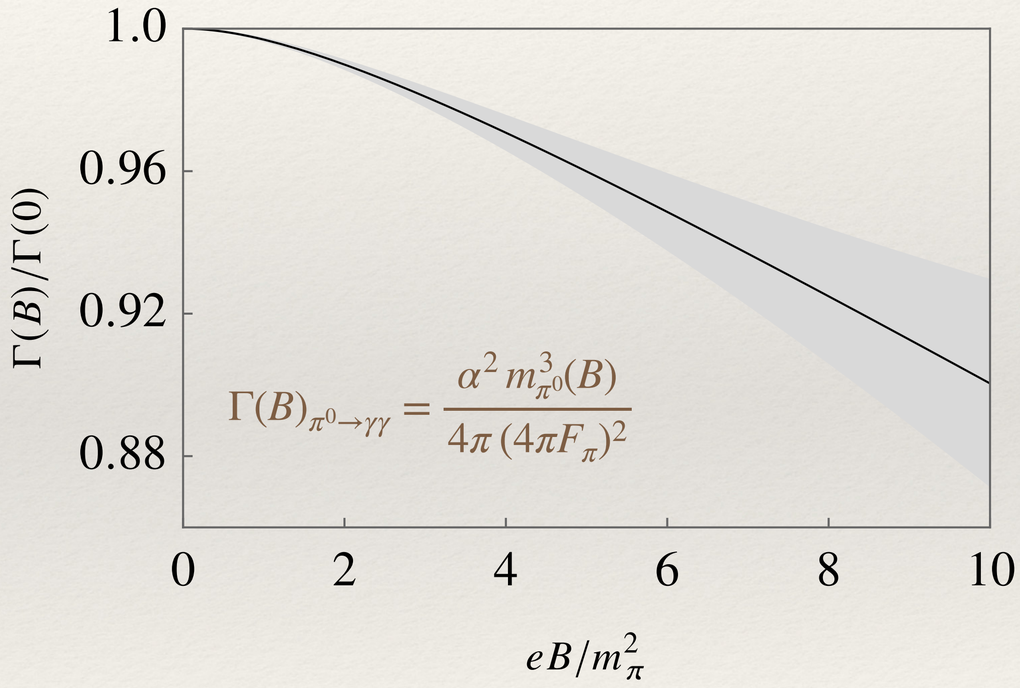


Tiburzi, Nucl. Phys. A ***814****, 74 (2008); Andersen, Phys. Rev. D* ***86*** *025020 (2012)*

Pion Decay Constants (!Heuristic!)

- At zero magnetic field the pion-to-vacuum matrix elements is

$$\langle\, 0 \,|\, \hat{J}_{A\mu}(0) \,|\, \pi^{Q}(P) \,\rangle \sim i \left[F_\pi P_\mu \right]$$

$$\hat{J}_A(x) = \hat{J}_L(x) - \hat{J}_R(x)$$
$$\hat{J}_V(x) = \hat{J}_L(x) + \hat{J}_R(x)$$

- In a magnetic field, there are two further pion decay constants

$$\langle\, 0 \,|\, \hat{J}_{A\mu}(0) \,|\, \pi^{Q}(\widetilde{P}) \,\rangle \sim i \left[F^{(A1)}_{\pi^Q}(B)\, \widetilde{P}_\mu + F^{(A2)}_{\pi^Q}(B)\, eQF_{\mu\nu}\, \widetilde{P}^{\,\nu} + F^{(A3)}_{\pi^Q}(B)\, eF_{\mu\nu}\, eF^{\nu\alpha}\, \widetilde{P}_{\,\alpha} \cdots \right]$$

$$\langle\, 0 \,|\, \hat{J}_{V\mu}(0) \,|\, \pi^{Q}(\widetilde{P}) \,\rangle \sim i \left[F^{(V)}_{\pi^Q}(B) e \widetilde{F}_{\mu\nu} \widetilde{P}^{\,\nu} + \cdots \right]$$

  - The structure is anticipated through Lorentz index and parity matching.
  - $F^{(A3)}_{\pi^Q}$ depends on higher order unknown LECs

$$F^{(A1)}_{\pi^\pm}(B) = F_\pi \left[1 - \frac{eB}{2(4\pi F_\pi)^2} \mathcal{J}\left(\frac{m_\pi^2}{eB}\right) + \mathcal{O}\left(\frac{eB}{(4\pi F_\pi)^2} \right) \right]$$

$$F^{(A1)}_{\pi^0}(B) = F_\pi \left[1 - \frac{eB}{(4\pi F_\pi)^2} \mathcal{J}\left(\frac{m_\pi^2}{eB}\right) + \mathcal{O}\left(\frac{eB}{(4\pi F_\pi)^2} \right) \right]$$

$$F^{(A2)}_{\pi^\pm} = \frac{F_\pi}{(4\pi F_\pi)^2} \bar{l} \left[1 + \mathcal{O}\left(\frac{eB}{(4\pi F_\pi)^2} \right) \right]$$

$$F^{(A2)}_{\pi^0} = 0$$

$$F^{(V)}_{\pi^Q}(B) = \frac{1}{8\pi^2 F_\pi} \left[1 + \mathcal{O}\left(\frac{eB}{(4\pi F_\pi)^2} \right) \right]$$

*Modified argument based on Bali, Brandt, Endrodi and Gläßle, PRL **121**, 072001 (2018)*

Charged Pion Decay Width

- The decay width naturally conserved good momentum

$$\Gamma_\ell(B,\,P_z) = \frac{1}{2E_{N,P_z}} \sum_n \int \frac{d^3q\, d^2\widetilde{p}}{2E_{\mathbf{q}}\, 2E_{n,p_z}(2\pi)^2} \langle\, |\mathscr{M}_n|^2 \,\rangle\, \delta^{(3)}\left(\widetilde{P} - \widetilde{p} - \widetilde{q}\right)$$

- Unlike for zero field, the amplitude squared is complicated

$$\langle\, |\mathscr{M}_n|^2 \,\rangle = 8G_F^2 V_{ud}^2\, e^{-\zeta}\left[\frac{\zeta^n}{n!}\left\{\frac{\mathscr{A}^2}{2}P^-q^+P^-p^+ + eB\,\mathscr{C}^2 q^- p^+ \frac{(n-\zeta)^2}{\zeta} + 2eB\mathscr{A}\mathscr{C}\,P^-p^+(n-\zeta)\right\} + \right.$$

$$\left. +(1-\delta_{n0})\frac{\zeta^{n-1}}{(n-1)!}\left\{\frac{\mathscr{B}^2}{2}P^+q^-P^+p^- - 2eB\mathscr{A}\mathscr{B}P^+P^-\zeta - 2eB\mathscr{B}\mathscr{C}P^+q^-(n-\zeta)\right\}\right]$$

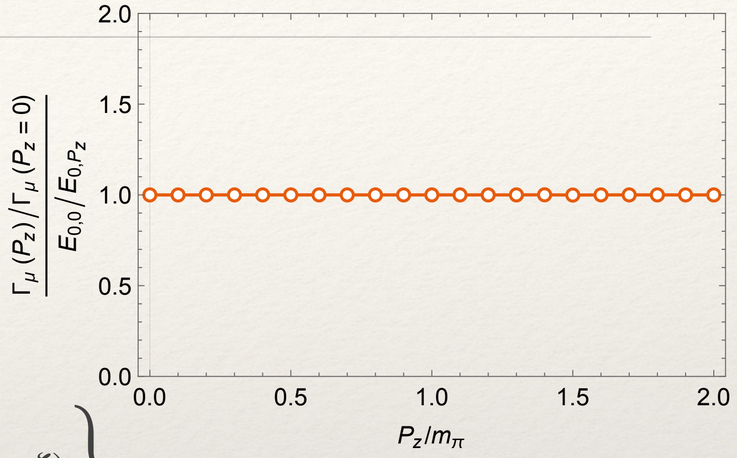
$$\mathscr{A} = F_{\pi^\pm}^{(A1)} + eB\,F_{\pi^\pm}^{(V)}$$

$$\mathscr{B} = F_{\pi^\pm}^{(A1)} - eB\,F_{\pi^\pm}^{(V)}$$

$$\mathscr{C} = F_{\pi^\pm}^{(A1)} + eB\,F_{\pi^\pm}^{(A2)}$$

$$\zeta = \frac{\mathbf{q}_\perp^2}{2eB}$$

$$a^\pm = a^0 \pm a^3$$



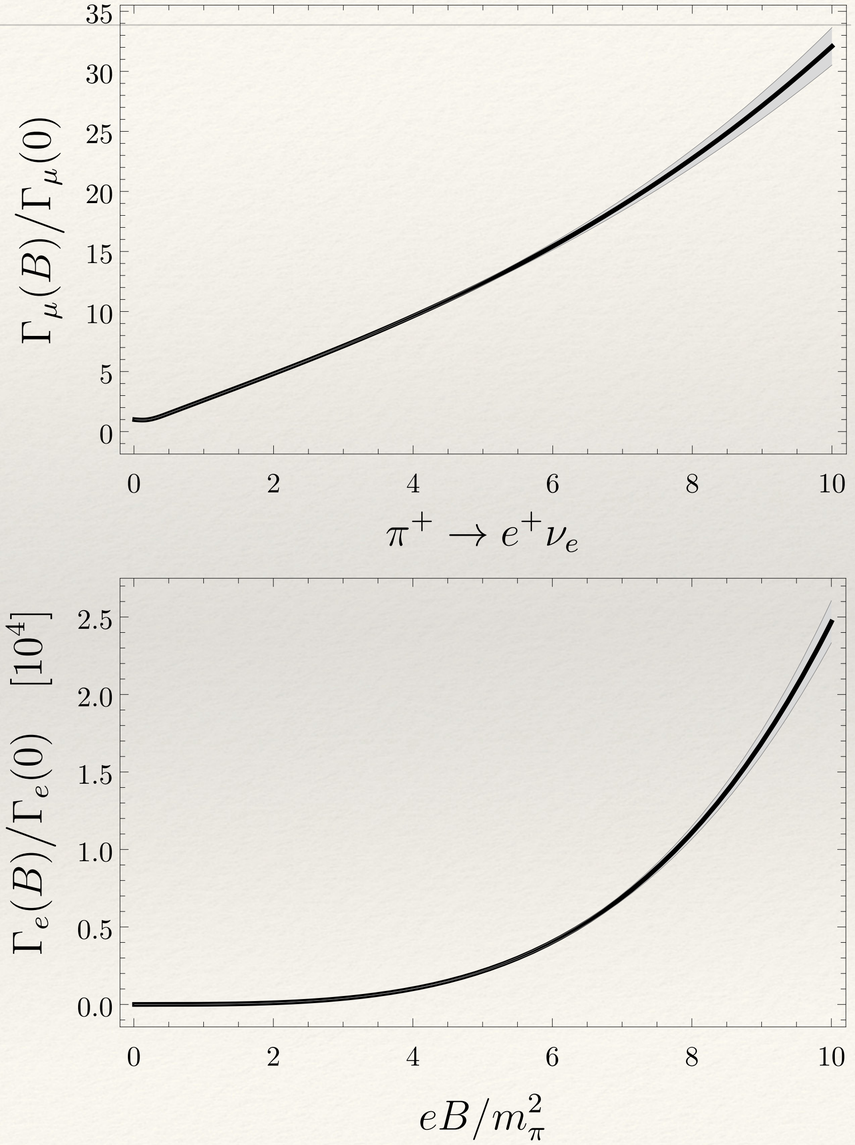
- Integral is invariant under boosts in the z-direction, the coefficient, however, is not.
- The ratio $\dfrac{\Gamma_\ell(B,\,P_z)/\Gamma_\ell(B,P_z=0)}{E_{0,0}/E_{0,P_z}}$ is invariant

Charged Pion Decay Width

- The sum is restricted to the floor of the maximum value of n permitted by energy conservation

$$n_{max} = \frac{1}{2}\left(\frac{m_\pi^2(B) - m_\ell^2}{eB} + 1\right)$$

- The total decay width in the zero field limit is dominated by the muon channel
- At finite B, the decay width in the e^+ channel increases more rapidly
  - Effective mass of e^+ increases at a higher rate than μ^+
- *Predictions are **model-independent,** the bands indicate expected corrections due to LEC uncertainties and higher order operators*



Branching Ratio

- The sum is restricted to the floor of the maximum value of n permitted by energy conservation

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- The total decay width in the zero field limit is dominated by the muon channel
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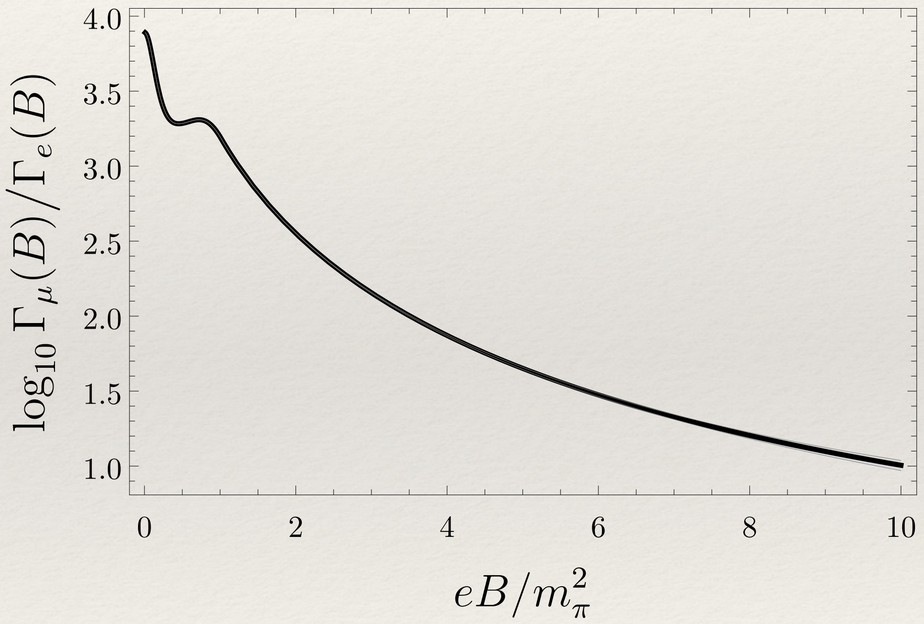
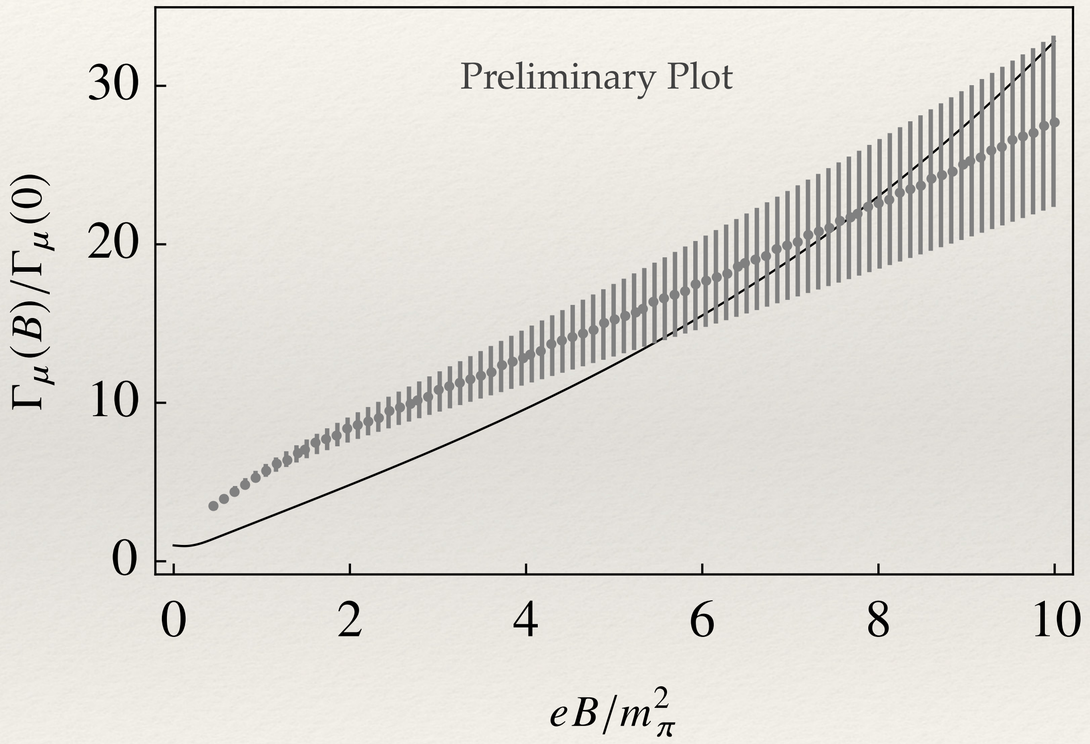


Concluding Remarks

- ChPT decay width complements lattice QCD
  - Decay width prediction is better constrained particularly at weak fields
  - ChPT provides access to $F^{(A2)}$ for which lattice QCD data is absent
  - $\lfloor n_{max} \rfloor = 1$ for $eB \gtrapprox 0.43 m_\pi^2$ in the μ^+ channel and $eB \gtrapprox 1.015 m_\pi^2$ for the e^+ channel
    - Difference between lattice and ChPT arises through pion decay constants
- Vector pion decay constant exhibits modest tension with lattice
  - $F^{(V)}_{\pi^\pm}(B=0)/F_\pi = 1.2(3)\ \text{GeV}^{-2}$ [Wilson, $m_\pi = 415$ MeV]
  - $F^{(V)}_{\pi^\pm}(B=0)/F_\pi = 0.8(2)\ \text{GeV}^{-2}$ [Staggered, physical mass]
  - $F^{(V)}_{\pi^\pm}(B=0)/F_\pi = 1.49(3)\ \text{GeV}^{-2}$ [Anomaly, ChPT]



Thank you !

Questions?

Acknowledgement of Support

U.S. Dept. of Energy, Office of Science, Office of Nuclear Physics and Quantum Horizons Program

KITP Scholars Program, Kavli Institute for Theoretical Physics, University of California, Santa Barbara