Obtaining Bounds of Cold Nuclear Matter While Evading the Sign Problem

TDC arXiv:2510.07124

An Outline of the talk

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- **Introduction**
  - **Cold Nuclear Matter is a central problem**
  - **Lattice QCD is intractable due to the sign problem**
  - **Rather than solving the problem can we learn anything useful without solving the problem; bounds on the cold mater EOS**
- **An old approach via QCD Inequalities**
- **An novel approach via relativity**

Introduction

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  - Basic idea was turn of Coulomb force and just look at strong interactions
  - My experimental friends have difficulty with doing this in the lab

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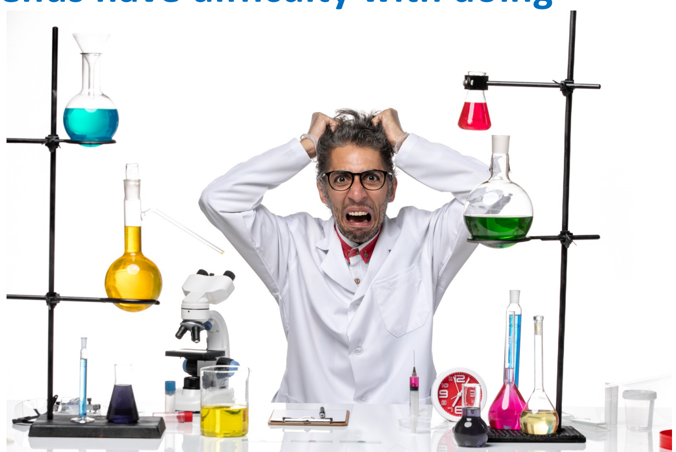
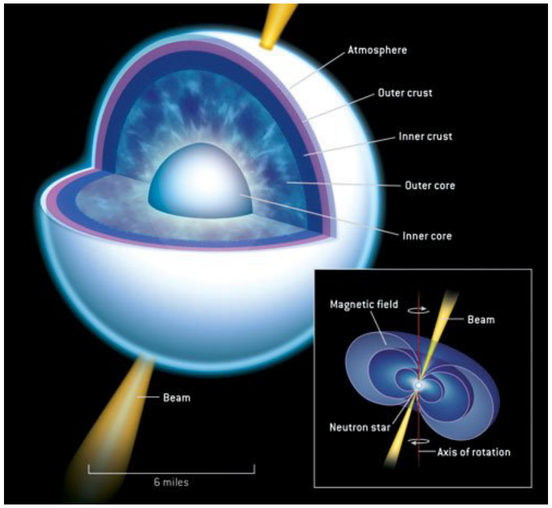


  - but it is both important in describing real nuclei and import to understand from a theoretical perspective.

- Ideally this could be computed directly from QCD.
  - This is particularly important in the context of the cold equation of state (EOS) of neutron matter
    - The neutron matter equation of state is a key input into neutron star physics, but the EOS is essentially unconstrained from terrestrial experiments.



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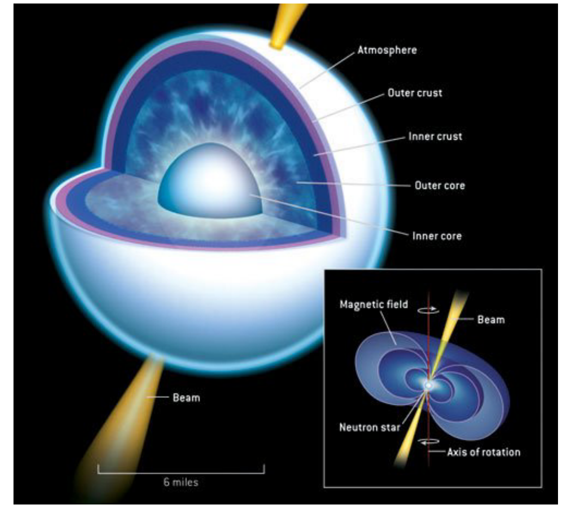

- **Unfortunately lattice QCD—the numerical evaluation of the Euclidean space functional integral—which is the best developed *ab intio* approach to solving QCD numerically is incapable of such calculations due to the fermion sign problem—which is intractable.**
  - **Once a chemical potential is added, the functional determinant develops a phase which leads to exponentially large cancellations**

- **Can the sign problem be evaded?**
  - **It is very unlikely that there is a general polynomial time solution to generic sign problems as that would imply *P=NP* ; if you found a general solution you would solve a Millennium Math Problem and will $1,000,000.**
  - **There have been numerous attempt to develop methods approaches to evade or minimize sign problems that do work for specific models, but so far none work for QCD**
  - **Perhaps functional methods will develop to the point that there can attack the cold dense matter problem—but this is a long-term prospect where the time horizon could well be infinite.**
  - **Perhaps quantum computation (hardware and algorithms) will develop to the point that there can attack the cold dense matter problem—but this is a long-term prospect where the time horizon could well be infinite.**

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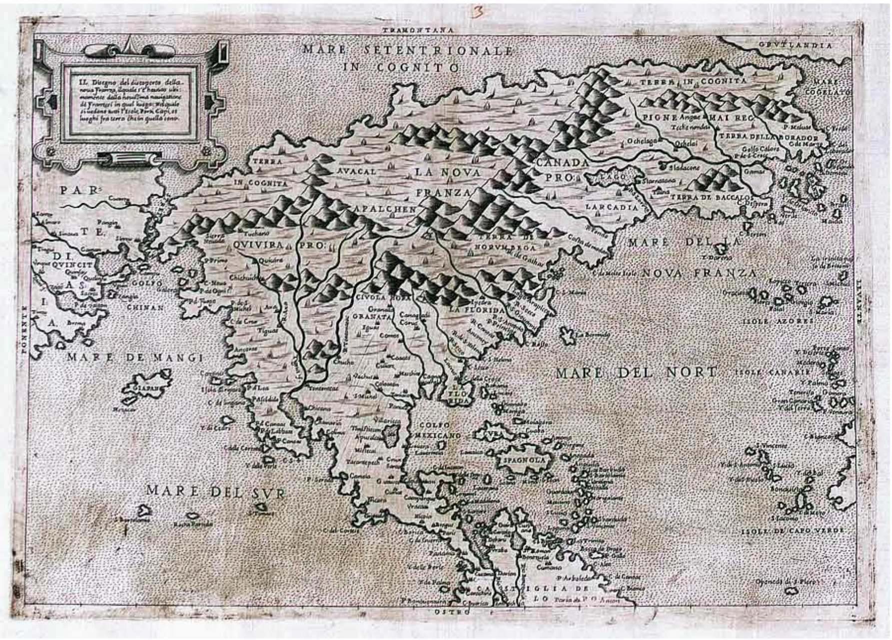
For the foreseeable future, it is unlikely that the cold matter EOS will be directly calculable from QCD.

- **From a QCD perspective the EOS of cold matter is essentially *Terra Incognita.***
- **It will not be fully mapped for some time to come—if ever.**
- **But this does not mean that we cannot learn *something* about it.**



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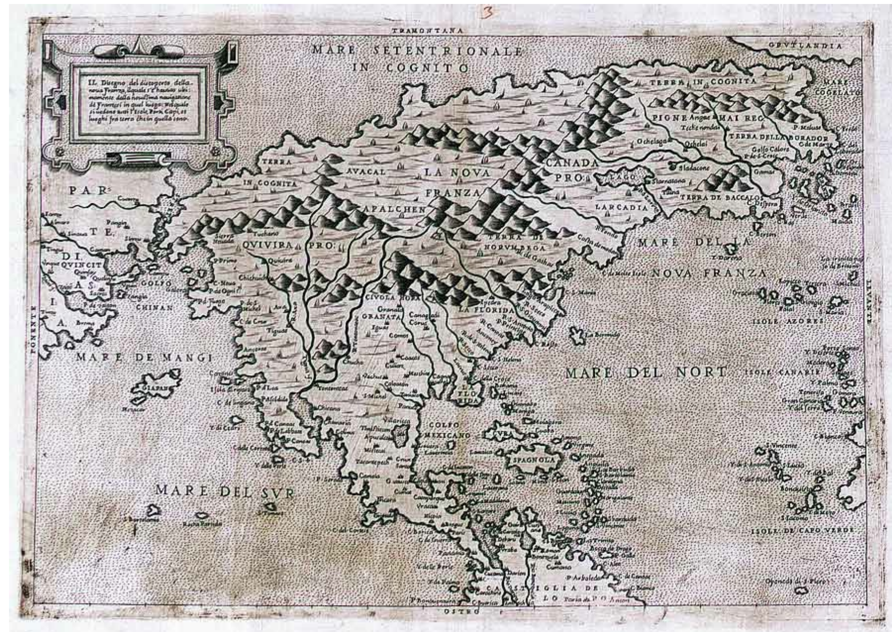

In trying to chart this new land the explorer must rely on the tools at his deposal although they may be crude.

An old crude approach via QCD Inequalities

TDC, PRL 91 (2003) 032002

- **While we may not be able to compute the EOS, we may be able to reliably determine bounds that the EOS must satisfy. QCD inequalities provide one way to determine bounds**
- **QCD inequalities are based on positivity of certain quantities in Euclidean space functional integrals.**
- **Key result: For QCD with 2-mass degenerate mass quark flavors: $P_I\left(T, \mu_I = \frac{2}{3}\mu_B\right) \geq P\left(T, \mu_I = \frac{2}{3}\mu_B\right)$ where P_I and P_B are the pressures for systems with an isospin or baryon chemical potential**

Nice formal result, but who cares? How can we make money with this?

- **It is useful!** P_I **can be computed without a sign problem providing a constraint on** P_B **and it works at any temperature.**
  - **Recently the NPLQCD collaboration calculated** F_I **(indirectly) and from this bounded EOS of baryonic matter.** Abbot *et. al*. Rev. Lett. **134**, (2025) 011903; Abbot et. al. Phys. Rev. D 108 no. 11, (2023)

- **Derivation of** $P_I\left(T, \mu_I = \frac{2}{3}\mu_B\right) \geq P_B(T, \mu_B)$ **follows from a simple property of functional integrals that is easy to prove:**
  $Det[\not{D} + m + \mu\gamma_0] = Det[\not{D} + m - \mu\gamma_0]^*$
- **This implies that for two degenerate flavors**

$$Z_I(\mu_I) = \int d[A] \underbrace{\left|\mathrm{Det}\left(\not{D} + m - \tfrac{1}{2}\mu_I\gamma_0\right)\right|^2 e^{-S_{\mathrm{YM}}}}_{\text{Its real \& positive: No sign problem}}$$

Its real & positive:
No sign problem

$$Z_B(\mu_B) = \int d[A]\, \underbrace{\mathrm{Det}\left(\not{D} + m - \tfrac{1}{3}\mu_B \gamma_0\right)^2}_{\text{Generally complex: a sign problem}} e^{-S_{\mathrm{YM}}}$$

So $Z_B(\mu_B) \leq \int d[A] \left|\mathrm{Det}\left(\not{D} + m - \tfrac{1}{3}\mu_B \gamma_0\right)\right|^2 e^{-S_{\mathrm{YM}}} = Z_I(\mu_I = \tfrac{2}{3}\mu_B)$

The factors of 1/2 (for isospin) & 1/3 (for baryon) is due to relation of quark chemical potential to isospin or baryon chemical potential

Recall that $\boldsymbol{Z = e^{-\beta F}}$ where *F* is the free energy and the fact that at fixed chemical potential, the free energy density is -*P* to obtain

$$\boldsymbol{P_I\left(T, \mu_I = \tfrac{2}{3}\mu_B\right) \geq P_B(T, \mu_B)}$$

- **Our crude tool provides some limited information: a bound on the cold matter equation of state, rather than the EOS itself**
  - **This bound is an upper bound for the pressure l** ***.***

- **An advantage to this approach: while it is valid at T=0, it is not restricted to T=0**
- **Limitations of this approach:**
  - **This is only an upper bound; no information on a lower bound**
  - **While the EOS in terms of the pressure allows one to compute the energy density via a Legendre transformation, a** ***bound*** **on the EOS for** ***P*** **does not yield a bound on** ε**.**

An novel approach to bounding the EOS via Relativity

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- **A limitation of this approach: it is valid at T=0, and only T=0.**
- **Advantages of this approach**
  - **It gives complementary information to the QCD inequality approach, providing bounds on the EOS expressed in terms of the energy density, ε, rather than P.**
  - **It gives both upper and lower bounds.**

- **Approach is based on a standard method for evading the sign problem**
  - **Use imaginary chemical potential—which have no sign problem**

- Of course, imaginary chemical potentials are not directly physical
  - Traditionally used to extract EOS at high temperatures via analytically continuing to real (and thus physical) values.
  - Unfortunately this fails at T=0 as the function is nonanlytic due to the Silverblaze phenomenon and analytic continuation is not viable.
- But imaginary chemical in Euclidean space calculations do correspond to something physical—but not directly connected to number density!
  - In Euclidean space temporal and spatial directions are equivalent, but translations of quantities from Minkowski space can pick up signs—or factors of i.
  - A Lagrange multiplier enforcing a fixed current denisity in Minkowski space, when translated to Euclidean space is mathematically equivalent to an imaginary chemical potential translated from Minkowski to Euclidean

- **There is no sign problem in calculating properties of QCD with a Lagrange multiplier for a current density rather than the number density!**
  - **In principle it is possible to use this to compute the properties of QCD matter with a Lagrange multiplier fixing baryon current density—i.e at *T=0* the lowest energy state with fixed j_z (assume current in z direction)**
    - ***Quantities that can be computed (at least in principle) without a sign problem***

j_z — **Current density; components of current 4-vector**

$$\left.\begin{matrix} T_{tt} \\ T_{zz} \end{matrix}\right\}$$ **Components of stress energy 4-tensor**

- **There is no sign problem in calculating properties of QCD with a Lagrange multiplier for a current density rather than the number density!**
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    - ***Quantities that can be computed (at least in principle) without a sign problem***

j_z — Current density; components of current 4-vector

T_{tt}, T_{zz} — Components of stress energy 4-tensor

Nice fact, but who cares? How can we make money with this?

- QCD with a fixed current density is of no particular interest in & of itself.
- Fortunately along with Lorentz transformations it can be used to provide upper & lower bounds on the energy density of QCD with a fixed number density.

How does this work?

Start with upper bound:

- There is no sign problem to compute the properties of the ground state of QCD with a fixed current density; denote this state $|g, j_z\rangle$.

- $|\boldsymbol{g}, \boldsymbol{j_z}\rangle$ has a fixed current density but zero number density (since number is ***C**-odd* Lagrange multiplier is **C**-even) and a diagonal stress energy

$$\langle g, j_z | j^{(4)} | g, j_z \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ j_z \end{pmatrix}$$

Superscript $^{(4)}$ indicates a 4-vector or 4-tensor

$$\langle g, j_z | T^{(4)} | g, j_z \rangle = \begin{pmatrix} T_{tt} & 0 & 0 & 0 \\ 0 & T_{xx} & 0 & 0 \\ 0 & 0 & T_{yy} & 0 \\ 0 & 0 & 0 & T_{zz} \end{pmatrix}$$

Note: in a covariant theory a quantum state under a Lorentz boost of β remains a valid quantum state but one with boosted properties $|g, j_z\rangle \rightarrow |g, j_z;\ \beta\rangle$

$$\langle g, j_z, \beta | j^{(4)} | g, j_z, \beta \rangle = \begin{pmatrix} \beta\gamma\, j_z \\ 0 \\ 0 \\ \gamma\, j_z \end{pmatrix}$$

Note that $|g, j_z;\ \beta\rangle$ is no longer a ground state of the system with any usual constraint

$$\langle g, j_z; \beta | T^{(4)} | g, j_{z;\beta} \rangle = \begin{pmatrix} \gamma^2\left(T_{tt} + \beta^2 T_{zz}\right) & 0 & 0 & 0 \\ 0 & T_{xx} & 0 & 0 \\ 0 & 0 & T_{yy} & 0 \\ 0 & 0 & 0 & \gamma^2\left(\beta^2 T_{tt} + T_{zz}\right) \end{pmatrix}$$

- **Implication:** there exists a set of states in the system $\{|g, j_z;\ \beta\rangle\}$ whose for which the energy density and number density are calculable without a sign problem

$$\langle g, j_z, \beta|\varepsilon|g, j_z, \beta\rangle = \gamma^2\left(T_{tt}(j_z) + \beta^2 T_{zz}(j_z)\right)$$
$$\langle g, j_z, \beta|n|g, j_z, \beta\rangle = \beta\gamma\, j_z$$

- On the other hand $\varepsilon(n)$, by definition is the lowest energy state with fixed n. Thus, for any j_z and any choice of β, there is an upper bound on $\varepsilon(n)$ for some value of n :

$$\boxed{\varepsilon(\beta\gamma\, j_z) \leq \gamma^2\left(T_{tt}(j_z) + \beta^2 T_{zz}(j_z)\right)}$$

- **Some comments:**
  - **While calculations $T_{tt}(j_z)$ and $T_{zz}(j_z)$ using a single value of j_z yields bounds for all values of values of *n* by varying β , for any given *n* the strongest bound will come from a particular combination of j_z and β.**
    - **Bounds are likely to be weak when $\beta\gamma \ll 1$ or $\beta\gamma \gg 1$**
    - **An obvious strategy for realistic calculations:**
      - **Compute $T_{tt}(j_z)$ and $T_{zz}(j_z)$ using a several values of j_z**
      - **For each use $\gamma^2 \left(T_{tt}(j_z) + \beta^2 T_{zz}(j_z)\right)$ and scan over β to find an upper bound of ε for each value of $n = \beta\gamma\, j_z$ and choose the combination of of *n* by varying β that yields the smallest upper bound.**
  - **A restriction: This only works for $T = 0$ – for $T \neq 0$ one could use a density matrix rather than a quantum state and boost that, but, boosting that would would correspond to a boosted heat bath would not yield standard thermal properties.**

- **What about a lower bound?**
  - **Start with a counterfactual assumption that you know $\boldsymbol{R(n)=\frac{P(n)}{\varepsilon(n)}}$ the ratio of the pressure to the energy density at a number density of interest. For that situation**

$$\boldsymbol{j}^{(4)}=\begin{pmatrix} n \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad T^{(4)}=\begin{pmatrix} \varepsilon(n) & 0 & 0 & 0 \\ 0 & R(n)\varepsilon(n) & 0 & 0 \\ 0 & 0 & R(n)\,\varepsilon(n) & 0 \\ 0 & 0 & 0 & R(n)\,\varepsilon(n) \end{pmatrix}$$

With $\boldsymbol{\varepsilon(n)}$ unknown

- **One can Lorentz boost this by β to obtain $j_z=\beta\gamma\,\varepsilon(n)$ and $T_{tt}=\varepsilon(n)\gamma^2(1+R(n)\beta^2)$ for the boosted system.**
- **But $T_{tt}(j_z)$, which is calculable without a sign problem is the lowest value of the the energy density for fixed j_z !**

So $T_{tt}(n\,\beta\gamma) \leq \varepsilon(n)\gamma^2(1+R(n)\beta^2)$ and $\varepsilon\left(\frac{j_z}{\beta\gamma}\right) \geq \dfrac{T_{tt}(j_z)}{\gamma^2\left(1+\beta^2 R\left(\frac{j_z}{\beta\gamma}\right)\right)}$

- **This may appear useless as a bound since it depends on a counterfactual knowledge of R.**
- **But causality requires $\mathbf{1} \geq \boldsymbol{R(n)}$ for all *n*(otherwise the speed of sound would be greater than the speed of light for some $\boldsymbol{n}$:** $V_{sound}^2 = \frac{\partial P}{\partial e}$ **so that** $P(n) = \int_0^n dn'\, V_{sound}^2(n')\varepsilon'(n')$ **where** ε' **indicated derivative wrt argument**

$$\boxed{\boldsymbol{\varepsilon\left(\frac{j_z}{\beta\gamma}\right) \geq \frac{T_{tt}(j_z)}{\gamma^2\left(1+\beta^2 R\left(\frac{j_z}{\beta\gamma}\right)\right)} \geq \frac{T_{tt}(j_z)}{\gamma^2(1+\beta^2)}}}$$

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Stronger bound; but requires knowledge of $R(n) = \frac{P(n)}{\varepsilon(n)}$.

This does not make it useless: one use it to constrain possible models of the EOS which give both ε and P.

Weaker bound; but reliable and does not require knowledge of $R(n) = \frac{P(n)}{\varepsilon(n)}$

- It will take a major effort of the lattice community to implement a program to constrain the EOS using this. This is likely to be sometime in the future—recall the method based on QCD inequalities took more than 2 decades.
  - It is likely will not be done at all unless there is some evidence that meaningful constraints are possible.
  - Thus it is useful to test the approach for situations where reliable tests of how constraining the approach is likely to be.
    - Theoretical analysis of worst cases us to conclude that the approach is clearly capable of constraining ε(n) to better—and typically much better—than an order of magnitude.
    - Since the underlying argument is relativity—not quantum mechanics—the approach can be tested with classical field theories that are tractable and capture at least certain features of QCD.
    - One can test how well the theory works for solvable QFTs (which are typicialy in 1-d an solvable via Bethe Ansatz & related methods)
    - The approach works for an any conserved $U(1)$ current; so it works for QCD with an isopsin current; this is solvable via χ-pt in the regime

Worst case analysis

$$\gamma^2\left(T_{tt}(j_z) + \beta^2 T_{zz}(j_z)\right) \geq \varepsilon(\beta\gamma j_z)$$

$$\varepsilon\left(\frac{j_z}{\beta\gamma}\right) \geq \frac{T_{tt}(j_z)}{\gamma^2(1+\beta^2)}$$

$$T_{zz}(j_z\,) \leq T_{tt}(j_z\,)$$

Worst case lower bound

Worst case upper bound

$$T_{tt}(j_z)\,\gamma^2\left(1+\beta^2\right) \geq \varepsilon(\beta\gamma j_z)$$

- Both bounds hold for all values of β, but only at $\beta = \frac{1}{\sqrt{2}}$, $\gamma = \sqrt{2}$, $\beta\gamma = 1$ do the upper & lower bonds occur at the same value of *n* for a given value of j_z .

$$3\,T_{tt}(j_z = n) \;\geq \varepsilon(n) \geq \frac{1}{3}\,T_{tt}(j_z = n)$$

Worst case bounds:

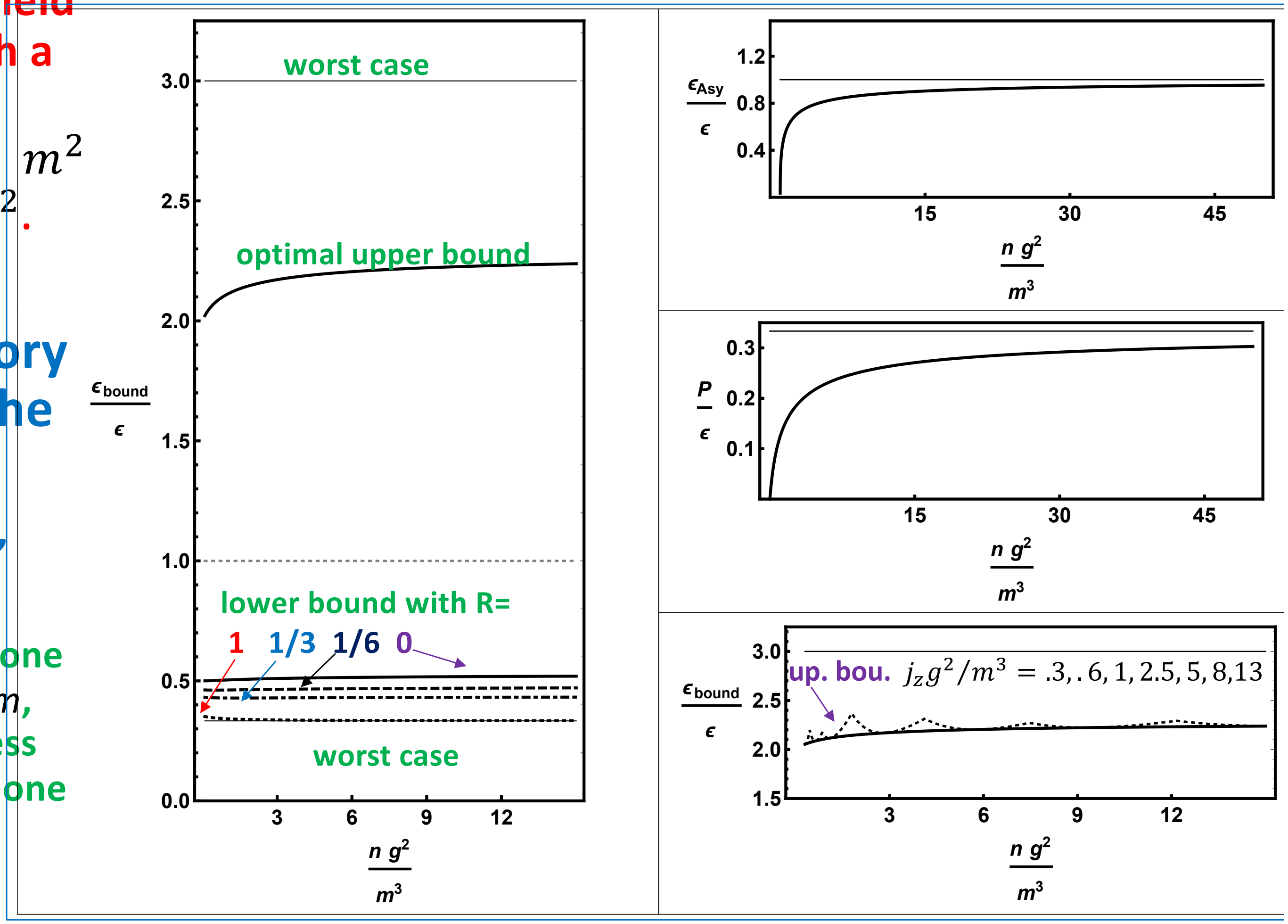
$$3\, T_{tt}(j_z = n) \geq \varepsilon(n) \geq \frac{1}{3} T_{tt}(j_z = n)$$

- There is at worst a factor of 9 between the upper and lower bounds for the energy density.
- In general one would expect the best bounds to be significantly better.
- **If one takes $\varepsilon^{\text{est}}(n) = T_{tt}\,(j_z = n)$ as an estimate for the true EOS one is off by no more than a factor of 3 and typically much better**

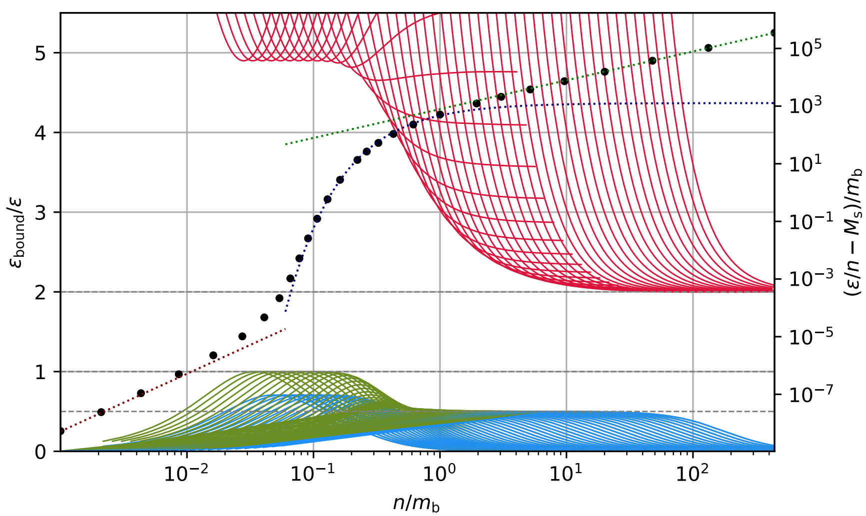
It is important to test how constraining the bounds are in general and also the extent to which $T_{tt}(j_z = n)$ follows $\varepsilon(n)$.

- **Example a classical field ϕ^4 field theory with a complex scalar :** $\mathcal{L} = \partial^\mu\phi\partial_\mu\phi^* - m^2 \phi^*\phi - g^2(\phi^*\phi)^2$.

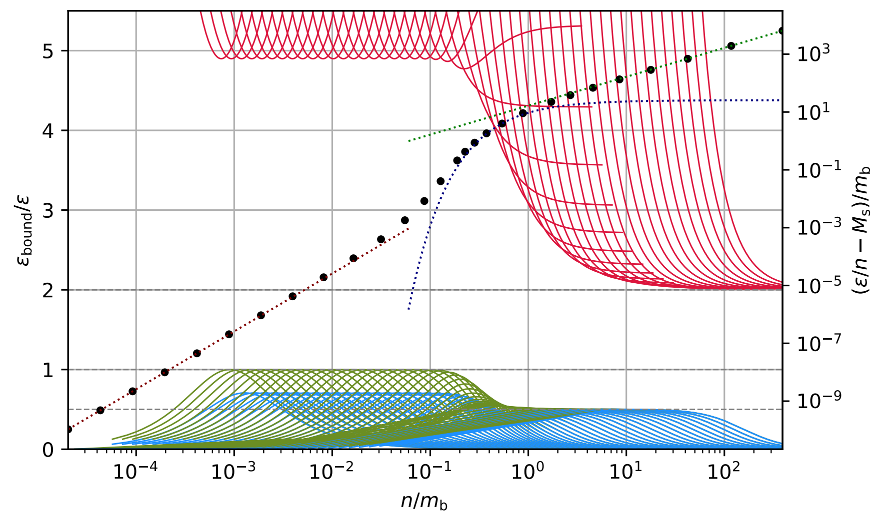
- **Note that this theory shares with QCD the property that at aseptically large *n*, $\varepsilon \sim n^{4/3}$.**
  - Classical theory has one massive parameter, *m*, and one dimensionless one, *g*, which allows one to rescale ε and *n*.



What about real quantum field theories? Eric Overmann (U. Jena) & I have preliminary results for the Massive Thirring/Sine-Gordon model solved via Bethe Ansatz; *κ=0* is uncoupled Sine-Gordon model and *κ=1* is uncoupled MTM, *κ >1* MTM with repulsive interactions

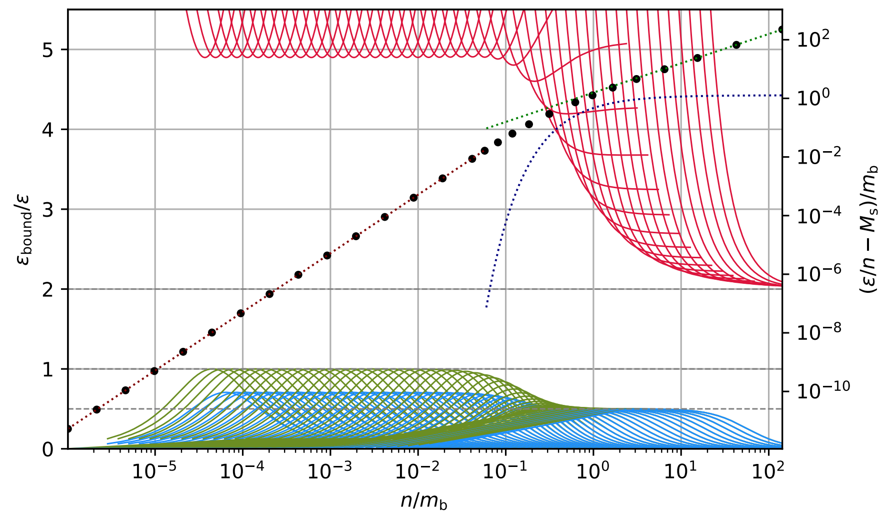


κ=0.001

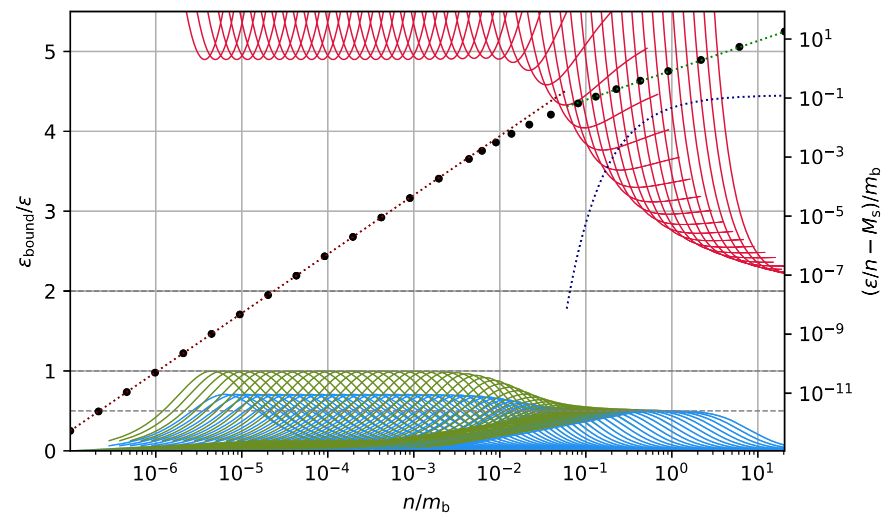


κ=0.05

Red is upper bound. Green is lower bound with actual "pressure". Blues is lower bound with *P=ε*



κ=1.0



κ=10.0

For all cases except for low *n* (and which has *P*=*ε*) bound is significantly better than worst case ratio of upper to lower bound factor of 9).

- **Some open questions:**
  - **Will the constraints obtained with this method be stronger enough to justify the computational costs?**
    - **Studies with "toy" models (classical, 1+1 dimensions etc.) may give clues**
  - **Will calculations with fixed J_z give useful qualitative/semi-quantative estimates of the EOS beyond the boumds**
    - **Studies with "toy" models (classical, 1+1 dimensions etc.) may give clues**
  - **While there are no sign problems preventing implementation of the approach, it remains possible that some other technical issues may render the approach nonviable.**
    - **Lattice studies of QCD in 1+1 & 2+1 dimensions may help answer this**

Conclusions

- The EOS for cold nuclear matter is an important quantity which remains in intractable with the current state of the art due to the sign problem

- Sign problems can be, and have been, evaded to establish an upper bound for P for using QCD inequalities .
- Sign problems can be evaded using relativity to establish both upper and lower bounds for ε .
    - **Tests of the approach have begun with toy models; more tests are needed.**