Three-loop Jet Function for Boosted Heavy Quarks

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Overview

1. **Introduction**

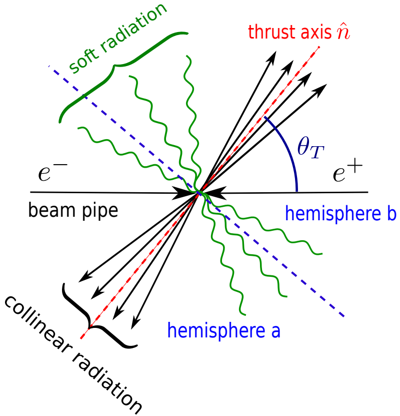
2. **Jet function computation**

3. **Main results**

4. **Summary**

Introduction

e^+e^- collision producing top quarks



Taken from [Mateu and Rodrigo, 2013]

- Incoming massless e^+e^- with $E_{\mathrm{CM}} = Q$.
- **Event shapes** in e^+e^- have been used to learn about hadronization, check group structure, determine α_s, etc.
- Outgoing $t, \bar{t}$ with mass m and decay width Γ_t. We are interested in $Q \gg m \gg \Gamma_t$.
- Event shapes probing the invariant jet mass M_i can be used to determine the **top mass** with uncertainty $< \Lambda_{\mathrm{QCD}}$ [Fleming, Hoang, Mantry and Stewart, 2008a].

Invariant hemisphere mass in e^+e^- collisions

- The invariant mass is defined as

$$P_i^\mu \equiv \sum_{j \in \text{hem. } i} p_j^\mu, \qquad M_i^2 \equiv P_i^\mu P_{i,\mu}.$$

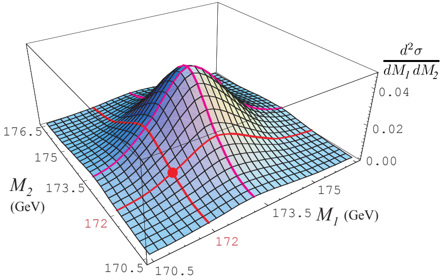
- Maximal sensitivity to m is attained in the peak region,

$$\hat{s} \equiv \frac{M_i^2 - m^2}{m} \sim \Gamma_t \quad (\text{implies } M_i \sim m)\,.$$

- EFT setup for peak region:

$$\text{QCD} \to \text{SCET} \to (2\times)\text{bHQET}$$

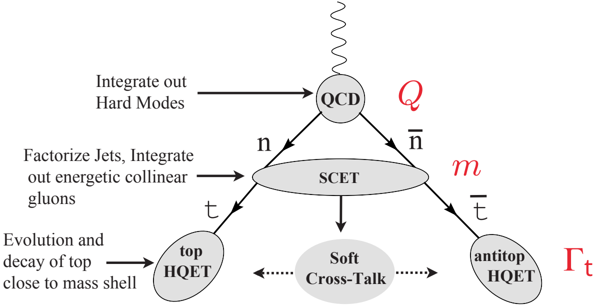
$$Q \gg m \sim M_i \gg \Gamma_t \sim \hat{s} \gg \Lambda_{\text{QCD}}$$



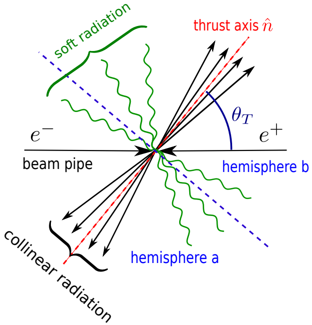
[Fleming, Hoang, Mantry and Stewart, 2008b]

EFT approach in the peak region

$$Q \gg m \gg \Gamma_t \sim \hat{s} \equiv \frac{M_i^2 - m^2}{m} \gg \Lambda_{\rm QCD} \;\Rightarrow$$ Event-shapes factorize at leading power in $\frac{m}{Q}, \frac{\Gamma_t}{m}, \frac{\hat{s}}{m}$



[Fleming, Hoang, Mantry and Stewart, 2008b]



Factorization theorem

[Fleming, Hoang, Mantry and Stewart, 2008a]

For example, the **double hemisphere invariant mass distribution**

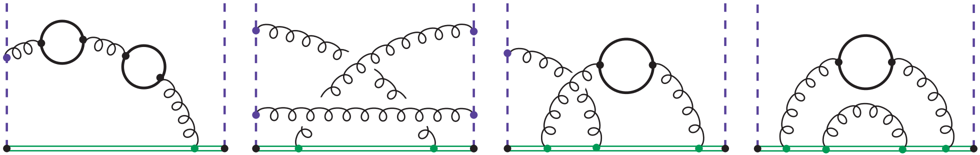
$$\frac{\mathrm{d}^2\sigma^{(\mathrm{dijet})}}{\mathrm{d}M_1^2\mathrm{d}M_2^2} = \sigma_0\, H_Q^{(n_\ell+1)}(Q,\mu)\, H_m\Big(m, \frac{Q}{m}, \mu\Big) \int \mathrm{d}\ell^+ \mathrm{d}\ell^-\, S^{(n_\ell)}(\ell^+,\ell^-,\mu)$$
$$\times\, B_n^{(n_\ell)}\Big(\frac{M_1^2 - Q\ell^+}{m} - m, \Gamma_t, \mu\Big)\, B_{\bar{n}}^{(n_\ell)}\Big(\frac{M_2^2 - Q\ell^-}{m} - m, \Gamma_t, \mu\Big).$$

For $\hat{s} = 2\, v\cdot r$ and m in the pole scheme the **jet function B** takes the form

$$B(\hat{s}, 0, \Gamma_t) \equiv \mathbf{Im}\left[\mathcal{B}(\hat{s} + i\Gamma_t)\right], \quad \mathcal{B}(\hat{s}) \equiv \frac{-i}{4\pi N_c m} \int \mathrm{d}^d x\, e^{i\, r\cdot x}\, \langle 0| T\{\bar{h}_v(0) W_n(0) W_n^\dagger(x) h_v(x)\}|0\rangle$$

The three-lop jet function

- Already known **analytically** at two loops [Jain, Scimemi and Stewart, 2008].
  We increase the precision by one order reaching three-loop accuracy.



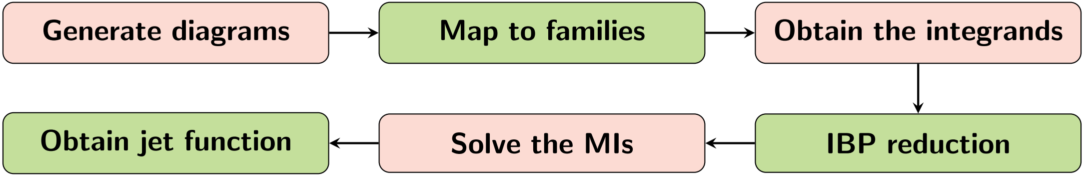
- Necessary to obtain the m at an e^+e^- collider and to improve the calibration between m^{MC} and m^{MSR} at $\mathrm{N}^3\mathrm{LL}'$ [Butenschoen et al., 2016, Dehnadi, Hoang, Jin and Mateu, 2023].

- The expression of B in terms of Wilson lines (and thus the computational strategy) resembles that of the soft function for heavy-to-light quark decays [Brüser, Liu and Stahlhofen, 2020].

Jet function computation

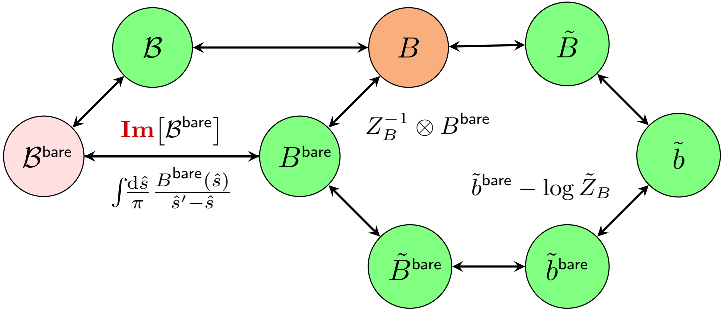
The complexity of a multiloop computation

	1 loop	2 loops	3 loops	4 loops
Feynman diagrams	4	$\sim$50	$\sim$1100	$\sim$31000
Scalar integrals	3	$\sim$70	$\sim$5400	?
Master integrals	1	3	20	?



From $\mathcal{B}^{\rm bare}$ to B (and vice versa)

We compute $\mathcal{B}^{\rm bare}(\hat{s}) \equiv \dfrac{-i}{4\pi N_c m}\int \mathrm{d}^d x\, e^{i\,r\cdot x}\langle 0|T\{\bar{h}_v(0)W_n(0)W_n^\dagger(x)h_v(x)\}|0\rangle$, then obtain B.



$$\tilde{B}(x) = \int \mathrm{d}\hat{s}\, e^{-ix\hat{s}} B(\hat{s})$$

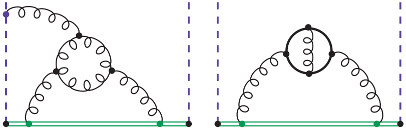
$$\tilde{B}(x) = \exp\left[\tilde{b}(x)\right]$$

We derived **relations between coefficients**, that enable efficient code implementation (e.g. in SCETLIB).

$$\tilde{B}_{in} = \frac{(-1)^n}{n!}\sum_{j=k-2}^{l-1} B_{ij}\,|j|!\,\kappa_{j+1-n}$$

Generating Feynman diagrams

- Diagrams are generated with `qgraf` [Nogueira, 1993].
- ~ 1100 diagrams are obtained.



Processing the diagrams with `Looping` [Brüser, np]

- Integral families are identified using the algorithm in [Pak, 2012].
- Remove linear dependences: Apply partial fraction using Gröbner basis [Pak, 2012].
- Color factor, Lorentz indices and Dirac traces: `FORM` [Ruijl, Ueda and Vermaseren, 2017] (`color.h` [van Ritbergen, Schellekens and Vermaseren, 1999]).
- Symmetries related to momentum shifts are exploited and scaleless integrals are discarded.

We obtained ~ 5400 non-vanishing scalar integrals.

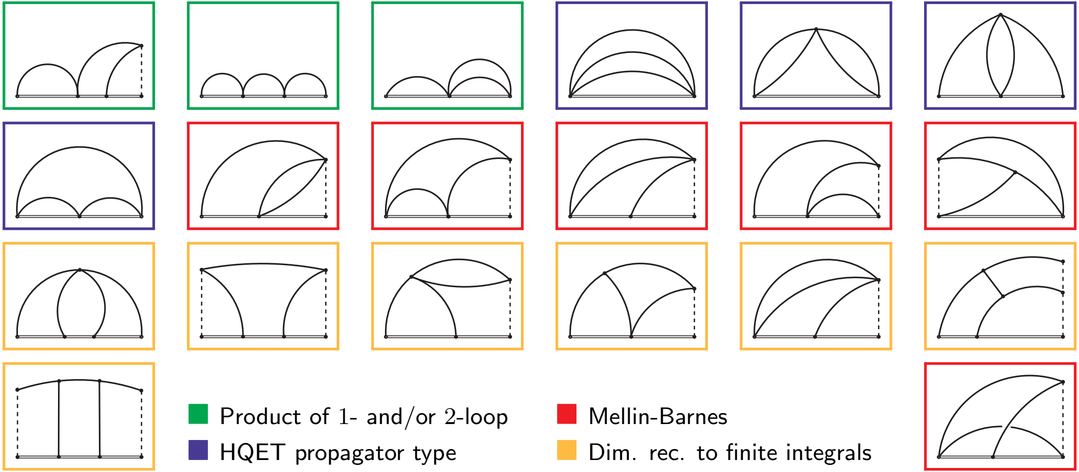
Integral reduction

- We **IBP** reduce [Chetyrkin and Tkachov, 1981] each integral family separately to obtain minimal set of **Master Integrals** (MIs).
- We use `FIRE6` [Smirnov and Chuharev, 2020] and `LiteRed` [Lee, 2014].
- Map across families with `Looping` to identify new relations. We are left with 21 MI candidates.
- **Dimensional recurrence relations** [Tarasov, 1996]: we write the MIs in d dimensions as a linear combination of MIs in $d \pm 2$,

$$\vec{g}^{\,(d)} = T_{\mp}^{(d\pm2)} \vec{g}^{\,(d\pm2)} \quad \Rightarrow \quad \vec{g}^{\,(d)} = T_{\mp}^{(d\pm2)} T_{\pm}^{(d)} \vec{g}^{\,(d)} \,.$$

The number of MIs is reduced to 20.

Master integrals



Computing the Master Integrals

We are left with a basis of 20 integrals that need to be **computed analytically**.

- 7 are found in [Grozin, 2000] or can be written as products of one- and two-loop MIs.
- The remaining 13 MIs require special techniques.

Mellin-Barnes techniques

- Based on the equality

$$(A+B)^{-\lambda} = \int_{\mathcal{C}} \frac{\mathrm{d}s}{2\pi i} \frac{\Gamma(s)\Gamma(\lambda - s)}{\Gamma(\lambda)} A^{s-\lambda} B^{-s}$$

- The integral in $\mathcal{C}$ can often be computed using Cauchy's theorem.

Dim. rec. to finite integrals

[von Manteuffel, Panzer and Schabinger, 2015]

- Look for finite integrals in $d + 2n$ using `Reduze2` [von Manteuffel and Studerus, 2012].
- The finite integrals can be computed with `HyperInt` [Panzer, 2015].
- Relate to the MIs via IBP and dimensional recurrence relations [Tarasov, 1996].

Main results

Result for the three-loop jet function

- In order to obtain the renormalized three-loop jet function, we recomputed the bare **one- and two-loop jet function for arbitrary d**.
- The $\mathcal{O}(\alpha_s^3)$ non-logarithmic part of the renormalized $\tilde{\boldsymbol{b}} = \log \tilde{\boldsymbol{B}}$

$$\begin{aligned}\tilde{b}_{30} &= \left(\frac{203\pi^2\zeta_3}{27} - \frac{105398\zeta_3}{243} + \frac{236\zeta_3^2}{9} + \frac{902\zeta_5}{9} + \frac{31952191}{26244} + \frac{93821\pi^2}{8748} - \frac{3023\pi^4}{4860} + \frac{1031\pi^6}{10206}\right)C_F C_A^2 \\ &+ \left(\frac{3488\zeta_3}{243} + \frac{846784}{6561} - \frac{8\pi^2}{243} + \frac{52\pi^4}{1215}\right)C_F T_F^2 n_\ell^2 + \left(\frac{10760\zeta_3}{81} + \frac{8\pi^2\zeta_3}{9} + \frac{224\zeta_5}{9} - \frac{124717}{486} - \frac{55\pi^2}{54} + \frac{148\pi^4}{405}\right)C_F^2 T_F n_\ell \\ &+ \left(\frac{1664\zeta_3}{81} - \frac{76\pi^2\zeta_3}{27} - \frac{88\zeta_5}{3} - \frac{5273287}{6561} - \frac{12793\pi^2}{2187} - \frac{421\pi^4}{1215}\right)C_F C_A T_F n_\ell\,.\end{aligned}$$

$$\tilde{b}_{l1} = \gamma^B_{l-1} + 2\sum_{j=1}^{l-1} j\beta_{l-j-1}\,\tilde{b}_{j0} \qquad \tilde{b}_{l2} = \Gamma^c_{l-1} + \sum_{j=1}^{l-1} j\beta_{l-j-1}\,\tilde{b}_{j1} \qquad \tilde{b}_{lk} = \frac{2}{k}\sum_{j=k-2}^{l-1} j\beta_{l-j-1}\tilde{b}_{j,k-1}$$

Checking the result: anomalous dimension

- The three-loop anomalous dimension was already known
  [Korchemsky and Radyushkin, 1987, Moch, Vermaseren and Vogt, 2004]
  [Hoang, Pathak, Pietrulewicz and Stewart, 2015, Brüser, Liu and Stahlhofen, 2020, Becher and Schwartz, 2008]
- From the RGE, we derived **universal** closed (all-order) form for the Z-factors

$$z(\gamma, \alpha_s) \equiv \int_0^{\alpha_s} \frac{\mathrm{d}x}{x} \frac{\gamma(x)}{\varepsilon + \hat{\beta}(x)}, \qquad \hat{\beta}(\alpha_s) = -\frac{1}{2\alpha_s}\beta_{\mathrm{QCD}}(\alpha_s).$$

- For example:
  - $\alpha_s^{\mathrm{bare}} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_\alpha[\alpha_s(\mu)]\,\alpha_s(\mu) \quad \Rightarrow \quad Z_\alpha(\alpha_s) = \exp\left[-z\left(\hat{\beta}, \alpha_s\right)\right]$
  - $\tilde{B}^{\mathrm{bare}}(x) = \tilde{Z}_B(x,\mu)\,\tilde{B}(x,\mu) \quad \Rightarrow \quad \tilde{Z}_B(x,\mu) = C_B[\alpha_s(\mu)]\,(ix\mu e^{\gamma_E})^{z_B[\alpha_s(\mu)]}$:

$$C_B(\alpha_s) = \exp\left[\frac{1}{2} z(\gamma_B^{\mathrm{nc}} + z_B, \alpha_s)\right], \qquad z_B(\alpha_s) = z(\Gamma^{\mathrm{c}}, \alpha_s).$$

Checking the result

- ✓ Our calculation reproduces the known three-loop anomalous dimension.
- ✓ All MI numerically checked with `FIESTA` [Smirnov, Shapurov and Vysotsky, 2022] and/or `PySecDec` [Heinrich et al., 2024] up to $\mathcal{O}(\varepsilon^3)$.
- ✓ We perform the calculation in a general covariant gauge. The gauge parameter drops out in $\mathcal{B}^{\text{bare}}$ once expressed in terms of MIs.
- ✓ The T_F^2 piece matches the large-β_0 computation in [Gracia and Mateu, 2021].
- ✓ Our analytic result of $\tilde{b}_{30}$ agrees with the estimate in [Gracia and Mateu, 2021].
- ✓ $\tilde{B}$ obeys non-Abelian exponentiation [Gardi, Smillie and White, 2013].
  - ✓ The terms proportional to C_F^3 and $C_F^2 C_A$ vanish.
  - ✓ Only diagrams involving a fermion loop with four gluon attachments contribute to the $C_F^2 T_F n_\ell$ term.

Jet function in a short-distance mass scheme

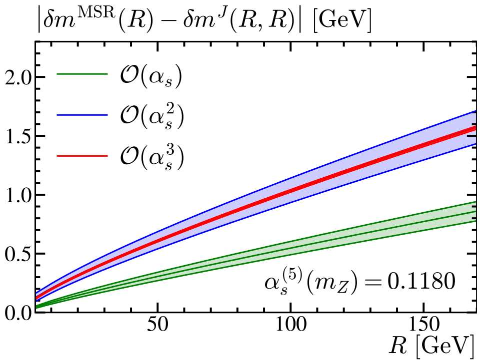
- The $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon in B is canceled by switching from the pole mass m to a short-distance scheme $m^{\rm SD}$ like the MSR mass [Hoang, Jain, Scimemi and Stewart, 2008, Hoang et al., 2018]

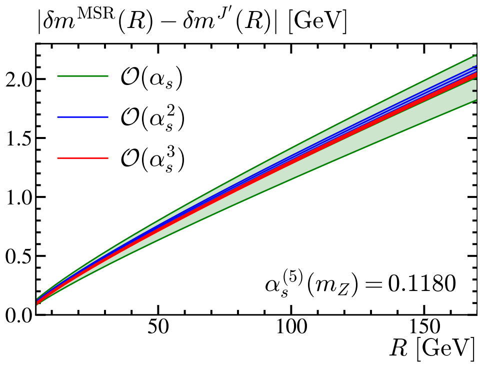
$$B(\hat{s},\delta m,\Gamma_t,\mu) = \mathrm{Im}\big[\mathcal{B}(\hat{s}-2\delta m + i\Gamma_t,\mu)\big] = \exp\biggl(-2\delta m\frac{\partial}{\partial\hat{s}}\biggr)B(\hat{s},0,\Gamma_t,\mu)\,, \qquad \delta m \equiv m - m^{\rm SD}.$$

- We studied how two different short-mass schemes approach their asymptotic limit
  - The *jet-mass*, defined in [Jain, Scimemi and Stewart, 2008].
  - The *non-derivative jet-mass*, defined in [Gracia and Mateu, 2021].

- The non-derivative scheme can be used to numerically estimate the four-loop jet function.

Derivative and non-derivative mass schemes





$$\delta m^{J}(\mu,R) = \frac{Re^{\gamma_E}}{2}\left\{\frac{\mathrm{d}}{\mathrm{d}\log(ix)}\log\!\left[m\tilde{B}(x,\mu)\right]\right\}_{ixe^{\gamma_E}=1/R}$$

$$\delta m^{J'}(R) = \frac{Re^{\gamma_E}}{2}\log\!\left[m\tilde{B}\!\left(\frac{1}{iRe^{\gamma_E}},R\right)\right]$$

Four-loop estimate

- We can obtain a **prediction** of $\tilde{b}_{40}(n_\ell)$ based on the $u = 1/2$ renormalon dominance [Hoang et al., 2018].

- Using the R-evolution formalism applied to the non-derivative jet-mass,

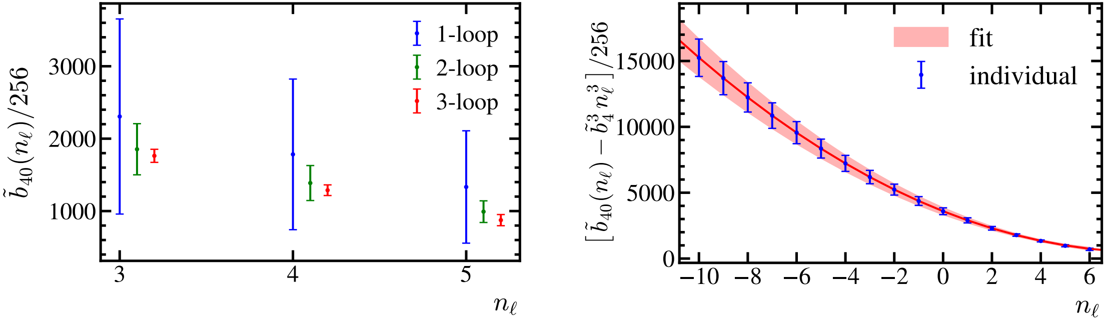
$$\tilde{b}_{l0} = (2\beta_0)^l \sum_{k=0}^{l-1} S_k \sum_{j=0}^{l-1-k} g_j \left(1 + \frac{\beta_1}{2\beta_0^2} + l\right)_{k-j-l}$$

  where S_k depends on the coefficients of the position-space jet function up to $(k+1)$-loops and β_{QCD}. g_j only depends on β_{QCD}.

- By setting the upper limit of the outer sum to a value smaller than $l-1$, we obtain an approximate formula for the jet function coefficients.

Four-loop estimate

Following the procedure in [Gracia and Mateu, 2021], we obtain predictions for the non-logarithmic pieces of the logarithm of the jet function:



Summary

Summary

- The bHQET jet function is a universal ingredient in the factorization theorems for observables that probe the invariant mass of heavy quark jets in the peak region.
- It takes a central role in determining the top quark mass in a renormalon-free scheme.
- Applying modern multi-loop techniques, **we have analytically computed the 3-loop jet function**: `arXiv:2412.06881`
- Our result paves the way to $\mathrm{N^3LL'}$ resummed cross sections.
- $\mathrm{N^3LL'}$ calibration of Monte-Carlo top mass already possible ($\mathrm{N^3LL}$: [Bachu et al., 2021]).
- Outlook: Allow for massive bottom quarks, push to $\mathrm{N^4LL}$...

Thank you for your attention!

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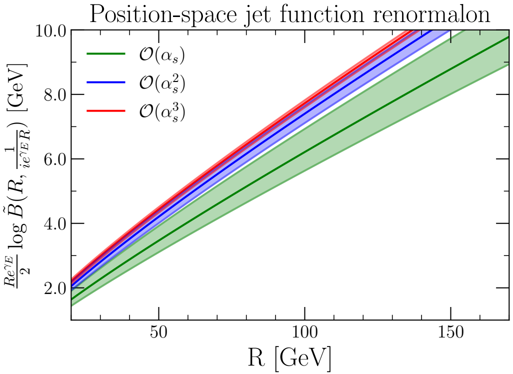
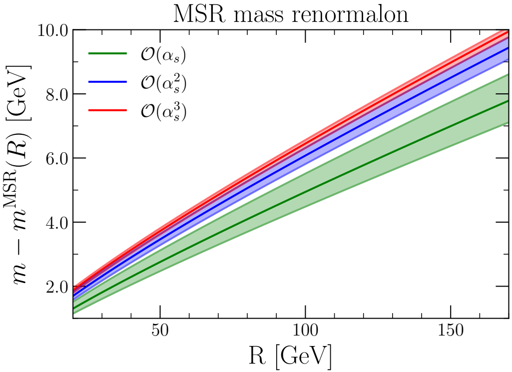
Position-space jet function renormalon

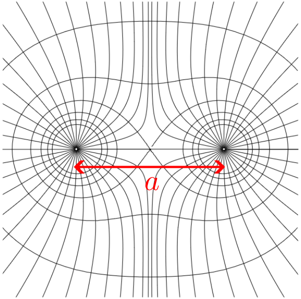
$$\frac{1}{2ix}\log\left[m\tilde{B}(x,\mu)\right] - \left(m - m^{\mathrm{SD}}\right) \equiv \text{renormalon free}$$

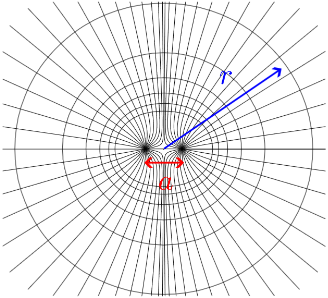




Basic concepts about EFTs

- Instead of using the full theory, compute quantities in powers of a small quantity λ.
- Analogy: electrostatic potential [Manohar, 2018]





$$V(\vec{r}, a) = \frac{1}{r} \sum_{\ell,m} c_{\ell m}(a) Y_{\ell m}(\Omega) \left(\frac{a}{r}\right)^{\ell}$$

$$O_{\mathrm{QCD}}(\hat{s}, Q) = \sum_{\ell m} C_{\ell m}(Q, \mu)\, O_{\mathrm{EFT},\ell m}(\hat{s}, \mu) \left(\frac{\hat{s}}{Q}\right)^{\ell}$$

Separation of scales in EFTs

$$O_{\text{QCD}}(\hat{s}, Q) = \sum_{\ell m} C_{\ell m}(Q, \mu)\, O_{\text{EFT},\ell m}(\hat{s}, \mu) \left(\frac{\hat{s}}{Q}\right)^{\ell}$$

$Q =$ hard scale.
$\hat{s} =$ soft scale.
$\mu =$ renormalization scale.

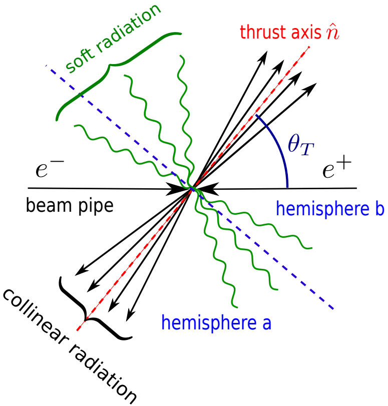
We can avoid large $\log$s in $\langle O_{\text{EFT},\ell m}\rangle$ with an appropriate choice of μ

$$\langle O_{\text{QCD},\ell m}\rangle \sim \log\left(\frac{\hat{s}}{Q}\right) \qquad C_{\ell m} \sim \log\left(\frac{\mu}{Q}\right) \qquad \langle O_{\text{EFT},\ell m}\rangle \sim \log\left(\frac{\hat{s}}{\mu}\right)$$

and resum the $\log$s in $C_{\ell m}(Q, \mu)$ with an RGE

$$C_{\ell m}(Q, \mu) O_{\text{EFT},\ell m}(s, \mu) = C_{\ell m}(Q, \mu) U(\mu, \mu') O_{\text{EFT},\ell m}(s, \mu').$$

Soft Collinear Effective Theory (SCET)



- Define $n^\mu = (1, \vec{n})$, $\bar{n}^\mu = (1, -\vec{n})$ such as

$$n^2 = \bar{n}^2 = 0 \qquad n \cdot \bar{n} = 2.$$

$$p^\mu = (p^+)\frac{\bar{n}^\mu}{2} + (p^-)\frac{n^\mu}{2} + p_\perp^\mu.$$

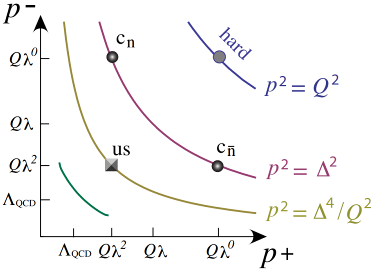
In SCET I, for $\Delta \ll Q$,

- collinear modes $p_n^\mu \sim Q\left(\frac{\Delta^2}{Q^2}, 1, \frac{\Delta}{Q}\right)$.
- soft modes $p_s^\mu \sim Q\left(\frac{\Delta^2}{Q^2}, \frac{\Delta^2}{Q^2}, \frac{\Delta^2}{Q^2}\right)$.

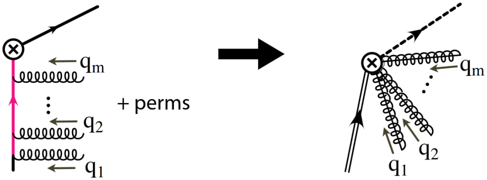
Separation of scales in SCET I

We can expand in $\lambda = \Delta/Q \ll 1$, and separate

- Hard modes $p_h^2 \sim Q^2$.
- Collinear modes $p_n^2 \sim Q^2(\Delta/Q)^2$.
- Soft modes $p_s^2 \sim Q^2(\Delta/Q)^4$.



[Stewart and Bauer, 2013]



$$W_n = \sum_k \sum_{\text{per}} \frac{(-g)^k}{k!} \frac{\bar{n} \cdot A_n(q_1) \ldots \bar{n} \cdot A_n(q_k)}{[\bar{n} \cdot q_1] \ldots [\bar{n} \cdot \sum_{i=1}^k q_i]}$$

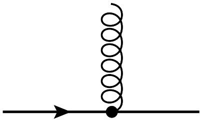
Heavy Quark Effective Theory (HQET)

Momentum of an off-shell top quark

$$p_t = mv + k, \qquad v^2 = 1$$

If $k^2 \ll m^2$, we can expand in $\lambda = k^2/m^2$.

$$i\frac{\not{p}_t + m}{p_t^2 - m^2 + i\,\eta}\delta_{\alpha\,\beta} \to i\frac{1}{2\,v\cdot k + i\,\eta}\delta_{\alpha\,\beta}$$



$$i\gamma^\mu T^a_{\alpha\,\beta} \to iv^\mu T^a_{\alpha\,\beta}$$