Small-x physics

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Outline

- Lecture 1a: DIS in dipole picture
- Lecture 1b: Light cone quantization
- Lecture 1c: The Color Glass Condensate: Wilson line
- Lecture 2a: The Color Glass Condensate: saturation
- Lecture 2b: MV model
- Lecture 2c: BK equation

Ideology

- Not infinitely many slides: intended as lectures, not a talk!
- Not summarizing recent work, but focusing on basic concepts
- Some calculations on the way
- Interrupt and ask!

Literature

Very good books, basis of these lectures

- Y. V. Kovchegov and E. Levin, "Quantum chromodynamics at high energy," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **33** (2012).
- V. Barone and E. Predazzi, "High-Energy Particle Diffraction," Springer 2002

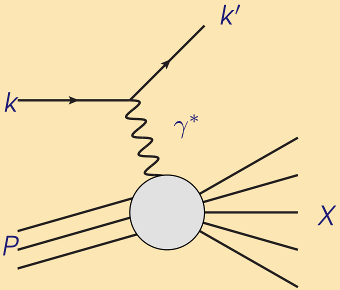
Reviews:

- E. Iancu and R. Venugopalan, "The Color glass condensate and high-energy scattering in QCD," In *Hwa, R.C. (ed.) et al.: Quark gluon plasma* 249-3363 `[hep-ph/0303204]`
- F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, "The Color Glass Condensate," Ann. Rev. Nucl. Pa rt. Sci. **60** (2010) 463 `[arXiv:1002.0333 [hep-ph]]`
- H. Weigert, "Evolution at small x(bj): The Color glass condensate," Prog. Part. Nucl. Phys. **55** (2005) 461 `[hep-ph/0501087]`
- J. Jalilian-Marian and Y. V. Kovchegov, "Saturation physics and deuteron-Gold collisions at RHIC," Prog. Part. Nucl. Phys. **56** (2006) 104 `[hep-ph/0505052]`
- F. Gelis, T. Lappi and R. Venugopalan, "High energy scattering in Quantum Chromodynamics," Int. J. Mod. Phys. E **16** (2007) 2595 `[arXiv:0708.0047 [hep-ph]]`

- Lecture note `https://users.jyu.fi/~tulappi/heqcd23/pruju.pdf`
  Plan: cover Chapter 4 and Sec 5.1

1 DIS in the dipole picture

DIS kinematics



$$
\begin{aligned}
s &= (k+P)^2 \\
q &= k - k' \quad q^2 \equiv -Q^2 \\
W^2 &= (P+q)^2 \\
\nu &= P \cdot q / m_N \\
x &= \frac{Q^2}{2P \cdot q} = \frac{Q^2}{2\nu m_N} = \frac{Q^2}{W^2 + Q^2 - m_N^2} \\
y &= \frac{2P \cdot q}{2P \cdot k} = \frac{W^2 + Q^2 - m_N^2}{s m_N^2}
\end{aligned}
$$

Topic of these lectures: high energy limit $x \to 0$

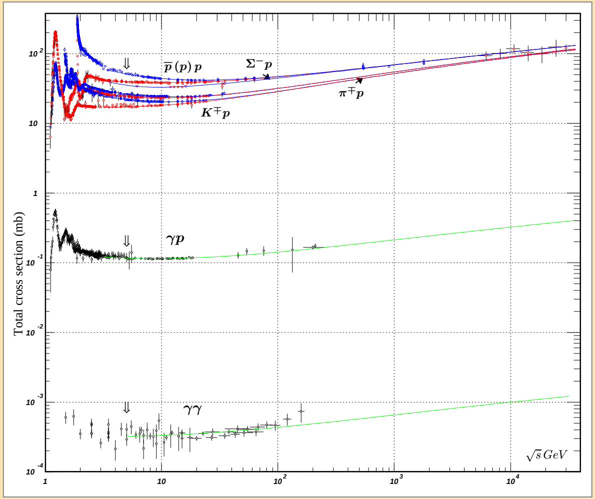
- This is when $W^2 \to \infty$; $\nu \to \infty$; i.e. the γ^*-target c.m.s. energy is high.
- Now Q^2 is "fixed". (DGLAP limit is x fixed, Q^2 large)
- Useful to think of W as longitudinal momentum, Q as transverse

Cross sections vs. energy

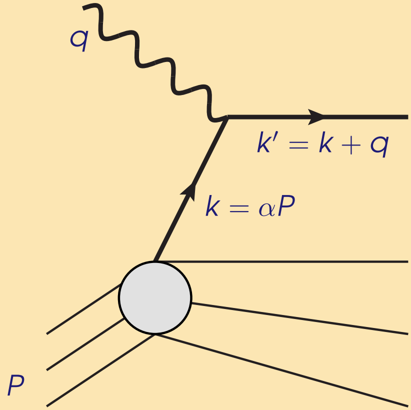
I want to convince you that the γ^* is the theorist's favorite hadron!

γ scattering behaves just like p scattering — apart from extra $\frac{1}{137}$

Same should be true for γ^*



Infinite momentum frame IMF



Assume photon hits quark carrying fraction α of target momentum. On-shell outgoing quark:

$$0 = (q + \alpha P) = -Q^2 + 2\alpha P \cdot q + \overbrace{\alpha^2 m_N^2}^{\approx 0}$$

$$\Longrightarrow \quad \alpha = \frac{Q^2}{2P \cdot q} = x$$

$x =$ momentum fraction

Identify kinematical variable x (measured from electron) with momentum fraction in IMF, @LO

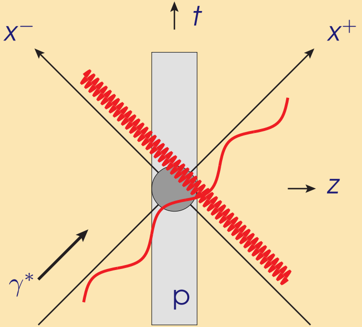
(In rest frame $k = \alpha P$ not good approx)

Target Rest Frame TRF kinematics

Light cone coordinates $x^{\pm} = \frac{1}{\sqrt{2}}(t \pm z)$ ($\mathbf{x}_\perp$ is 2d transverse, γ^* right-moving)

$$\text{Target } P^\mu = (\overset{0}{m}, \overset{\perp}{\mathbf{0}}, \overset{z}{0}) \Longrightarrow (\overset{+}{m/\sqrt{2}}, \overset{-}{m/\sqrt{2}}, \overset{\perp}{\mathbf{0}})$$

$$\text{Photon } q^\mu = (\overset{0}{\nu}, \overset{\perp}{\mathbf{0}}, \overset{z}{\sqrt{\nu^2 + Q^2}}) \Longrightarrow (\overset{+}{q^+}, \overset{-}{-Q^2/(2q^+)}, \overset{\perp}{\mathbf{0}})$$



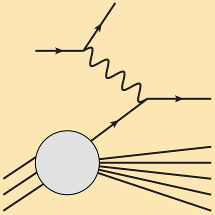
High energy: $q^+ \approx \sqrt{2}\nu$ **big**

Look at γ^* wavefunction $e^{-i(q^+x^- + q^-x^+)}$

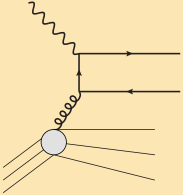
- Very accurate resolution in x^-
- No resolution in x^+
  Scattering instantaneous in x^+ compared to natural timescale of γ^*

Quantum fluctuations in γ^* before interaction, frozen **inside** target.

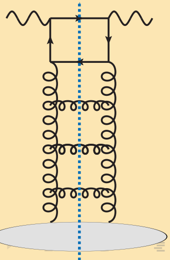
Inclusive DIS in collinear factorization



- Proton consists of partons
- Struck by γ^*: interaction time short
  $\Longrightarrow$ partonic interaction factorizes
- LO $\gamma^* + q \to q$ counts quarks
  $\Longrightarrow$ PDF's
- NLO: also gluon-initiated



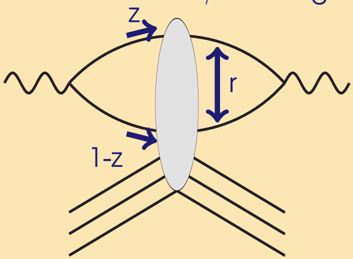
- High Q^2 logs resummed into DGLAP
- $Q^2 \gg \Lambda^2_{\mathrm{QCD}}$ and $W^2 \gg \Lambda^2_{\mathrm{QCD}}$



Nonperturbative physics parametrized by PDF's

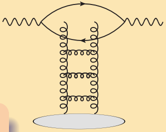
Inclusive DIS in dipole picture

Different frame: γ^* moving fast; not proton



- γ^* consists of partons
- Partons struck by target gluon field: interaction time short $\Longrightarrow$ structure of γ^* frozen
- LO: γ^* is only $q\bar{q}$ dipole
- NLO corrections: also include $\gamma^* \to q\bar{q}g$
- $q\bar{q}$ dipole of size $r \sim 1/Q$

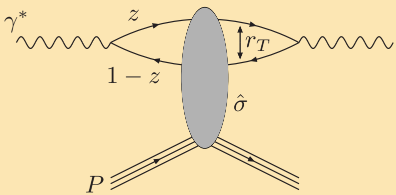
- High W^2 logs resummed into BK/BFKL
- $W^2 \gg Q^2$



Nonperturbative physics: q/g+p scattering amplitudes

Dipole picture of DIS

Simplest hadronic state in the interacting γ^*: quark-antiquark dipole.



$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{r}_\perp \, \mathrm{d}z \left| \psi^{\gamma^* \to q\bar{q}}(\mathbf{r}_\perp, z)_{T,L} \right|^2 2\mathrm{Im}\mathcal{A}$$

High energy: we assume (lifetime/timescale) factorization between

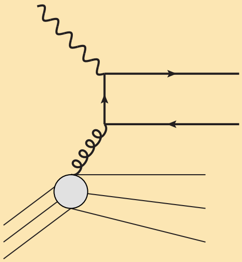
- $\left|\psi^{\gamma^* \to q\bar{q}}(\mathbf{r}_\perp, z)_{T,L}\right|^2$: probability for photon to fluctuate into $\bar{q}q$
- $\mathrm{Im}\mathcal{A}$ = imaginary part of the forward elastic scattering amplitude, i.e. the total cross section; optical theorem

Remarks: same process in IMF

Dipole picture diagram in collinear picture

- Looks like formally higher order (NLO DIS)
- Does not describe valence quarks

However: leading contribution at small-x



- The valence quark distribution is small
- DGLAP sea quarks come from gluons: $xq(x, Q^2) \sim \alpha_s \ln Q^2 xg(x, Q^2)$
- Same diagram contains both
  - $\gamma^* + g \to q + \bar{q}$: a NLO DIS process $\sim \alpha_s$
  - 1 DGLAP split $g \to q + \bar{q}, \sim \alpha_s$ + LO DIS process $\gamma^* + q \to q$
- Split between these two is scheme dependent
- Matching dipole and collinear pictures not very clear beyond leading log

  (where only $xg(x, Q^2)$ matters)

2 Light Cone Perturbation Theory

Quantizing the photon: LCPT

- Recall: want to understand the **partonic content of the photon**
- Unlike proton, γ^* is a perturbative object: calculable
- The theoretical tool of choice is Light Cone Perturbation Theory

What is Light Cone Perturbation Theory (LCPT)?

- Heisenberg quantization: time-dependent operators $\hat{A}(t)$, **equal-time** commutation relations $[\hat{A}(t), \hat{B}(t)]$ known.
- Then solve equation of motion $\partial_t \hat{A}(t) = -i[\hat{A}(t), \hat{H}(t)]$
- LCPT: choose the "time" variable to be light-like: x^+

Advantages and disadvantages

+ Only physical degrees of freedom $\Longrightarrow$ partonic interpretation
+ Maximal number of commuting Lorentz generators
+ Longitudinal boosts are easy $\Longrightarrow$ high energy
- 3d rotational invariance is hard
- Connection to lattice, hadron rest frame difficult

Idea of LCPT calculation

- Know free particle Fock states: $|\gamma^*\rangle_0, \quad |q\bar{q}\rangle_0, \quad |q\bar{q}g\rangle_0$ etc.
- **Interacting** states are superpositions of these:

$$|\gamma^*\rangle = (1+\dots)|\gamma^*\rangle_0 + \psi^{\gamma^*\to q\bar{q}} \otimes |q\bar{q}\rangle_0 + \psi^{\gamma^*\to q\bar{q}g} \otimes |q\bar{q}g\rangle_0 + \dots$$

- Calculate in QM perturbation theory, e.g. ground state $|0\rangle$ wavefunction:

$$\psi^{0\to n} = \sum_n \frac{\langle n|\hat{V}|0\rangle}{E_n - E_0} + \dots$$

  - Here $1/\Delta E$ is $\sim$ the lifetime of the quantum fluctuation from 0 to n
- "Energy" E is conjugate to "time", LC time is x^+ $\Longrightarrow$ LC energy k^-
- Note: energy not "conserved," only 3-momentum $\vec{k} = (k^+, \mathbf{k}_\perp)$ is

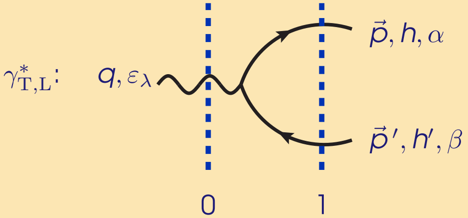
Connection to Feynman perturbation theory

- Matrix elements $\langle n|\hat{V}|m\rangle$ are vertices in Feynman rules
- LC energy denominators from propagators, integrating over k^- with pole

Let's calculate $\psi^{\gamma^* \to q\bar{q}}$

Steps for calculating light cone wave function:

$$\psi^{\gamma^*_{\mathrm{T/L}} \to q\bar{q}} = \frac{-ee_f \delta_{\alpha\beta}}{q^- - p^- - p'^-} \left[\bar{u}_h(p) \not{\varepsilon}_{\lambda,\mathrm{T/L}}(q) v_{h'}(p') \right]$$



- Color factor $\delta_{\alpha\beta}$ and electric charge ee_f
- Energy denominator $k_0^- - k_1^-$
- Matrix element $\bar{u} \not{\varepsilon} v_{h'}$
- Fourier transform to transverse coordinate space
  (Why coordinate space? Eikonal scattering, will come back to this in next lecture)

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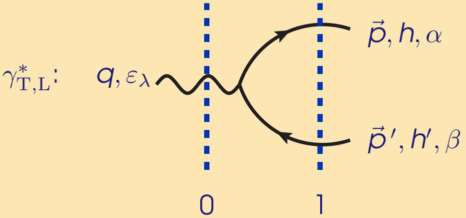

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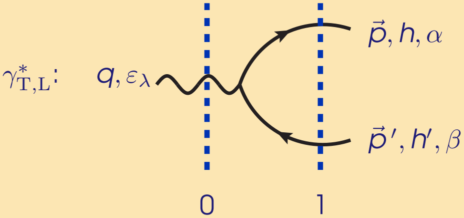

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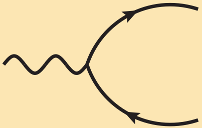
Matrix element

$\bar{u}_h(p)\not{\varepsilon}_{\lambda,\mathrm{T/L}}(q)v_{h'}(p')$ with



$$h, h' = \pm\frac{1}{2}; \quad \lambda = 0 = L, \quad \lambda = \pm 1 = T$$

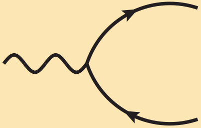
- Transv. polarization vector (LC gauge $\varepsilon^+ = 0$!!) $\varepsilon^\mu_{\lambda=\pm}(q) = (0, \frac{\mathbf{q}_\perp \cdot \boldsymbol{\varepsilon}_{\perp\lambda}}{q^+}, \boldsymbol{\varepsilon}_{\perp\lambda})$
- Circularly polarized 2d polarization vectors $\boldsymbol{\varepsilon}_{\perp\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp 1 \\ -i \end{pmatrix}$
- Longit. polarization effectively $\varepsilon^\mu_{\lambda=0}(q) = (0, \frac{\sqrt{Q^2 - \mathbf{q}_\perp{}^2}}{q^+}, \frac{\mathbf{q}_\perp}{\sqrt{Q^2 - \mathbf{q}_\perp{}^2}}) \xrightarrow{\mathbf{q}_\perp = \mathbf{0}} (0, \frac{Q}{q^+}, \mathbf{0})$,
- Using tables for matrix elements T polarization is

$$\bar{u}\not{\varepsilon}v = \frac{2\delta_{h,-h'}}{\sqrt{z(1-z)}}\left(z\delta_{\lambda,2h} - (1-z)\delta_{\lambda,-2h}\right)\boldsymbol{\varepsilon}_{\perp\lambda}\cdot\mathbf{k}_\perp + \delta_{h,h'}\delta_{\lambda,2s}\frac{\sqrt{2}m}{\sqrt{z(1-z)}}.$$

Remarks:

- Only relative center-of-mass momentum $\mathbf{k}_\perp = \mathbf{p}_\perp - z\mathbf{q}_\perp$
- Quark helicity conserving $\sim \mathbf{k}_\perp$ + helicity-flip $\sim m$

Energy denominator



Energy denominator $(q^- - k^- - k'^-)^{-1}$ (On-shell momenta!)

$$q^- - p^- - p'^- = -\left(\overbrace{\frac{Q^2}{2q^+}}^{-q^-} + \overbrace{\frac{\mathbf{p}_\perp{}^2 + m^2}{2zq^+}}^{p^-} + \overbrace{\frac{(\mathbf{p}_\perp - \mathbf{q}_\perp)^2 + m^2}{2(1-z)q^+}}^{p'^-} \right)$$

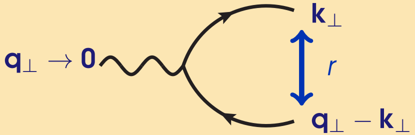
$$\frac{1}{q^- - p^- - p'^-} = \frac{-2q^+ z(1-z)}{Q^2 z(1-z) + m^2} \qquad Q^2 z(1-z) + m^2 \equiv \bar{Q}^2$$

- Also only relative center-of-mass momentum $\mathbf{k}_\perp = \mathbf{p}_\perp - z\mathbf{q}_\perp$

Fourier transform

High energy: **eikonal**, transverse **position** of parton does not change

$\Longrightarrow$ Fourier-transform $\mathbf{k}_\perp \to \mathbf{r}_\perp$ (Confusingly $\mathbf{r}_\perp$ is called $\mathbf{b}_\perp$ in TMD physics)



$$\mathrm{L:} \quad \int \mathrm{d}^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \frac{1}{{\mathbf{k}_\perp}^2 + \bar{Q}^2} \sim K_0(r\bar{Q})$$

$$\mathrm{T:} \quad \int \mathrm{d}^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \frac{k^i}{{\mathbf{k}_\perp}^2 + \bar{Q}^2} \sim \frac{r^i}{r^2} K_1(r\bar{Q})$$

- Recall $\bar{Q}^2 \equiv z(1-z)Q^2 + m^2$.
- Note asymptotics $K_{0,1}(x) \sim e^{-x} \Longrightarrow$ enforces $r \sim 1/Q$.

DIS dipole picture: summary

- High energy DIS: γ^* fluctuates into $q\bar{q}$, which scatters off target

$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{r}_\perp \, \mathrm{d}z \left| \psi^{\gamma^* \to q\bar{q}}(r,z)_{T,L} \right|^2 2\mathrm{Im}\mathcal{A}$$

Impact parameters

- Quark at $\mathbf{x}_\perp$ antiquark (or quark in c.c. amplitude) at $\mathbf{y}_\perp$
- TMD's: $\mathbf{r}_\perp = \mathbf{x}_\perp - \mathbf{y}_\perp$ (conjugate to gluon $\mathbf{k}_\perp$)
- GPD's $\mathbf{b}_\perp = z\mathbf{x}_\perp + (1-z)\mathbf{y}_\perp$ (conjugate to $\boldsymbol{\Delta}_\perp$, $t = -\boldsymbol{\Delta}_\perp{}^2$)

- Typical dipole size: $r \sim 1/Q$
- Used optical theorem: $2\mathrm{Im}\mathcal{A} = 2\int \mathrm{d}^2\mathbf{b}_\perp \mathcal{N}$ is total cross section
  - Can also take $|\mathcal{N}|^2$: elastic scattering (diffractive DIS)
- Assuming that fixed-size dipoles are basis that diagonalizes ImT
  - In general: high energy/eikonal approximation: $\mathbf{x}_\perp$ is fixed; (does not imply zero momentum transfer!)

3 CGC: the Wilson line

CGC basics

- Until now: $\gamma^* \to q\bar{q}$. But what is the target?
- CGC separation of scales:
  - small x: classical field
  - large x: color charge

1. A high energy particle scatters off a color field: Wilson line
2. An explicit model for the color field: the McLerran-Venugopalan model
3. Energy dependence of the color field from BK evolution

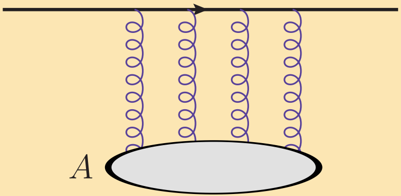
What is the target made of?

- So far we have not specified anything about the degrees of freedom in the target.
- At high energy the target consists dominantly of gluons
  - Experimentally: small x gluon distribution is larger than the quark one.
  - High $\sqrt{s}$: QCD radiation builds up the target by adding gluons to it.

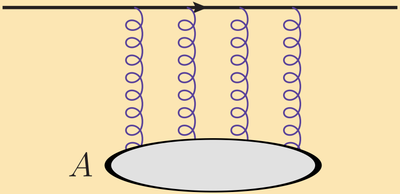
Color Glass Condensate (CGC)

We assume that there are so many gluons in the target, that it can be described by a classical gluon field. This is the heart of the CGC effective theory.

Many gluons = large color field A_μ
Sum all diagrams with n gluon lines
— but we can assume gluons are a classical field



Scattering off CGC target



Quark in classical color field: Dirac equation!

$$(i\partial\!\!\!/ - gA\!\!\!/)\psi(x) = 0$$

(Note: $A\!\!\!/ = A^{\mu}_{a}\gamma_{\mu}t^{a}$ is $N_c \times N_c \otimes 4 \times 4$-matrix)

High energy: **eikonal** approximation

- Gluon is spin 1: it couples to a vector: $\sim p^{\mu}A_{\mu}$
- For high energy particle the only vector available is p^{μ}
- p^{μ} has one large component: p^{+} $\implies$ $p^{\mu}A_{\mu} \sim p^{+}A^{-}$ $\implies$ only need A^{-}

Ansatz $\psi(x) = V(x)e^{-ip\cdot x}u(p) \implies \partial_{+}V(x^{+},x^{-},\mathbf{x}_{\perp}) = -igA^{-}(x^{+},x^{-},\mathbf{x}_{\perp})\mathbf{V}(x^{+},x^{-},\mathbf{x}_{\perp})$

This is solved by path-ordered exponential

$$V(x^{+},x^{-},\mathbf{x}_{\perp}) = \mathbb{P}\exp\left\{-ig\int^{x^{+}} dy^{+}A^{-}(y^{+},x^{-},\mathbf{x}_{\perp})\right\} \qquad N_c \times N_c\text{-matrix!}$$

Eikonal propagation

- Now we know the scattering S-matrix for quark states
- Incoming free quark $|q_i(\mathbf{x}_\perp)\rangle$ at $x^+ \to -\infty$ is, at $x^+ \to \infty$

$$\hat{S}(-\infty,\infty)\,|q_i(p^+,\mathbf{x}_\perp)\rangle = \left[\mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}y^+ A^-(y^+,x^-,\mathbf{x}_\perp)\right\}\right]_{ji} |q_j(p^+,\mathbf{x}_\perp)\rangle$$

  outgoing state linear superposition of color rotated quarks.
- In the high energy limit quark wavefunction oscillates like $e^{ip^+x^-}$ with large p^+ $\Longrightarrow$ x^--dependence negligible compared to this $\Longrightarrow$ approximate $x^- = 0$

glass in CGC

Scattering described by 2-d field of SU(N_c)-matrices

$$V(\mathbf{x}_\perp) \equiv \mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}x^+ A^-(x^+,x^-=0,\mathbf{x}_\perp)\right\}$$

— The **Wilson line**

Dipole amplitude and Wilson lines

Incoming dipole becomes (color neutral, normalized! ; $V(\mathbf{y}_\perp)^\dagger_{jk} = V(\mathbf{y}_\perp)^*_{kj}$ for antiquark)

$$|\mathrm{in}\rangle = \frac{\delta_{ii'}}{\sqrt{N_c}} |q_i(\mathbf{x}_\perp)\bar{q}_{i'}(\mathbf{y}_\perp)\rangle \qquad \hat{S}\,|\mathrm{in}\rangle = \frac{\delta_{ii'}}{\sqrt{N_c}} V_{ji}(\mathbf{x}_\perp)V^\dagger_{i'j'}(\mathbf{y}_\perp)\,|q(\mathbf{x}_\perp)_j\bar{q}(\mathbf{y}_\perp)_{j'}\rangle$$

Count outgoing dipoles in this state

$$S = \frac{1}{\sqrt{N_c}}\langle q_k(\mathbf{x}_\perp)\bar{q}_k(\mathbf{y}_\perp)|\,\mathrm{in}\rangle = \frac{\delta_{ii'}}{N_c}\delta_{kj}\delta_{kj'}V_{ji}(\mathbf{x}_\perp)V^\dagger_{i'j'}(\mathbf{y}_\perp) = \frac{1}{N_c}\mathrm{Tr}\,V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)$$

Dipole amplitude in the CGC

$q\bar{q}$ amplitude $\Longleftrightarrow$ **microscopical description of the target** $\quad (\mathbf{b}_\perp = z\mathbf{x}_\perp + (1-z)\mathbf{y}_\perp)$:

$$\mathcal{N}_{q\bar{q}}(\mathbf{r}_\perp = \mathbf{x}_\perp - \mathbf{y}_\perp) = 1 - \frac{1}{N_c}\mathrm{Tr}\,V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) \qquad \mathrm{Im}\mathcal{A}(\mathbf{r}_\perp) = \int \mathrm{d}^2\mathbf{b}_\perp \mathcal{N}_{q\bar{q}}(\mathbf{r}_\perp = \mathbf{x}_\perp - \mathbf{y}_\perp)$$

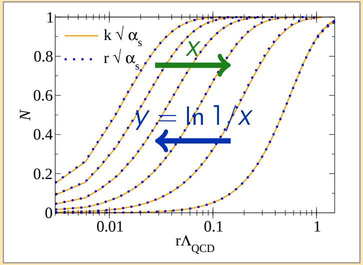
(Imaginary unit convention $S_{fi} = \langle f|\,\hat{S}\,|f\rangle = 1 + iT_{fi}$ $\quad \sigma_{\mathrm{tot}} = 2\mathrm{Im}T_{ii}$ $\quad \mathcal{N} \equiv \mathrm{Im}T_{ii}$ $\quad S_{ii} = \delta_{ii} - \mathcal{N} + \mathrm{imag}$)

4 CGC and saturation

Saturation of the dipole cross section

Basic features following from

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c} \operatorname{Tr} V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp)$$



- Small dipoles:
  - At $\mathbf{x}_\perp = \mathbf{y}_\perp$: $V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) = \mathbb{1}$.
  - At $r = 0$ color neutral $\Longrightarrow$ no scattering!
  - $\mathcal{N}_{q\bar{q}}(r) \sim r^2 \Longrightarrow$ perturbative limit
- Large dipoles
  - Uncorrelated Wilson lines: $\left\langle \operatorname{Tr} V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) \right\rangle = 0$
  - Large dipoles $r \gtrsim 1/Q_s$ scatter with $\mathcal{N}_{q\bar{q}} \lesssim 1$
- Know that $\mathcal{N}_{q\bar{q}}$ grows at $x \to 0$

  $\Longrightarrow$ turnover from $\mathcal{N} \ll 1$ to $\mathcal{N} \sim 1$ happens at smaller $r \sim R_s$ when $\sqrt{s} \to 0$

Nonperturbative weak coupling unitarization = Saturation

Saturation scale $Q_s = 1/R_s$ = inverse distance for turnover

Gluons in the target

- High energy proton/nucleus as a classical field: $A^- \implies$ Wilson line
- What does this imply for the partonic content of the nucleus?
- The physical picture of "gluons as partons" requires two things
  1. Infinite momentum frame: nucleus moving fast
     Change direction: nucleus moves now in $+z$**-direction** with large p^+.
     Means we have large A^+ component.
  2. Light cone gauge: have to gauge transform to $A^+ = 0$

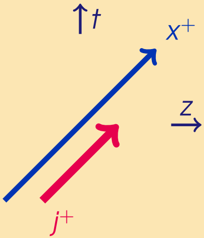
Classical field $\equiv$ from equation of motion

$$[D_\mu, F^{\mu\nu}] = J^\mu$$

What remains is

$$\nabla_\perp{}^2 A^+ = J^+$$

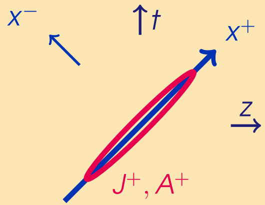
Nice, big $+$-field $\iff$ **color** current J^+



Spacetime structure of the field: covariant gauge

I Current lives on the light cone.

1. Naive explanation: Nucleus is Lorentz-contracted to $\Delta z \sim 2R_A m_A/\sqrt{s}$
2. Real explanation: Current represents large x degrees of freedom
   - Current: large p^+, field small p^+
   - Current more localised in x^- than field.



II Current independent of LC time x^+; **glass!**

1. Time is dilated for the nucleus
2. Probe has larger k^- than color current $\Longrightarrow$ oscillates faster in x^+, sees J^+ as static.

Extreme approximation:

$$J^+(x^-, \mathbf{x}_\perp) \approx \delta(x^-)\rho(\mathbf{x}_\perp)$$

$$A^+(x^-, \mathbf{x}_\perp) \approx \delta(x^-)\frac{1}{\boldsymbol{\nabla}_\perp^{\,2}}\rho(\mathbf{x}_\perp)$$

Spacetime picture in LC gauge

Recall: partonic interpretation needs LC gauge — Gauge transform:

$$A^+ \Rightarrow V^\dagger(\mathbf{x}_\perp, x^-) A^+ U(\mathbf{x}_\perp, x^-) - \frac{i}{g} V^\dagger(\mathbf{x}_\perp, x^-)\partial_- V(\mathbf{x}_\perp, x^-) = 0$$

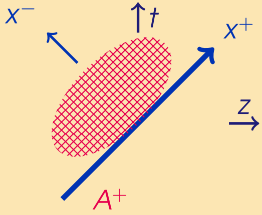
$$A^- \Rightarrow -\frac{i}{g} V^\dagger(\mathbf{x}_\perp, x^-)\partial_+ V(\mathbf{x}_\perp, x^-) = 0, \text{ still, no } x^+ \text{ dependence}$$

$$A^i \Rightarrow \frac{i}{g} V^\dagger(\mathbf{x}_\perp, x^-)\partial_i V(\mathbf{x}_\perp, x^-)$$

transverse pure gauge

This is solved by familiar Wilson line

$$V(\mathbf{x}_\perp, x^-) = \mathbb{P}\exp\left[-ig\int^{x^-} \mathrm{d}y^- A^+\right]$$



$A^+_{\mathrm{cov}} \sim \delta(x^-) \Longrightarrow A^i_{\mathrm{LC}} \sim \theta(x^-)$
$\Longrightarrow$ delocalized in x^- $\Longleftrightarrow$ small k^+

Weizsäcker-Williams gluon distribution

In LC quantization (Now of nucleus, not γ^*) number distribution of gluons:

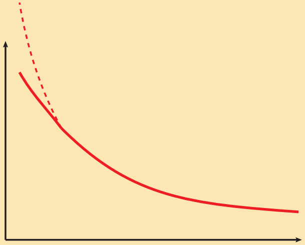
$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}_\perp\,\mathrm{d}y} \sim \left\langle A^i_a(\mathbf{k}_\perp) A^i_a(-\mathbf{k}_\perp) \right\rangle$$

- $A^i_a(\mathbf{k}_\perp)$ is obtained from the Wilson line
- Wilson line is related to DIS dipole cross section, BK equation
- This is **Weizsäcker-Williams** gluon distribution $\varphi^{\mathrm{WW}}(\mathbf{k}_\perp)$

$$\varphi^{\mathrm{WW}}(\mathbf{k}_\perp) = \frac{C_\mathrm{F}}{2\pi^3}\frac{1}{\alpha_\mathrm{s}} \int_{\mathbf{b}_\perp} \int_{\mathbf{r}_\perp} \frac{e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}}{\mathbf{r}_\perp{}^2} \widetilde{\mathcal{N}}(\mathbf{b}_\perp, \mathbf{r}_\perp)$$

($\widetilde{\mathcal{N}}$ is the adjoint representation Wilson line dipole correlator)

- Gluon saturation in $\varphi^{\mathrm{WW}}(\mathbf{k}_\perp)$ at $\mathbf{k}_\perp \lesssim Q_\mathrm{s}$
- $\varphi^{\mathrm{WW}}(\mathbf{k}_\perp) \sim 1/\alpha_\mathrm{s} \implies$ gluon "**condensate**"



Now we have a **C**olor **G**lass **C**ondensate.

5 McLerran-Venugopalan model

McLerran-Venugopalan model

Basic idea:

- Want explicit model for color charges, easy to calculate things
- Simple argument: independent, uncorrelated color charges (CLT: Gaussian)
- Single phenomenological parameter: color charge density μ^2

MV model charge density $\rho(\mathbf{x}_\perp)$

- stochastic, Gaussian random variable
- local in x^- and $\mathbf{x}_\perp$ (nucleus still moving in $+$ direction)

$$\left\langle \rho^a(\mathbf{x}_\perp, x^-)\rho^b(\mathbf{y}_\perp, y^-)\right\rangle = g^2\delta^{ab}\mu^2(x^-)\delta(x^- - y^-)\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)$$

(Although $\mu^2(x^-) \sim \delta(x^-)$, need to spread out a bit to make sense of the path ordering.)

(My conventions: μ charge carrier number density, $g\mu$ charge density $\sim$ field A_μ, thus what appears in exponent of Wilson line is $\sim gA_\mu \sim g^2\mu \sim Q_s$. But convention varies.)

Dipole amplitude in MV model

Start with definition of dipole operator

$$S(\mathbf{x}_\perp - \mathbf{y}_\perp) \equiv \frac{1}{N_c}\left\langle \mathrm{Tr}\, V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp)\right\rangle \quad V(\mathbf{x}_\perp, x^-) = \mathbb{P}\exp\left[-ig\int_{-\infty}^{x^-} \mathrm{d}y^- A^+\right]$$

- Remember $\nabla_\perp{}^2 A_a^+(\mathbf{x}_\perp, x^-) = \rho(\mathbf{x}_\perp, x^-)$
- Denote $\langle A_a^+(\mathbf{x}_\perp, x^-) A_b^+(\mathbf{y}_\perp, y^-)\rangle = g^2\delta^{ab}\mu^2(x^-)\delta(x^- - y^-)L(\mathbf{x}_\perp - \mathbf{y}_\perp)$
- Discretize $x^- = n\Delta^-$ $\Longrightarrow$ $\langle A_{ma}^+(\mathbf{x}_\perp) A_{nb}^+(\mathbf{y}_\perp)\rangle = g^2\delta^{ab}\mu_n^2 \frac{1}{\Delta^-}\delta_{mn} L(\mathbf{x}_\perp - \mathbf{y}_\perp)$

$$S(r) = \lim_{N\to\infty}\frac{1}{N_c}\mathrm{Tr}\Big\langle e^{-ig\Delta^- A_N^+(\mathbf{x}_\perp)} \dots e^{-ig\Delta^- A_n^+(\mathbf{x}_\perp)} \dots e^{-ig\Delta^- A_{-N}^+(\mathbf{x}_\perp)}$$

$$\times\, e^{ig\Delta^- A_{-N}^+(\mathbf{y}_\perp)} \dots e^{ig\Delta^- A_n^+(\mathbf{u}_\perp)} \dots e^{ig\Delta^- A_N^+(\mathbf{y}_\perp)}\Big\rangle$$

(Note: in V ordering is large x^- to small, in $V^\dagger$ inverse)

Infinitesimal Wilson line

Steps in x^- independent and next to each other in product:

$\Longrightarrow$ expectation value for each separately:

$$\left\langle e^{-ig\Delta^- A_n^+(\mathbf{x}_\perp)} e^{ig\Delta^- A_n^+(\mathbf{y}_\perp)} \right\rangle \approx \Big\langle 1 - ig\Delta^- A_n^+(\mathbf{x}_\perp) + ig\Delta^- A_n^+(\mathbf{y}_\perp)$$

$$+ g^2(\Delta^-)^2 A_n^+(\mathbf{x}_\perp) A_n^+(\mathbf{y}_\perp) - \frac{1}{2} g^2(\Delta^-)^2 \left((A_n^+(\mathbf{x}_\perp))^2 + (A_n^+(\mathbf{y}_\perp))^2 \right) \Big\rangle + \mathcal{O}(\Delta^-)^{3/2})$$

$\langle A^+ \rangle = 0$, and thus (Remember Δ^- infinitesimal)

$$\left\langle e^{-ig\Delta^- A_n^+(\mathbf{x}_\perp)} e^{ig\Delta^- A_n^+(\mathbf{y}_\perp)} \right\rangle$$

$$\approx 1 + g^4(\Delta^-)^2 \frac{\mu_n^2}{\Delta^-} \left[L(\mathbf{x}_\perp - \mathbf{y}_\perp) - \frac{1}{2} L(\mathbf{x}_\perp - \mathbf{x}_\perp) - \frac{1}{2} L(\mathbf{y}_\perp - \mathbf{y}_\perp) \right] \overbrace{t^a t^a}^{= C_F \mathbb{I}_{N_c \times N_c}}$$

$$\approx e^{g^4 \Delta^- C_F \mu_n^2 [L(\mathbf{x}_\perp - \mathbf{y}_\perp) - \frac{1}{2} L(\mathbf{x}_\perp - \mathbf{x}_\perp) - \frac{1}{2} L(\mathbf{y}_\perp - \mathbf{y}_\perp)]}$$

Put pieces together: dipole operator

Color $\sim \mathbb{I}_{N_c\times N_c}$ & $\langle\cdot\rangle$ done $\Longrightarrow$ only commuting numbers left.

$$S(r) = \lim_{N\to\infty} \overbrace{\frac{1}{N_c}\,\mathrm{Tr}}^{=1} \prod_{n=-N}^{N} e^{g^4 \Delta^- C_F \mu_n^2 [L(\mathbf{x}_\perp - \mathbf{y}_\perp) - L(\mathbf{0})]}$$

$$= \exp\left\{ g^4 C_F \int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-) \overbrace{[L(\mathbf{x}_\perp - \mathbf{y}_\perp) - L(\mathbf{0})]}^{\equiv -\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)} \right\}$$

The MV model is defined by two things:

1. The $\rho\rho$ correlators are Gaussian
2. The $\rho\rho$ correlators are delta functions in tranverse coordinate

- Until now we have only used property 1.
- Choosing some desired $\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) \Longrightarrow$ "Nonlinear Gaussian".

Functional form in MV model

$$\langle \rho(\mathbf{x}_\perp)\rho(\mathbf{y}_\perp)\rangle \sim \delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp) \Longrightarrow L(\mathbf{x}_\perp - \mathbf{y}_\perp), \Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) = L(\mathbf{0}) - L(\mathbf{x}_\perp - \mathbf{y}_\perp)\ ?$$

$$A^+(k_T) = \frac{1}{\mathbf{k}_\perp{}^2}\rho(\mathbf{k}_\perp)$$

$$\left\langle \rho^a(\mathbf{k}_\perp)\rho^b(\mathbf{p}_\perp)\right\rangle = g^2\delta^{ab}\mu^2 \int \mathrm{d}^2\mathbf{x}_\perp\, \mathrm{d}^2\mathbf{y}_\perp\, e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp + i\mathbf{p}_\perp\cdot\mathbf{y}_\perp}\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)$$
$$= g^2\delta^{ab}\mu^2(2\pi)^2\delta^{(2)}(\mathbf{k}_\perp + \mathbf{p}_\perp)$$

$$\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{p}_\perp}{(2\pi)^2}(2\pi)^2\frac{\delta^{(2)}(\mathbf{k}_\perp + \mathbf{p}_\perp)}{p_T{}^2 k_T{}^2}\left[1 - e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp - i\mathbf{p}_\perp\cdot\mathbf{y}_\perp}\right]$$

$$= \frac{1}{2\pi}\int \mathrm{d}k \frac{1 - J_0(kr)}{k^3} \approx \frac{1}{8\pi} r^2 \ln \overbrace{\frac{1}{r\Lambda}}^{\gg 1}$$

Dipole operator in the MV model

Final result

$$S(r) \equiv \frac{1}{N_c} \left\langle \operatorname{Tr} V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp) \right\rangle = \exp\left\{ -\frac{g^4 C_F}{8\pi} \left[\int_{-\infty}^{\infty} dx^- \mu^2(x^-) \right] r^2 \ln \frac{1}{r\Lambda} \right\}$$

From which we identify (up to log) the saturation scale as

$$Q_s^2 \sim \frac{g^4 C_F}{4\pi} \left[\int_{-\infty}^{\infty} dx^- \mu^2(x^-) \right]$$

- Spread out delta function $\delta(x^-)$, in the end only $\int dx^-$ matters.
- We can also Fourier-transform this to get the "dipole gluon distribution"

$$\varphi_{\text{dip.}}(k_T) \sim {k_T}^2 \int d^2\mathbf{r}_\perp e^{i\mathbf{r}_\perp \cdot \mathbf{k}_\perp} S(r)$$

- Because of $\log 1/r$ it behaves like $1/{k_T}^2$ at large k_T.

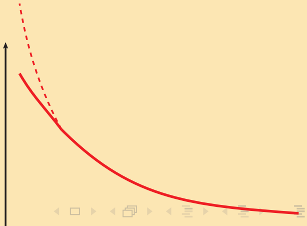
Weizsäcker-Williams distribution

Similar, but more complicated, calculation $\Longrightarrow$ also calculate the WW-distribution

$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}_\perp\,\mathrm{d}y} = \varphi^{\mathrm{WW}}(\mathbf{k}_\perp) \sim \left\langle A^i_a(\mathbf{k}_\perp)A^i_a(-\mathbf{k}_\perp)\right\rangle \qquad A^i_a(\mathbf{x}_\perp) = \frac{i}{g}V^\dagger(\mathbf{x}_\perp)\partial_i V(\mathbf{x}_\perp)$$

$$\left\langle A^i_a(\mathbf{x}_\perp)A^i_b(\mathbf{y}_\perp)\right\rangle = \delta^{ab}\frac{2C_\mathrm{F}}{g^2({N_\mathrm{c}}^2-1)}\frac{{\boldsymbol{\nabla}_\perp}^2\Gamma(\mathbf{x}_\perp-\mathbf{y}_\perp)}{\Gamma(\mathbf{x}_\perp-\mathbf{y}_\perp)}$$
$$\times\left(\exp\left\{-g^4N_\mathrm{c}\int_{-\infty}^{\infty}\mathrm{d}x^-\mu^2(x^-)\Gamma(\mathbf{x}_\perp-\mathbf{y}_\perp)\right\}-1\right)$$

$$\varphi_{WW}(k_T) \sim \begin{cases} \frac{1}{\alpha_\mathrm{s}}\ln k_T, & k_T \ll Q_\mathrm{s}^2 \\ \frac{1}{\alpha_\mathrm{s}}\frac{Q_\mathrm{s}^2}{{k_T}^2}, & k_T \gg Q_\mathrm{s}^2. \end{cases}$$



Two different **TMD**'s from the same $\Gamma(\mathbf{x}_\perp-\mathbf{y}_\perp)$

Some comments: two gluon distributions (TMD's)

- WW distribution: adjoint Casimir C_A, compared to $C_F = ({N_c}^2 - 1)/(2N_c)$ in the fundamental representation dipole. Expected for gluons.
- Both distributions $\sim 1/{k_T}^2$ at large $k_T \gg Q_s$ $\Longrightarrow$

$$xG(x, Q^2) = \int^{Q^2} d^2\mathbf{k}_\perp \varphi(k_T) \sim \ln Q^2$$

Natural. Result of starting from independent point charges

- Dipole and WW distributions different at at small k_T.
  Saturation: the behavior changes at $k_T \lesssim Q_s$ (dipole $\sim {k_T}^2$ and WW$\sim \ln k_T$).
- Different TMD operator definitions, different processes.
  - In traditional pQCD factorization separate.
  - MV model / Gaussian allows to calculate one from the other.

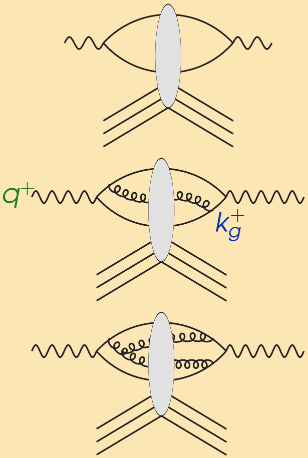
6 Balitsky-Kovchegov equation

Power counting

- Leading order (LO)
- Add one **soft** gluon $\Longrightarrow$ Leading Log (LL), resum by BK evolution
- Add one gluon, but **not** necessarily soft: Next-to-Leading Order (NLO) (need to subtract the soft gluon!)
- Add two gluons, one of them soft: NLL $\Longrightarrow$ resum by NLO BK equation

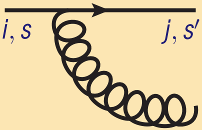
$$\sigma \sim \overbrace{\mathcal{O}(1)}^{\text{LO}} + \overbrace{\mathcal{O}(\alpha_s \ln 1/x)}^{\text{LL}} + \overbrace{\mathcal{O}(\alpha_s)}^{\text{NLO}} + \overbrace{\mathcal{O}(\alpha_s^2 \ln 1/x)}^{\text{NLL}}$$

- Current research: NLO & NLL
- Previous lectures: LO
- This lecture: LL



What happens if one radiates a gluon?

$$\psi^{q\to qg}(z,\mathbf{k}_\perp) = \frac{1}{\frac{\mathbf{p}_\perp{}^2}{2p^+} - \frac{\mathbf{k}_\perp{}^2}{2k^+} - \frac{\mathbf{p}'_\perp{}^2}{2p'^+}} \times \bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p)$$



Full matrix element (take quark mass $m = 0$)

$$\bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p) = \frac{-2gt^a_{ji}}{z\sqrt{1-z}}(\delta_{\lambda,2s} + (1-z)\delta_{\lambda,-2s})\mathbf{q}_\perp \cdot \varepsilon_\perp{}^*_\lambda, \qquad \mathbf{q}_\perp = \mathbf{k}_\perp - z\mathbf{p}_\perp$$

Focus on the **soft** (or slow) gluon limit $z \to 0$:

$$\psi^{q\to qg}(k^+,\mathbf{k}_\perp) \approx \frac{-2zp^+}{\mathbf{k}_\perp{}^2}\frac{-2gt^a_{ji}\delta_{ss'}}{z}\mathbf{k}_\perp \cdot \varepsilon_\perp{}^*_\lambda = \frac{4gt^a_{ji}p^+}{\mathbf{k}_\perp{}^2}\mathbf{k}_\perp \cdot \varepsilon_\perp{}^*_\lambda$$

IR divergences in gluon emission

Squaring the wave function: "probability for gluon emission"

$$\mathrm{d}P_{q\to qg} = \left|\psi^{q\to qg}(k^+,\mathbf{k}_\perp)\right|^2 \frac{\mathrm{d}k^+\,\mathrm{d}^2\mathbf{k}_\perp}{2k^+(2\pi)^3} \sim \frac{\mathrm{d}z}{z}\frac{\mathrm{d}^2\mathbf{k}_\perp}{\mathbf{k}_\perp{}^2} \qquad \left(\sum_{\lambda=\pm1}\varepsilon_i\varepsilon_j^* = \delta_{ij}\right)$$

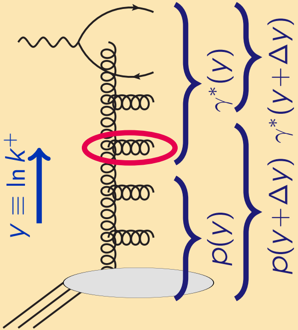
collinear divergence $\int_0 \frac{\mathrm{d}^2\mathbf{k}_\perp}{\mathbf{k}_\perp{}^2}$ Cancels in emission from color neutral dipole.

soft divergence $\int_{\sim 0} \frac{\mathrm{d}z}{z}$ Not really divergence, but needs to be resummed

- The limit $z \to 0$: large $q\bar{q}g$ invariant mass: $M_{q\bar{q}g} \to \infty$
- We are working in the "$s = \infty$" eikonal approximation
- Physically, one must however have $M^2_{q\bar{q}g} < s \implies z \gtrsim [\perp]/s$
  (where "$\perp$" is some relevant transverse momentum scale.)
- Thus "divergence" is a sign of a large log $\sim \alpha_s \ln s \sim \alpha_s \ln 1/x$

Resum this large log using the Balitsky-Kovchegov equation

Soft gluons and large logs, idea of RGE



- Emitted gluons have z between 1 and x: each gluon contributes $\sim \alpha_s \ln 1/x$
- For x small $\alpha_s \ln 1/x \sim 1 \implies$ all n gluon emissions contribute same $\implies$ resum
- Resummation by renormalization

Is the gluon at y a part of γ^* or of p?

Physical cross section is the same.

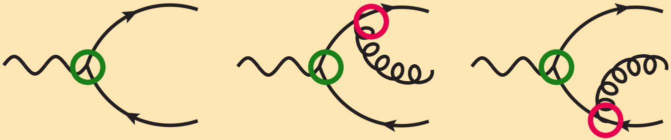
$$\sigma^{\gamma^* p} = \overbrace{\left|\psi^{\gamma^*\to q\bar q}\right|^2_y \otimes 2N_y^{q\bar q+p} + \left|\psi^{\gamma^*\to q\bar q g}\right|^2_y \otimes 2N_y^{q\bar q g+p} + \dots}^{\text{gluons up to } y \text{ are part of proton}}$$

$$= \underbrace{\left|\psi^{\gamma^*\to q\bar q}\right|^2_{y+\Delta y} \otimes 2N_{y+\Delta y}^{q\bar q+p} + \left|\psi^{\gamma^*\to q\bar q g}\right|^2_{y+\Delta y} \otimes 2N_{y+\Delta y}^{q\bar q g+p} + \dots}_{\text{gluons up to } y+\Delta y \text{ are part of proton}}$$

Quick derivation of the BK equation

- Calculate $\psi^{\gamma^* \to q\bar{q}g}(z)$
- Take soft gluon limit $z \to 0$
- Reabsorb the gluon to become a part of the target
- Get evolution equation for $q\bar{q}$ cross section

We need:



Soft gluon limit simpler $\quad z \to 0 \implies \psi^{\gamma^* \to q\bar{q}g} \approx \psi^{\gamma^* \to q\bar{q}} \left(\psi^{q \to qg} + \psi^{\bar{q} \to \bar{q}g} \right)$

This is true only for limit $z \to 0$ where $k^-_{q\bar{q}g} - k^-_{\gamma^*} \approx k^-_g \sim 1/z$

(In the full kinematics the gluon emission knows about the γ^*, not just the emitting parent $q/\bar{q}$

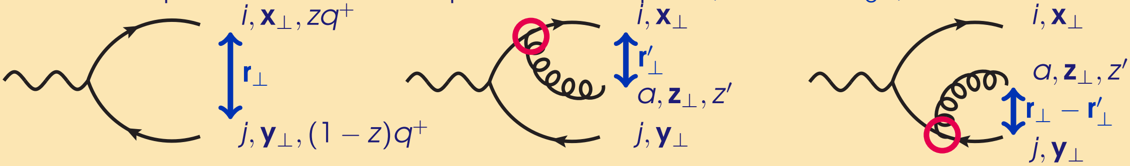
$\implies$ full NLO cross section computation much complex)

Gluon emission from coordinate space dipole

First step: Fourier-transform the gluon emission wavefunction to coordinate space

$$\psi^{q\to qg}(k^+,\mathbf{r}_\perp) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\psi^{q\to qg}(k^+,\mathbf{k}_\perp) = -i2p^+\frac{2gt^a_{ji}}{2\pi}\frac{\varepsilon_\perp\cdot\mathbf{r}_\perp}{\mathbf{r}_\perp{}^2}\delta_{s,s'}$$

Second step: sum emission from quark and antiquark (note relative sign!)



$$|\gamma^*\rangle_{\text{int}} = |\gamma^*\rangle + \int_{z,\mathbf{r}_\perp} C(\mathbf{r}_\perp)\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\,|q_i(\mathbf{x}_\perp,z)\bar{q}_i(\mathbf{y}_\perp,1-z)\rangle$$

$$+\int_{z,\mathbf{r}_\perp,\mathbf{r}'_\perp}\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\int\frac{\mathrm{d}z'}{4\pi z'}\frac{-i2g}{2\pi}t^a_{ji}\left[\frac{(\mathbf{x}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_\perp}{(\mathbf{x}_\perp-\mathbf{z}_\perp)^2}-\frac{(\mathbf{y}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_\perp}{(\mathbf{y}_\perp-\mathbf{z}_\perp)^2}\right]|q_i(\mathbf{x}_\perp)\bar{q}_j(\mathbf{y}_\perp)g_a(\mathbf{z}_\perp)\rangle$$

Virtual term

(Take a "unitarity" shortcut instead of calculating loop diagram)

$$|\gamma^*\rangle_{\text{int}} = |\gamma^*\rangle + \int_{z,\mathbf{r}_\perp} C(\mathbf{r}_\perp)\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\,|q_i(\mathbf{x}_\perp,z)\bar{q}_i(\mathbf{y}_\perp,1-z)\rangle + \dots$$

Gluon emission $\Longrightarrow$ correct the normalization of $|q\bar{q}\rangle$ by $C(\mathbf{r}_\perp) = 1 + \mathcal{O}(\alpha_s)$

$$N_c\,|C(\mathbf{r}_\perp)|^2 =$$

$$N_c - \frac{(2g)^2}{(2\pi)^2}\frac{1}{4\pi}t^a_{ij}t^a_{ji}\int\frac{dz'}{z'}\int d^2\mathbf{r}'_\perp \sum_{\lambda=\pm1}\left|\frac{(\mathbf{x}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_{\perp\lambda}}{(\mathbf{x}_\perp-\mathbf{z}_\perp)^2} - \frac{(\mathbf{y}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_{\perp\lambda}}{(\mathbf{y}_\perp-\mathbf{z}_\perp)^2}\right|^2$$

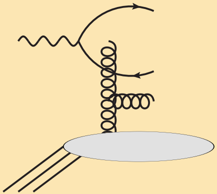
$$= N_c - \frac{\alpha_s}{\pi^2}\frac{N_c{}^2-1}{2}\Delta y\int d^2\mathbf{r}'_\perp\frac{\mathbf{r}_\perp{}^2}{\mathbf{r}'_\perp{}^2(\mathbf{r}_\perp-\mathbf{r}'_\perp)^2}$$

$$\sum_{\lambda=\pm1}\varepsilon_i^{(\lambda)}\varepsilon_j^{(\lambda)*} = \delta_{ij}$$

Crucial step: move the gluon to the target

Absorb corrections from gluon to a redefinition of the $q\bar{q}$ amplitude

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2}\frac{N_c{}^2-1}{2N_c}\int_y^{y+\Delta y} \mathrm{d}\ln 1/z' \int \mathrm{d}^2\mathbf{r}'_\perp \frac{\mathbf{r}_\perp{}^2}{\mathbf{r}'_\perp{}^2(\mathbf{r}_\perp-\mathbf{r}'_\perp)^2}\left[\mathcal{N}_{q\bar{q}g}^{\ln 1/z'} - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}\right]$$



Dipole scattering on new target $\mathcal{N}_{q\bar{q}}^{y+\Delta y}$ is

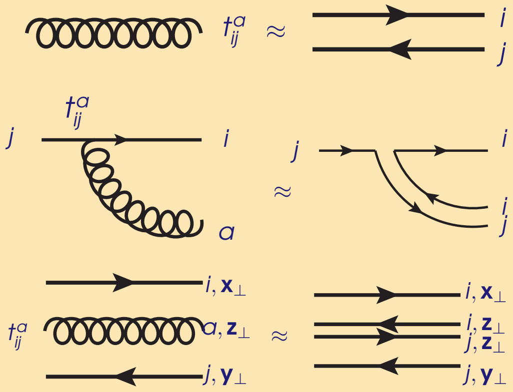
- Dipole scattering off original target $\mathcal{N}_{q\bar{q}}^{y}$
- Dipole emits a gluon into rapidity interval $[y, y+\Delta y]$, which scatters off target
- Normalization of original dipole is corrected
  (There are now less dipoles in γ^*)

Almost there

Want equation for $\mathcal{N}_{q\bar{q}}$: but enocuntered new quantity $\mathcal{N}_{q\bar{q}g}$.

$\Longrightarrow$ Relate to $\mathcal{N}_{q\bar{q}}$ in large N_c approximation

Gluon at large N_c



- At large $N_c \implies$ gluon = $q\bar{q}$ pair
  (not dipole!)
- $N_c{}^2 - 1$ gluon colors $\approx N_c{}^2$ quark-antiquark pair colors.
- Had $|q(\mathbf{x}_\perp)\bar{q}(\mathbf{y}_\perp)g(\mathbf{z}_\perp)\rangle$
- Approximate by
  $|q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)\rangle$

Now, instead of $\mathcal{N}_{q\bar{q}g}$, we need $\mathcal{N}_{q\bar{q}q\bar{q}}$;
amplitude for simultaneous scattering of two dipoles.

(Note: the gluon is not becoming a new dipole, but the common end of two new dipoles.)

Two dipole scattering amplitude

- $\mathcal{N}$ is really **scattering probability**;
- $S = 1 - \mathcal{N}$ is probability **not to scatter**

For two dipoles:

- No scattering: neither dipole scatters $\Longrightarrow S_{q\bar{q}q\bar{q}} = S_{q\bar{q}} S_{q\bar{q}}$
- Scattering probability $\mathcal{N}_{q\bar{q}q\bar{q}} = 1 - S_{q\bar{q}q\bar{q}} = 1 - (1 - \mathcal{N}_{q\bar{q}})(1 - \mathcal{N}_{q\bar{q}})$

$$\Longrightarrow \mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{y}_\perp)g(\mathbf{z}_\perp)} \approx \mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)} + \mathcal{N}_{q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)} - \mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)} \mathcal{N}_{q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)}$$

and our equation is

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2} \frac{{N_c}^2 - 1}{2N_c} \int_y^{y+\Delta y} \mathrm{d} \ln 1/z' \int \mathrm{d}^2\mathbf{z}_\perp \frac{(\mathbf{x}_\perp - \mathbf{y}_\perp)^2}{(\mathbf{x}_\perp - \mathbf{z}_\perp)^2 (\mathbf{z}_\perp - \mathbf{y}_\perp)^2}$$

$$\times \left[\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{z}_\perp) + \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z}_\perp, \mathbf{y}_\perp) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{z}_\perp) \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z}_\perp, \mathbf{y}_\perp) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{y}_\perp) \right]$$

Which is easy to write differentially in y

Summary

Balitsky-Kovchegov equation (∼1995)

$$\partial_y \mathcal{N}(\mathbf{r}_\perp) = \frac{\alpha_s N_c}{2\pi^2} \int_{\mathbf{r}'_\perp} \frac{{\mathbf{r}_\perp}^2}{{\mathbf{r}'_\perp}^2 (\mathbf{r}'_\perp - \mathbf{r}_\perp)^2} \left[\mathcal{N}(\mathbf{r}'_\perp) + \mathcal{N}(\mathbf{r}_\perp - \mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r}'_\perp)\mathcal{N}(\mathbf{r}_\perp - \mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r}_\perp)\right]$$

This is the basic tool of modern small-x physics.

- Given initial condition $\mathcal{N}(\mathbf{r}_\perp)$ at $y = y_0$ the equation predicts the scattering amplitude at larger y = smaller x = higher $\sqrt{s}$.
- Drop nonlinear term: BFKL equation
- Divergences at $\mathbf{r}'_\perp \to 0$ and $\mathbf{r}'_\perp \to \mathbf{r}_\perp$ ok because $\mathcal{N}(0) = 0$
- Enforces black disk limit $\mathcal{N} < 1$
- Practical work: coupling runs $\alpha_s(\mathbf{r}_\perp, \mathbf{r}'_\perp, \mathbf{r}_\perp - \mathbf{r}'_\perp)$ (Much ink spilled on precisely how.)
- Assumed $\langle \mathcal{N}\mathcal{N} \rangle = \langle \mathcal{N} \rangle \langle \mathcal{N} \rangle = \mathcal{N}^2 \Longrightarrow$ JIMWLK generalizes this

What the solution of BK looks like

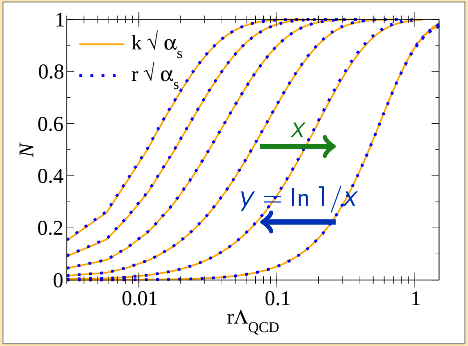
BK can be solved numerically

- Small dipoles $r \lesssim 1/Q_s$: $\mathcal{N} \ll 1$
- Large dipoles $r \gtrsim 1/Q_s$: $\mathcal{N} \to 1$
- **Saturation** scale grows with $\ln 1/x$.

For F_2, F_L convolute this with the $\psi^{\gamma^* \to q\bar{q}}$

$$\sigma^{\gamma^* p}_{T,L} = \int_{\mathbf{b}_\perp, \mathbf{r}_\perp} \left| \psi^{\gamma^* \to q\bar{q}}(r,z)_{T,L} \right|^2 2\mathcal{N}(\mathbf{r}_\perp, \mathbf{b}_\perp, x)$$

$\Longrightarrow$ confront with HERA, EIC



(Actually cheating, this plot is a solution of JIMWLK, which generalizes BK)