# Energy-momentum tensor and GPDS

# The Energy-Momentum Tensor

Energy Density  Momentum Density

$$T^{\mu\nu} = \begin{matrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{matrix}$$

shear forces

pressure

Energy Flux  Momentum Flux

$$\langle N(p')|T^{\mu\nu}_{q,g}|N(p)\rangle$$

- Where does the spin of the proton come from?
- What are the mechanical properties (pressure, shear forces) inside the proton ?
- What is the origin of the proton mass?

# Canonical Energy Momentum Tensor

Emmy Noether (1882-1935)

If a system with conservative forces has a continuous symmetry property, then there is a corresponding quantity whose value is conserved in time

Translation invariance $\longrightarrow$ Conservation of the canonical EMT $T_C^{\mu\nu}(x)$

Lorentz invariance $\longrightarrow$ Conservation of the generalized Angular Momentum (AM) density

$$J_C^{\mu\alpha\beta}(x) = L_C^{\mu\alpha\beta} + S_C^{\mu\alpha\beta} \qquad L_C^{\mu\alpha\beta}(x) = x^\alpha T_C^{\mu\beta}(x) - x^\beta T_C^{\mu\alpha}(x)$$

Space components: $J_C^i(x) = \frac{1}{2}\epsilon^{ijk} J_C^{0jk}(x)$

$$\underbrace{\vec{J}_C = \vec{L}_C + \vec{S}_C}$$

$\vec{L}_C \downarrow$ Orbital AM $\quad$ $\vec{S}_C \downarrow$ Spin

$T_C^{\mu\nu}$ is in general neither gauge-invariant nor symmetric

$$T_C^{\mu\nu} - T_C^{\nu\mu} = -\partial_\lambda S_C^{\lambda\mu\nu}$$

*Belinfante, Rosenfeld (1940)*

## Belinfante improved EMT

$$T_{\text{Bel}}^{\mu\nu}(x) = T_C^{\mu\nu}(x) + \partial_\lambda G^{\lambda\mu\nu}(x) \qquad \text{with } G^{\lambda\mu\nu} = -G^{\mu\lambda\nu}$$

## Belinfante generalized AM

$$J_{\text{Bel}}^{\mu\alpha\beta}(x) = J_C^{\mu\alpha\beta}(x) + \partial_\lambda[x^\alpha G^{\lambda\mu\beta}(x) - x^\beta G^{\lambda\mu\alpha}(x)]$$

with the super-potential

$$G^{\lambda\mu\nu}(x) = \frac{1}{2}[S_C^{\lambda\mu\nu}(x) - S_C^{\mu\nu\lambda}(x) - S_C^{\nu\mu\lambda}(x)] = -G^{\mu\lambda\nu}(x)$$

$$J_{\text{Bel}}^{\mu\alpha\beta}(x) = x^\alpha T_{\text{Bel}}^{\mu\beta}(x) - x^\beta T_{\text{Bel}}^{\mu\alpha}(x)$$

The total charges do not change:

$$\int T_C^{0\nu}\, d^3x = \int T_{\text{Bel}}^{0\nu}\, d^3x \qquad \int J_C^{0\alpha\beta}\, d^3x = \int J_{\text{Bel}}^{0\alpha\beta}\, d^3x$$

## Canonical $\longleftrightarrow$ Belinfante

in general not symmetric

$$T_C^{[\mu\nu]}(x) = -\partial_\alpha S^{\alpha\mu\nu}(x) \neq 0$$

$$[\mu\nu] = \mu\nu - \nu\mu$$

symmetric

$$T_{\text{Bel}}^{[\mu\nu]}(x) = 0$$

clear distinction between OAM and spin at the density level

$$J_C^{\mu\alpha\beta}(x) = L_C^{\mu\alpha\beta}(x) + S_C^{\mu\alpha\beta}(x)$$

$$L_C^{\mu\alpha\beta}(x) = x^\alpha T_C^{\mu\beta}(x) - x^\beta T_C^{\mu\alpha}(x)$$

purely OAM density

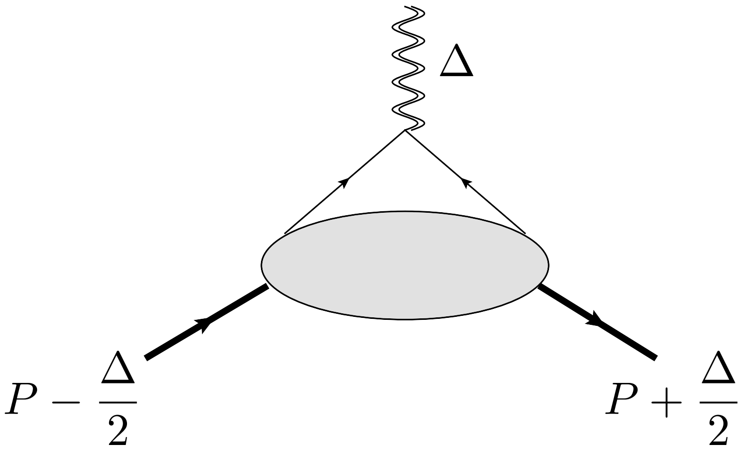
$$J_{\text{Bel}}^{\mu\alpha\beta}(x) = x^\alpha T_{\text{Bel}}^{\mu\beta}(x) - x^\beta T_{\text{Bel}}^{\mu\alpha}(x)$$

The total charges do not change:

$$\int T_C^{0\nu}\, d^3x = \int T_{\text{Bel}}^{0\nu}\, d^3x$$

$$\int J_C^{0\alpha\beta}\, d^3x = \int J_{\text{Bel}}^{0\alpha\beta}\, d^3x$$

# Form Factors of the EMT




Nucleon in external classical gravitational field

couples to energy-momentum tensor
(symmetric-Belinfante)

$$\langle P + \frac{\Delta}{2} | T^{\mu\nu} | P - \frac{\Delta}{2} \rangle$$

$$= \bar{u}(P + \frac{\Delta}{2}) \left\{ A(t) \frac{\gamma^{\{\mu} P^{\nu\}}}{2} + B(t) P^{\{\mu} i\sigma^{\nu\}\alpha} \frac{\Delta_\alpha}{4M} \right.$$

$$\left. + D(t)(\Delta^\mu \Delta^\nu - \Delta^2 g^{\mu\nu}) \frac{1}{4M} \right\} u(P - \frac{\Delta}{2})$$

$$\{\mu, \nu\} \equiv (\mu\nu + \nu\mu)$$

Gordon identity

$$\bar{u}' \gamma^\mu u = \bar{u}' \left( \frac{P^\mu}{M} + i\sigma^{\mu\alpha} \frac{\Delta_\alpha}{2M} \right) u$$

$$= \bar{u}(P + \frac{\Delta}{2}) \left\{ A(t) P^\mu P^\nu / M + (A(t) + B(t)) P^{\{\mu} i\sigma^{\nu\}\alpha} \frac{\Delta_\alpha}{4M} \right.$$

$$\left. + D(t)(\Delta^\mu \Delta^\nu - \Delta^2 g^{\mu\nu}) \frac{1}{4M} \right\} u(P - \frac{\Delta}{2})$$

# Momentum sum rule

$$\langle P|\hat{P}^\nu|P\rangle = \langle P|\int d^3\vec{x}\, T^{0\nu}(x)|P\rangle$$

$$= \lim_{\Delta\to 0}\langle P+\frac{\Delta}{2}|\int d^3\vec{x}\, T^{0\nu}(x)|P-\frac{\Delta}{2}\rangle$$

$$= \lim_{\Delta\to 0}\int d^3\vec{x}\, e^{-i\vec{x}\cdot\vec{\Delta}}\langle P+\frac{\Delta}{2}|\, T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle$$

$$= \lim_{\Delta\to 0}(2\pi)^3\delta^3(\vec{\Delta})\langle P+\frac{\Delta}{2}|\, T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle$$

$$= A(0)\, P^\nu (2P^0)(2\pi)^3\delta^3(0)$$

$$= A(0)\, P^\nu \langle P|P\rangle$$

Recall that:

$$\langle p', s'|T^{\mu\nu}_{\mathrm{Bel},a}(0)|p,s\rangle$$

$$= \bar{u}(p',s')\left[\frac{P^{\{\mu}\gamma^{\nu\}}}{2}A_a(t) + \frac{P^{\{\mu}i\sigma^{\nu\}\Delta}}{4M}B_a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{4M}D_a(t) + Mg^{\mu\nu}\bar{C}_a(t)\right]u(p,s)$$

# Momentum sum rule

$$\langle P|\hat{P}^\nu|P\rangle \quad = \quad A(0)P^\nu\langle P|P\rangle \qquad\qquad \langle P|\hat{P}^\nu|P\rangle \quad = \quad P^\nu\langle P|P\rangle$$

$$\Downarrow$$

$$\boxed{A(0) = 1}$$

- Polinomiality: $\int dx\, xH(x,\xi,t) = A(t) + D(t)\,\xi^2$
- Physical interpretation in terms of momentum carried by partons:

Quarks $\quad A^q(0) = \int dx\, xH^q(x,0,0)$

&

Gluons $\quad A^g(0) = \int dx\, xH^g(x,0,0)$

- Total system: Momentum conservation

$$\boxed{A^q(0) + A^g(0) = 1}$$

# Angular Momentum sum rule

Consider nucleon in the rest frame: $P^\mu = (M, 0, 0, 0) \quad S^\mu = (0, 0, 0, 1)$

Recall that: $\hat{J}^i = \frac{1}{2}\epsilon_{ijk} \int d^3\vec{x} J^{0jk}(x)$

$$J^{0jk}(x) = x^i T^{0j} - x^j T^{0i}$$

$$\langle P, \frac{1}{2}|\hat{J}^3|P, \frac{1}{2}\rangle = J \, \langle P, \frac{1}{2}|P, \frac{1}{2}\rangle$$

$$= \epsilon_{ij3}\langle P, \frac{1}{2}| \int d^3\vec{x}\, x^i T^{0j}(x)|P, \frac{1}{2}\rangle$$

$$= \epsilon_{ij3} \lim_{\Delta\to 0}\langle P + \frac{\Delta}{2}, \frac{1}{2}| \int d^3\vec{x}\, x^i T^{0j}(x)|P - \frac{\Delta}{2}, \frac{1}{2}\rangle$$

$$= \epsilon_{ij3} \lim_{\Delta\to 0} \int d^3\vec{x}\, x^i e^{-i\vec{x}\cdot\vec{\Delta}}\langle P + \frac{\Delta}{2}, \frac{1}{2}|T^{0j}(0)|P - \frac{\Delta}{2}, \frac{1}{2}\rangle$$

$$= \epsilon_{ij3} \lim_{\Delta\to 0} \left[ i\frac{\partial}{\partial\Delta^i}(2\pi)^3\delta^3(\vec{\Delta})\right] \langle P + \frac{\Delta}{2}, \frac{1}{2}|T^{0j}(0)|P - \frac{\Delta}{2}, \frac{1}{2}\rangle$$

# Angular Momentum sum rule

$$= \epsilon_{ij3} \lim_{\Delta\to 0} (2\pi)^3 \delta^3(\vec{\Delta}) \left(-i\frac{\partial}{\partial \Delta^i}\right) \Big\{ [A(t)+B(t)] \bar{u}(p', 1/2)\, P^{\{0} i\sigma^{j\}\alpha} \frac{\Delta_\alpha}{4M} u(p, \frac{1}{2})$$

$$+ \text{ terms independent of } \Delta$$

$$+ \text{ terms quadratic in } \Delta \Big\}$$

$$= \epsilon_{ij3} (2\pi)^3 \delta^3(0) [A(0)+B(0)] \left(-\frac{1}{4M}\right) \bar{u}(P, \frac{1}{2}) \{ \underset{\substack{\downarrow \\ M}}{P^0} \sigma^{ji} + \underset{\substack{\downarrow \\ 0}}{P^j} \sigma^{0i} \} u(P, \frac{1}{2})$$

in rest frame

$$= (2\pi)^3 \delta^3(0) [A(0)+B(0)] \left(\frac{1}{2M}\right) M\, \underbrace{\bar{u}(P, \frac{1}{2})\, \sigma^{12}\, u(P, \frac{1}{2})}_{2M \text{ in rest frame}}$$

$$= \frac{1}{2}[A(0)+B(0)] \langle P, \frac{1}{2} | P, \frac{1}{2} \rangle$$

# Angular Momentum sum rule

$$\langle P,\frac{1}{2}|\hat{J}^3|P,\frac{1}{2}\rangle \;=\; J\,\langle P,\frac{1}{2}|P,\frac{1}{2}\rangle \;=\; \frac{1}{2}[A(0)+B(0)]\langle P,\frac{1}{2}|P,\frac{1}{2}\rangle$$

$$J \;=\; \frac{1}{2}[A(0)+B(0)]$$

- Total system: Angular Momentum conservation

$$\boxed{A(0)+B(0)\;=\;1}$$

- Polinomiality: $\int dx\, x H(x,\xi,t) = A(t) + D(t)\,\xi^2$ $\quad\quad \int dx\, x E(x,\xi,t) = B(t) - D(t)\,\xi^2$

- Angular momentum sum rule (Ji's relation)

$$J^{q,g} = \frac{1}{2}[A^{q,g}(0)+B^{q,g}(0)] = \frac{1}{2}\int dx\, x\,\left(H^{q,g}(x,\xi,0)+E^{q,g}(x,\xi,0)\right)$$

- Momentum + Angular Momentum conservation $\longrightarrow$ Gravitomagnetic sum rule

$$\left\{\begin{array}{l} A(0)=1 \\ \\ A(0)+B(0)=1 \end{array}\right. \quad\longrightarrow\quad \boxed{B(0)=0}$$

# Kinetic EMT in QCD

*Ji, 1997*

$$T^{\mu\nu}_{\text{kin}}(x) = T^{\mu\nu}_{\text{kin},q}(x) + T^{\mu\nu}_{\text{kin},g}$$

- Quark contribution: $T^{\mu\nu}_{\text{kin},q}(x) = \frac{1}{2}\bar{\psi}(x)\gamma^{\mu} i \overleftrightarrow{D}^{\nu}\psi(x)$ $\qquad (D^{\mu} = \partial^{\mu} + igA^{\mu})$

$$\boxed{\frac{1}{2}T^{\{\mu\nu\}}_{\text{kin},q}(x) = T^{\mu\nu}_{\text{Bel},q}(x)}$$

$$\frac{1}{2}T^{[\mu\nu]}_{\text{kin},q}(x) = -\partial_{\lambda}S^{\lambda\mu\nu}_{q}(x)$$

$$S^{\lambda\mu\nu}_{q}(x) = \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}(x)\gamma_{\alpha}\gamma_{5}\psi(x)$$

- Gluon contribution: $T^{\mu\nu}_{\text{kin},g} = -2\,\text{Tr}[F^{\mu\lambda}(x)F^{\nu}_{\lambda}(x)] + \frac{1}{2}g^{\mu\nu}\,\text{Tr}[F^{\rho\sigma}(x)F_{\rho\sigma}(x)]$

$$\boxed{T^{\mu\nu}_{\text{kin},g}(x) = T^{\mu\nu}_{\text{Bel},g}(x)}$$

# Kinetic generalized AM

$$J^{\mu\alpha\beta}_{\text{kin},q}(x) = L^{\mu\alpha\beta}_{\text{kin},q}(x) + S^{\mu\alpha\beta}_{q}(x)$$

$$J^{\mu\alpha\beta}_{\text{Bel},q}(x) = J^{\mu\alpha\beta}_{\text{kin},q}(x) + \frac{1}{2}\partial_{\lambda}[x^{\alpha}S^{\lambda\mu\beta}_{q}(x) - x^{\beta}S^{\lambda\mu\alpha}_{q}(x)]$$

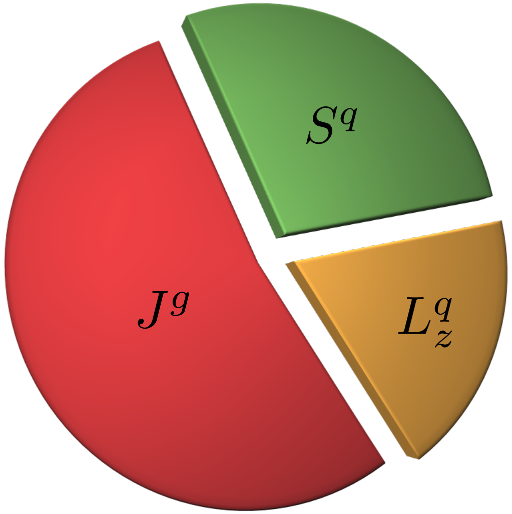
$$J^{i} = \frac{1}{2}\epsilon^{ijk}\int d^3x\, J^{0jk}$$

$$\int \vec{J}_{\text{Bel},q}\, d^3x = \int \vec{J}_{\text{kin},q}\, d^3x$$

equal total charge

# Angular Momentum Relation (``Ji's Sum Rule)

*X. Ji, PRL **78** (1997) 610*




Ji's relation: $J^{q,g}(t=0) = \frac{1}{2}\int_{-1}^{1} dx\, x\, \left(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0)\right) \qquad (\xi = -\frac{\Delta^+}{2P^+})$

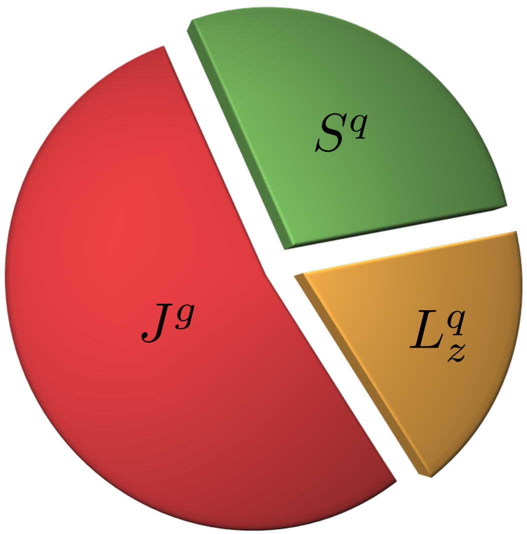
at $\xi = 0$ unpolarized PDF — not directly accessible

$J^q = L^q + S^q \qquad S^q = \frac{1}{2}\int_0^1 dx\, g_1(x)$

$J^g$ — no further gauge-invariant decomposition

- Requires extrapolation at t=0
- Requires spanning x at fixed values of $\xi$ ( $\xi = 0$ is the most convenient)
- $J^q(x) \neq \frac{1}{2}x(H^q(x,0,0) + E^q(x,0,0)$ $\longrightarrow$ not angular momentum density

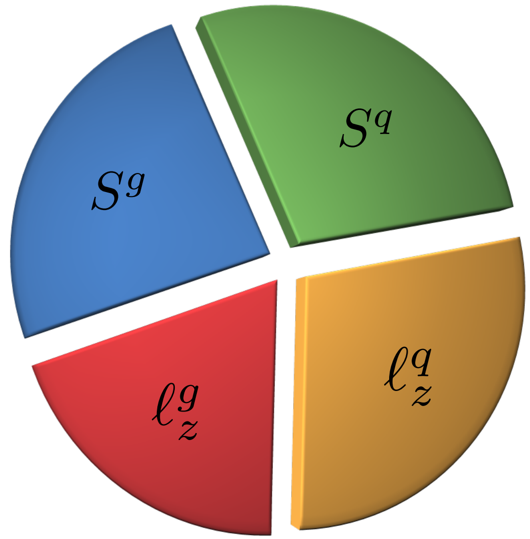
## Ji Sum Rule




$$\frac{1}{2} = \underbrace{S^q(\mu) + L^q_z(\mu)}_{J^q} + J^g(\mu)$$

- each term is gauge invariant
- No decomposition of gluon contribution
- $J^q$ and $J^g$ can be obtained from moments of GPDs

## Jaffe-Manohar Sum Rule



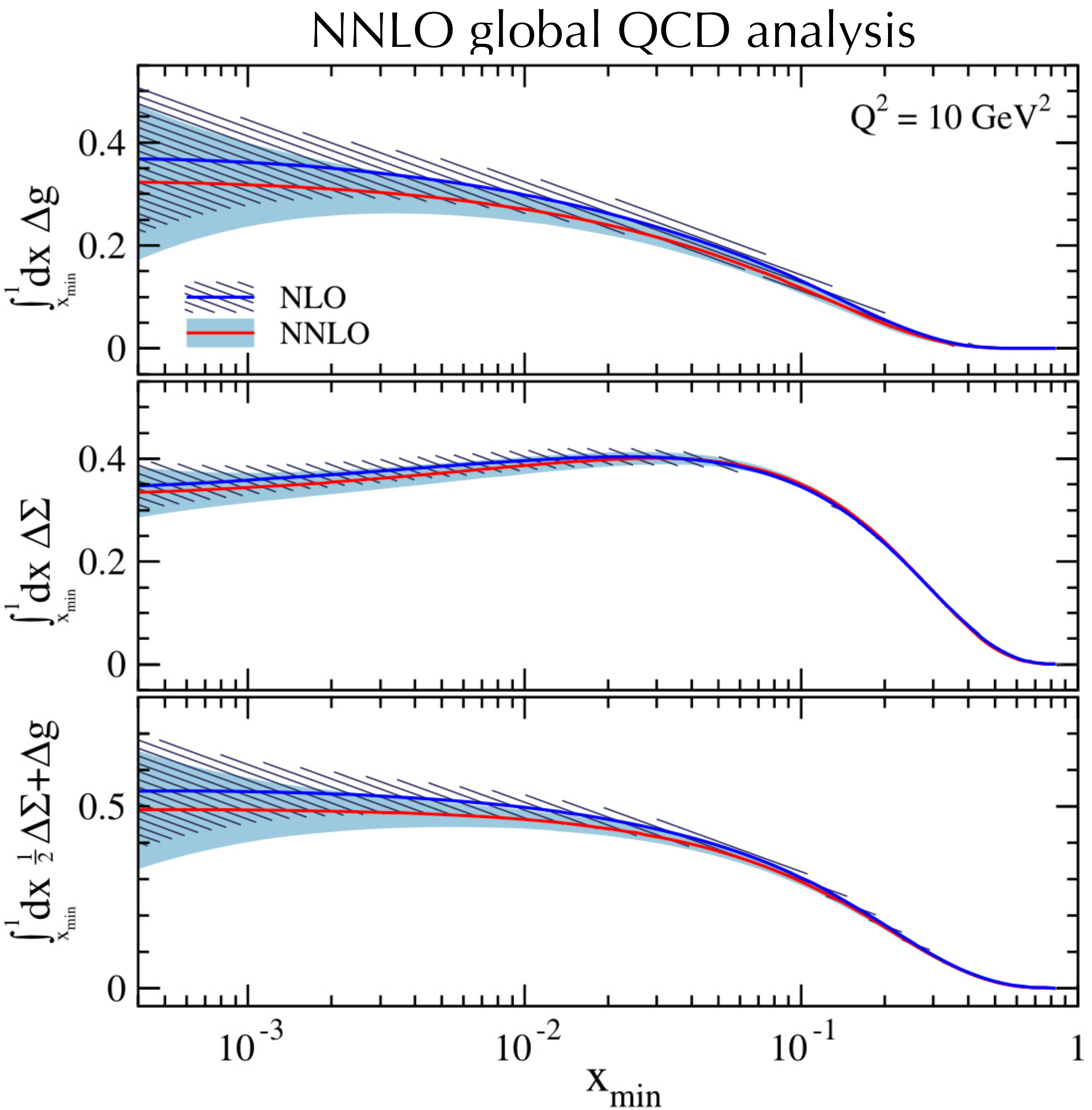

$$\frac{1}{2} = S^q(\mu) + \ell^q_z(\mu) + \ell^g_z(\mu) + S^g(\mu)$$

- $\ell^q_z,\ \ell^g_z,\ S^g$ are gauge dependent $(A^+ = 0)$, BUT measurable
- $S^q,\ S^g$ can be obtained from pol. PDFs
- $\ell^q_z,\ \ell^g_z$ can be obtained from Wigner distr.

$$\ell^q_z - l^{q,\mathrm{pot}}_z = L^q_z \longrightarrow \ell^g_z + S^g + l^{q,\mathrm{pot}}_z = J^g$$

Y. Guo, X. Ji, K. Shiells, NPB 96 (2021) 115440; X. Ji, F. Yuan, Y. Zhao, Nature Rev. Phys. 3 (2021) 1

# Quark and Gluon Spin contributions

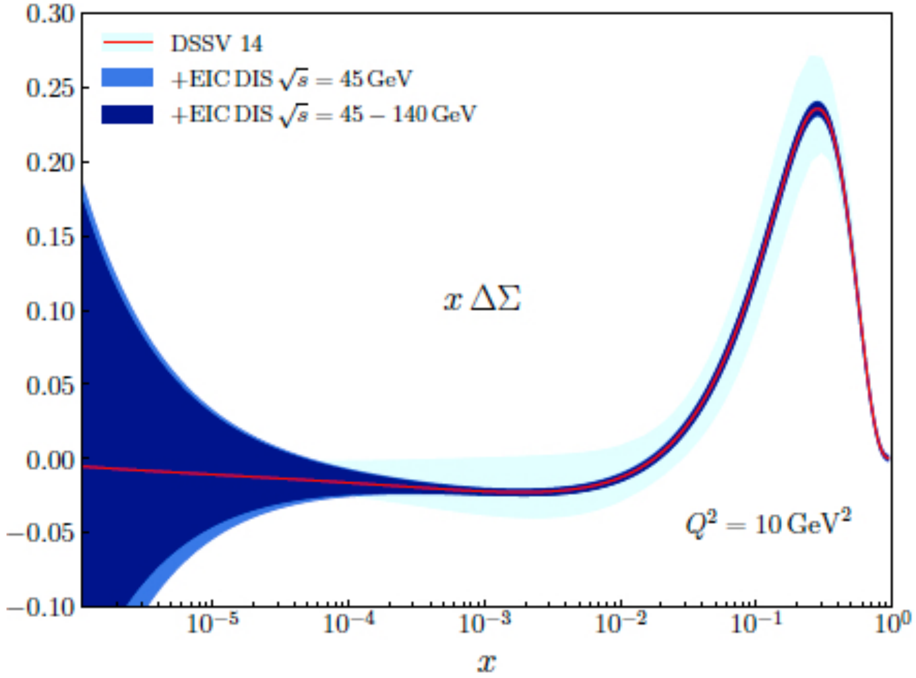



Gluon and Quarks almost saturate spin (JM) sum rule $\longrightarrow$ small contribution from OAM

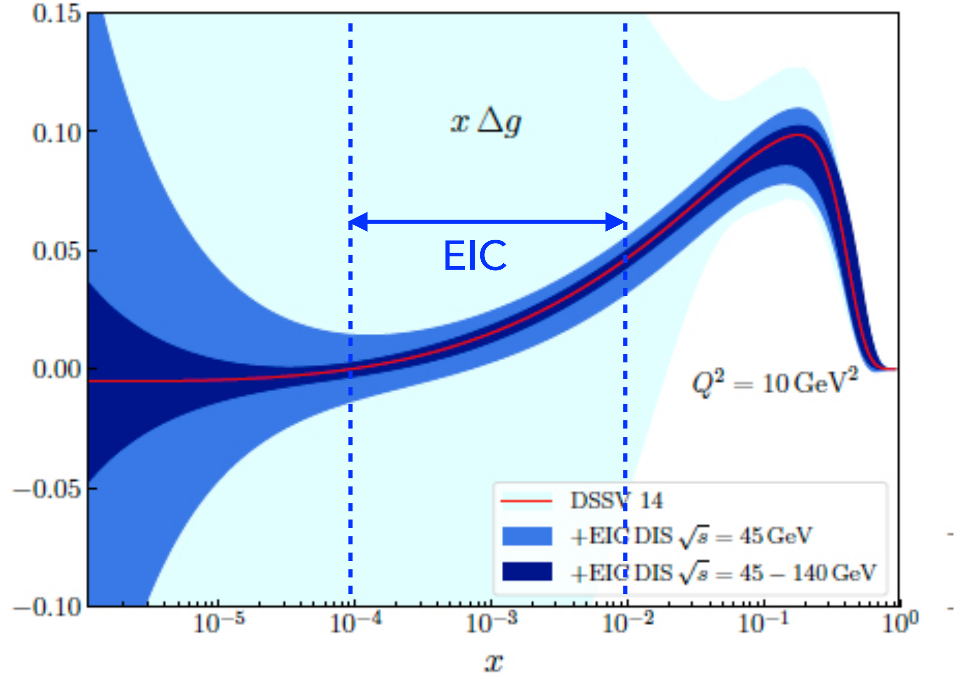
*I. Borsa, et al., PRL133,151901, 2024*

# Impact of future EIC
# for quark and gluon spin contributions

Quark Spin




Gluon Spin




*EIC Yellow Report: arXiv: 2103.05419*

We are constantly improving the knowledge of the contributions to the spin of the nucleon

However the details on the flavor and sea contributions are still sketchy

What about a direct measurement of orbital angular momentum?

# Orbital Angular momentum of the proton from available GPD measurements

$$J^{q,g} = \frac{1}{2}\int_{-1}^{1} dx\, x\, \left(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0)\right) \qquad L^q = J^q - S^q$$

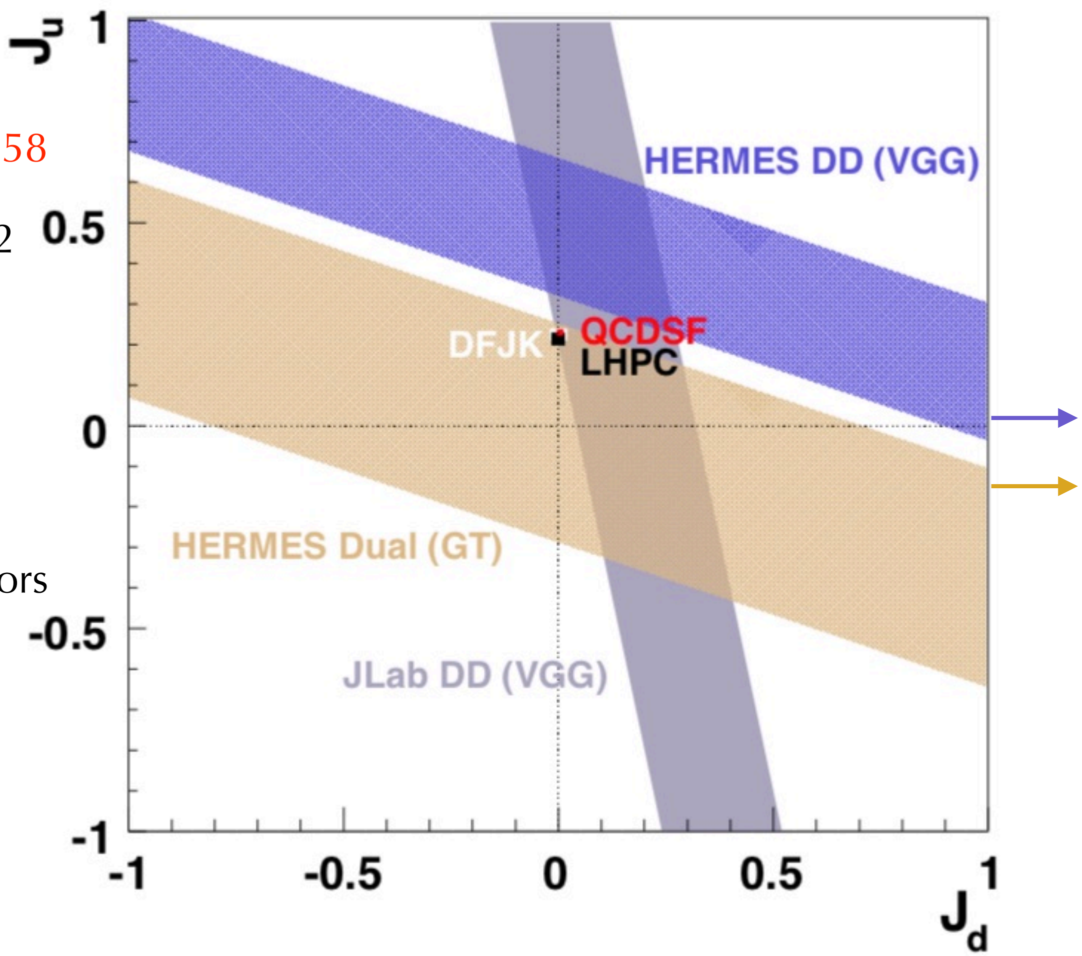
Lattice results

QCDSF: PoS (Lattice 2007) 158

LHPC: PRD77 (2008) 094502

GPDs extracted from form factors

DFJK, EPJC39 (2005) 1




extractions from HERMES data using two different models

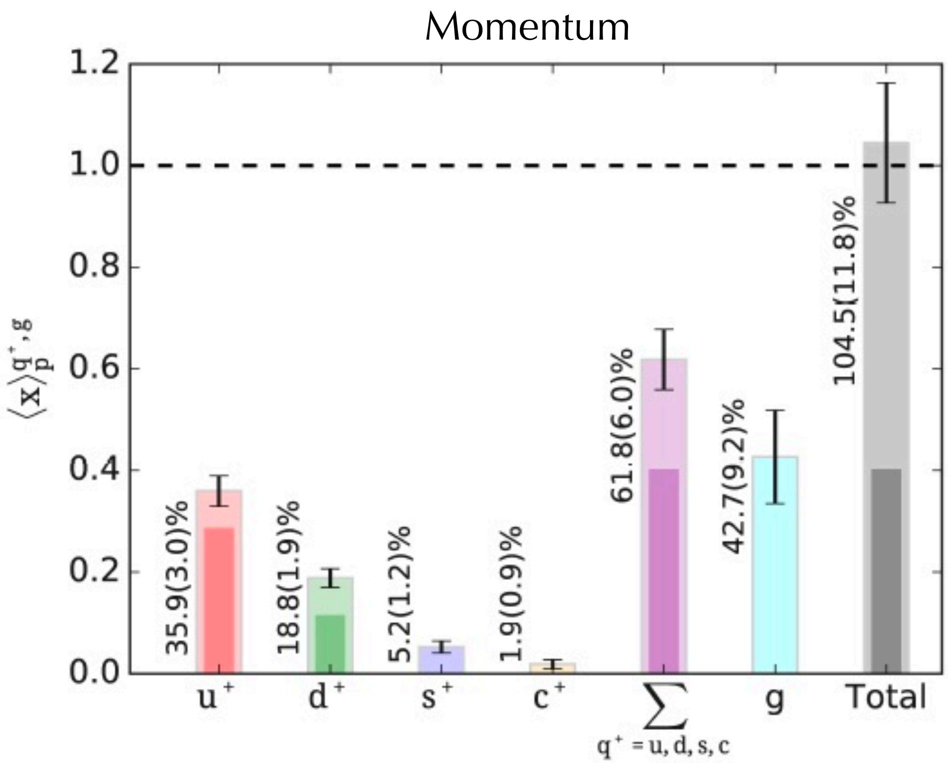
*JLab Hall A, Phys. Rev. Lett. 99 (2007) 242501*

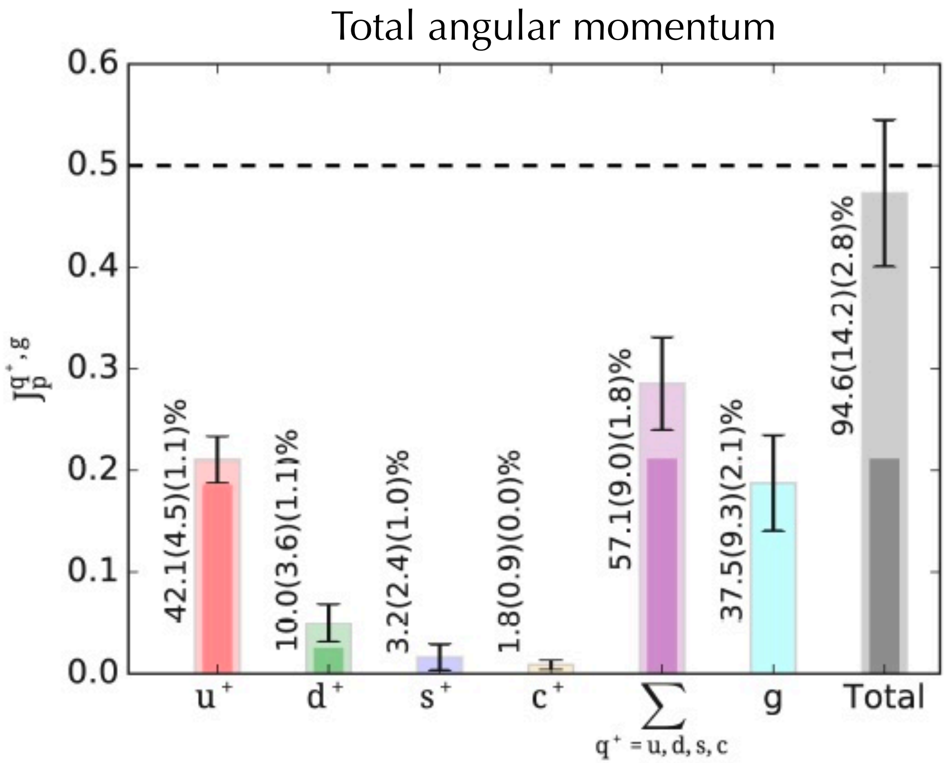
*Hermes Coll., JHEP 06 (2008) 066*

Improved accuracy with JLab12 and future EIC measurements!

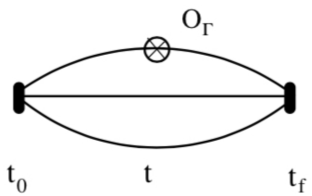
# Sum rules from lattice QCD

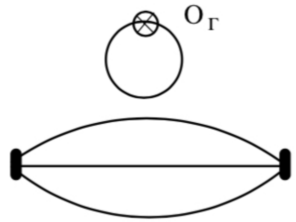
- Results at physical pion mass at the scale of 2 GeV




- dark bars: connected
- light bars: disconnected contributions (quarks & gluons)



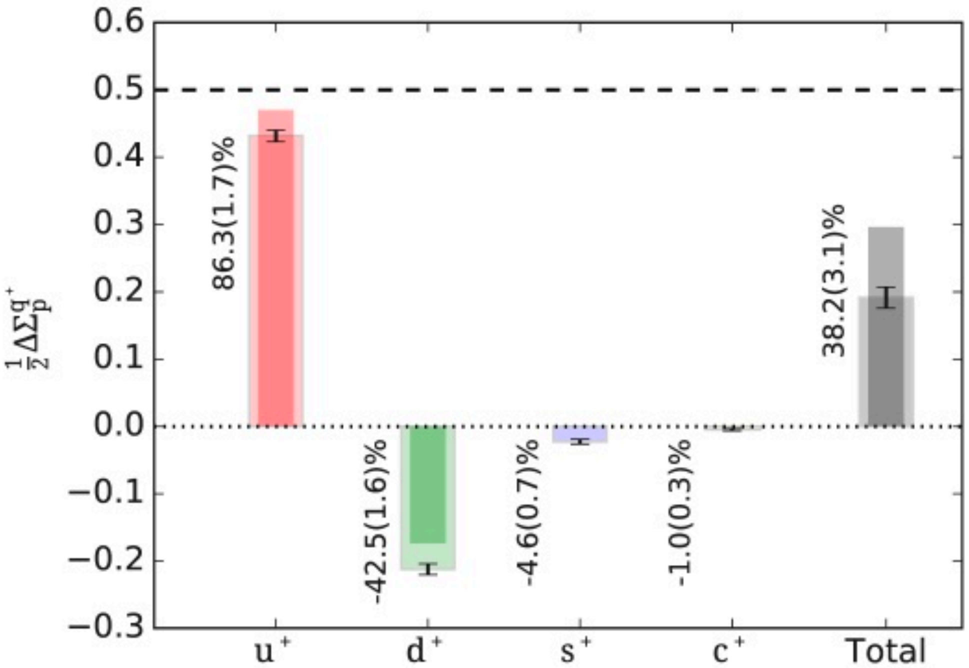




- gravitomagnetic sum rule: $\sum_{q=u,d,s} B^q(0) + B^g(0) = -0.099(91)(28)$

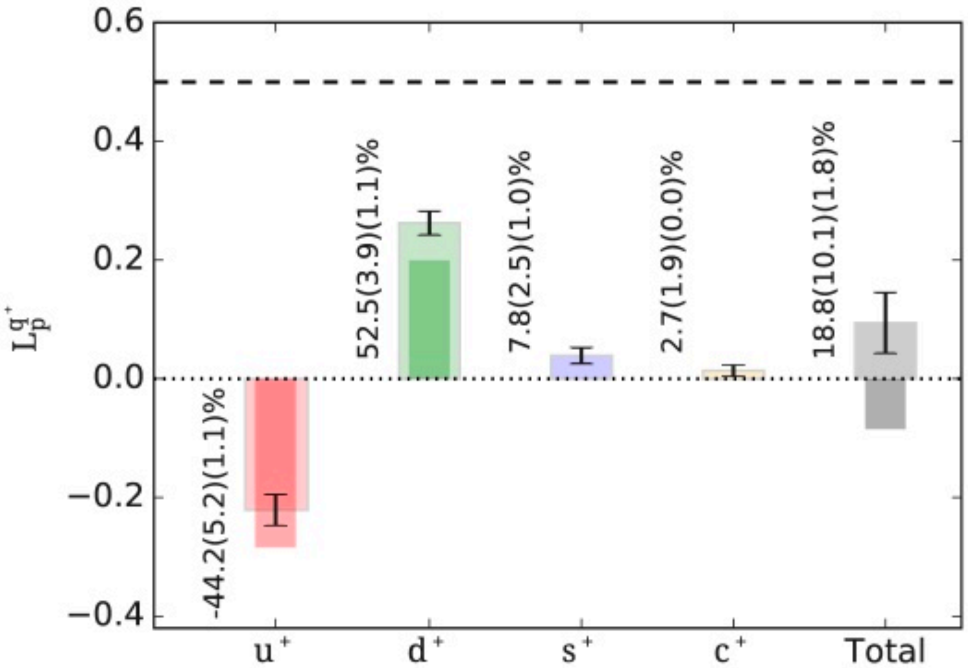
*Alexandrou et al. [Extended Twist Mass Coll.], Phys.Rev.D 101 (2020), 094513*

# Spin and orbital angular momentum from lattice QCD

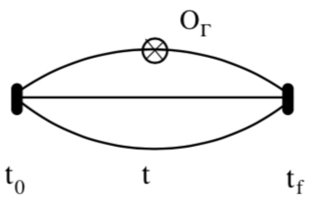
Quark spin



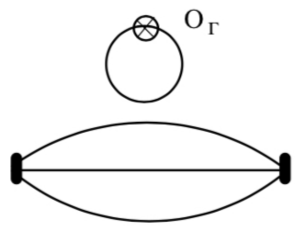

Quark orbital angular momentum $J^q - \frac{1}{2}\Delta\Sigma^q$




- dark bars: connected



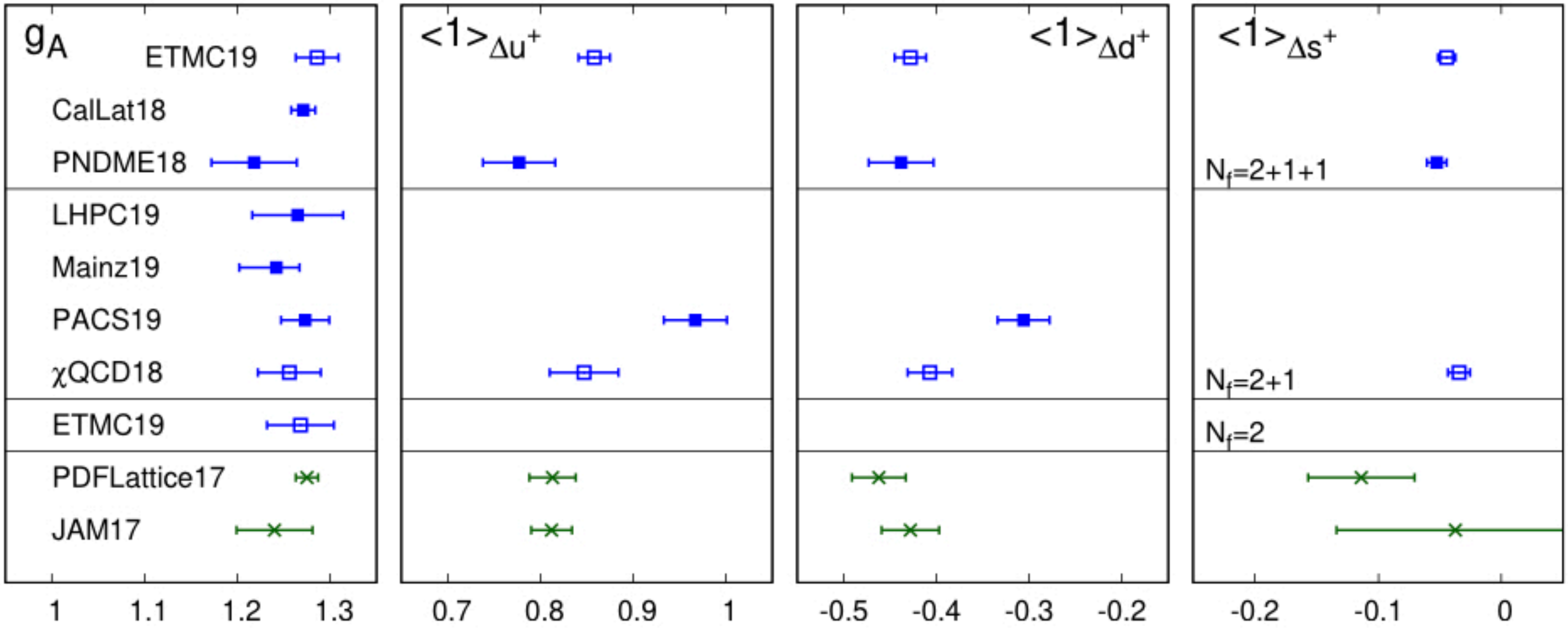

- light bars: disconnected contributions (quarks & gluons)




*Alexandrou et al., Phys.Rev.D 101 (2020), 094513*

# Comparison Lattice QCD and Phenomenology

$g_A \equiv \langle 1 \rangle_{\Delta u^+} - \langle 1 \rangle_{\Delta d^+}$ Quark spin $Q^2 = 4$ GeV$^2$




***PDFLattice17**: average of Lattice QCD results from the community white paper, Prog.Part.Nucl.Phys. 100 (2018) 107*

Overall fair agreement between lattice calculations and phenomenological fits

The uncertainties of the two have comparable size

Lattice QCD results could provide useful inputs to global fits of polarized PDFs

# Nucleon Structure Properties

| | | | | |
|---|---|---|---|---|
| em | $\partial_\mu J^\mu_{\text{em}} = 0$ | $\langle N'\|J^\mu_{\text{em}}\|N\rangle$ | $\longrightarrow$ | $Q,\ \mu$ |
| weak | $\partial_\mu J^\mu_{\text{weak}} = 0$ | $\langle N'\|J^\mu_{\text{weak}}\|N\rangle$ | $\longrightarrow$ | $g_A,\ g_p$ |
| gravity | $\partial_\mu T^{\mu\nu}_{\text{grav}} = 0$ | $\langle N'\|T^{\mu\nu}_{\text{grav}}\|N\rangle$ | $\longrightarrow$ | $M_N, J, D$ |

$Q_{\text{prot}} = 1.602176487(40) \times 10^{-19}\,\text{C}$

$\mu_{\text{prot}} = 2.792847356(23)\mu_N$

$M_{\text{prot}} = 938.272013(23)\,\text{MeV}$

$g_p = 8 - 12$

$g_A = 1.2694(28)$

$J = \frac{1}{2}$

$$D = \frac{4}{5} d_1 = ??$$

can be accessed from GPDs in hard exclusive reactions

*M. Polyakov and P. Schweitzer, Int. J. Mod. Phys. A33 (2018) 1830025*

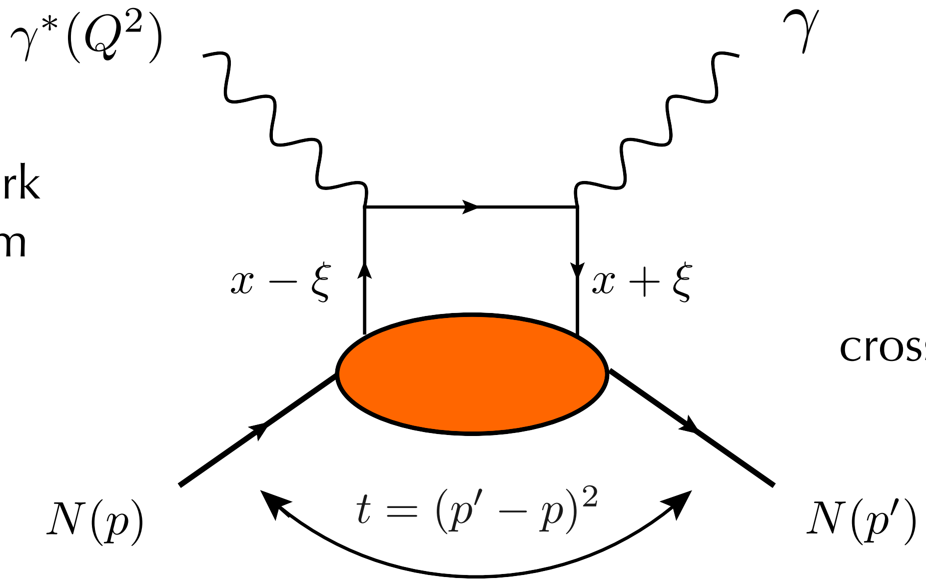
# Dispersion relation approach

# DVCS at leading twist

$x$ : average fraction of quark longitudinal momentum

$\xi$ : fraction of longitudinal momentum transfer

$t$ : nucleon momentum transfer




$t \ll \nu,\ Q^2$

$\downarrow$

crossing symmetric variable

$$\nu = \frac{s-u}{4M_N}$$

DVCS tensor at twist 2: $H^{\mu\nu} = \sum_{i=1}^{4} A_i(\nu, t, Q^2) O_i^{\mu\nu}$

unpolarized quark

$$A_1 = \mathcal{H} + \mathcal{E}$$

$$A_2 = -\mathcal{E}$$

long. polarized quark

$$A_3 = \tilde{\mathcal{H}}$$

$$A_4 = \tilde{\mathcal{E}}$$

# Twist-2 DVCS amplitudes

$$A_i(\xi,t) = \int_0^1 \mathrm{d}x\, F_i^+(x,\xi,t) \left[\frac{1}{x-\xi+i\epsilon} + \frac{1}{x+\xi-i\epsilon}\right] \qquad (i=1,\ldots,4)$$

- Involve singlet GPDs: $F_i^+ = \{H^+ + E^+, -E^+, \tilde{H}^+, \tilde{E}^+\}$

$$F_i^+(x,\xi,t) = F_i(x,\xi,t) - F_i(-x,\xi,t)$$

- Imaginary part: GPD at $x=\xi$

$$\mathrm{Im}\, A_i(\xi,t) = -\pi F_i^+(\xi,\xi,t)$$

- Real part involves convolution integral:

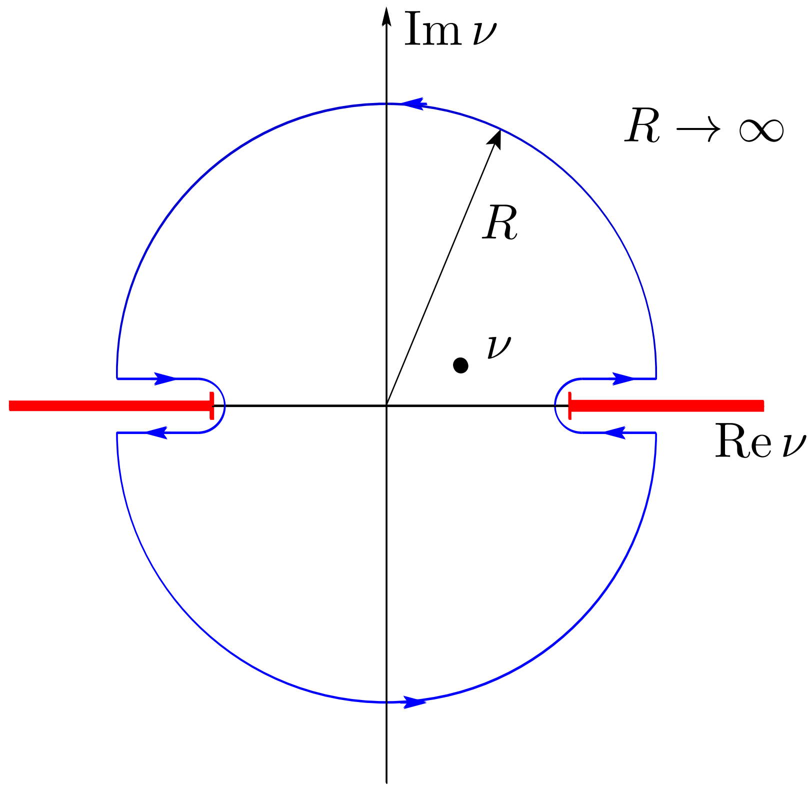
$$\mathrm{Re}\, A_i(\xi,t) = \mathcal{P} \int_0^1 \mathrm{d}x\, F_i^+(x,\xi,t) \left[\frac{1}{x-\xi} + \frac{1}{x+\xi}\right]$$

*the dependence on $Q^2$ is implicit

# Dispersion relations at fixed t and Q²

energy variables: $\nu = \dfrac{s-u}{4M_N} = \dfrac{Q^2}{4M_N\xi}$ and $\nu' = \dfrac{Q^2}{4M_N x}$

$A_i(\nu,t)$: analytical functions in the complex $\nu$ plane, with cuts on the real axis




- Cauchy integral formula

$$A_i(\nu,t) = \frac{1}{2\pi i}\oint_C d\nu' \frac{A_i(\nu',t)}{\nu'-\nu}$$

- Crossing symmetry and analyticity

$$A_i(\nu,t) = A_i(-\nu,t) \qquad\qquad A_i(\nu^*,t) = A_i^*(\nu,t)$$

# Unsubtracted Dispersion Relations

$$\mathrm{Re}\,A_i(\nu,t) = \frac{2}{\pi}\int_{\nu_0}^{\infty} \mathrm{Im}\,A_i(\nu',t)\frac{\nu'\mathrm{d}\nu'}{\nu'^2-\nu^2} \qquad (i=1,\ldots,4)$$

non-convergent integrals for $A_2$

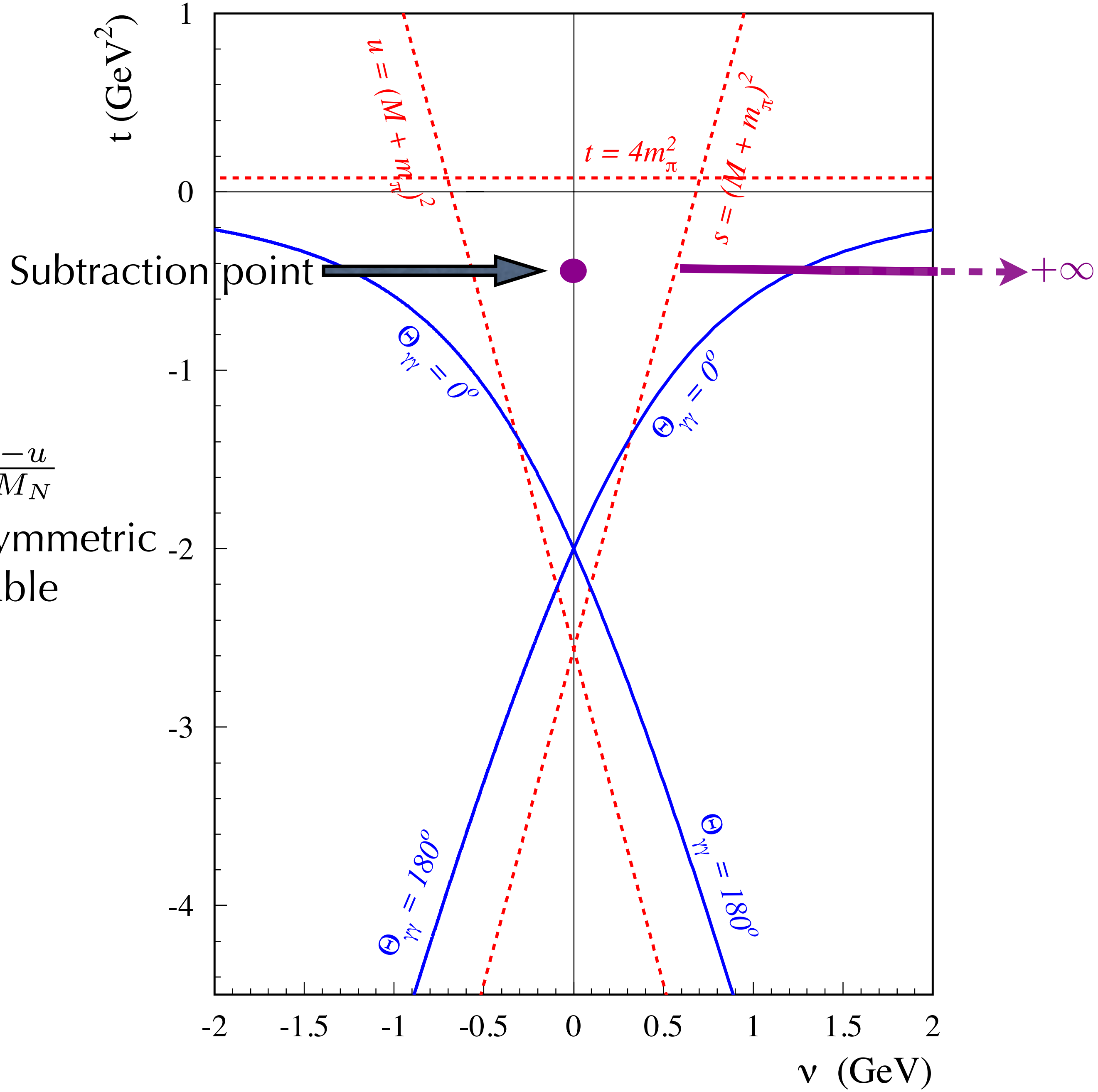
# Subtracted Dispersion Relations

$$\mathrm{Re}\,A_2(\nu,t) = A_2(0,t) + \frac{2}{\pi}\nu^2\mathcal{P}\int_{\nu_0}^{\infty} \mathrm{Im}\,A_2(\nu',t)\frac{\mathrm{d}\nu'}{\nu'(\nu'^2-\nu^2)}$$

subtraction at $\nu = 0$

Fixed

$Q^2 = 2\,\mathrm{GeV}^2$




DRs
in the s-channel
at fixed t

$\nu = \frac{s-u}{4M_N}$

crossing symmetric
variable

# Dispersion relations in terms of GPDs

once subtracted fixed-t DR in the variable x

$$\mathrm{Re}\, A_2(\xi, t) = \Delta(t) + \frac{2}{\pi}\mathcal{P}\int_0^1 \frac{dx}{x}\frac{\mathrm{Im}\, A_2(x,t)}{(\xi^2/x^2 - 1)}$$

- imaginary part in terms of GPDs: $\mathrm{Im}\, A_2(x,t) = \pi E^+(x, \xi = x, t)$

$$\mathrm{Re}\, A_2(\xi, t) = \Delta(t) - \mathcal{P}\int_0^1 dx E^+(x,x,t)\left[\frac{1}{x-\xi} + \frac{1}{x+\xi}\right]$$

- real part from convolution integral:

$$\mathrm{Re}\, A_2(\xi, t) = -\mathcal{P}\int_0^1 \mathrm{d}x\, E^+(x,\xi,t)\left[\frac{1}{x-\xi} + \frac{1}{x+\xi}\right]$$

- difference between convolution and dispersion integrals:

$$\Delta(t) = \mathcal{P}\int_0^1 \mathrm{d}x\left[E^+(x,x,t) - E^+(x,\xi,t)\right]\left[\frac{1}{x-\xi} + \frac{1}{x+\xi}\right]$$

*Anikin, Teryaev (2007); Kumericki-Passek, Mueller, Passek (2008); Diehl, Ivanov (2007); Polyakov, Vanderhaeghen (2008)*

# Subtraction Function

$$\Delta(t) = \mathcal{P}\int_0^1 \mathrm{d}x \left[E^+(x,x,t) - E^+(x,\xi,t)\right] \left[\frac{1}{x-\xi} + \frac{1}{x+\xi}\right]$$

⇨ Subtraction function is independent of $\xi$ $\longrightarrow$ formally put $\xi = 0$

$$\Delta(t) = 2\,\mathcal{P}\int_0^1 \mathrm{d}x \frac{1}{x}\left[E^+(x,x,t) - E^+(x,0,t)\right]$$

⇨ Time-Reversal invariance: GPD even in $\xi$ : $E(x,x,t) = E(x,-x,t)$

$$\boxed{\Delta(t) = 2\,\mathcal{P}\int_{-1}^1 \mathrm{d}x \frac{1}{x}\left[E(x,x,t) - E(x,0,t)\right]}$$

*Anikin, Teryaev (2007); Radyushkin (2012)*

# Subtraction Function: relation with D-term

$$\int_{-1}^{1} \frac{\mathrm{d}x}{x}[E(x,x+\xi,t)-E(x,\xi,t)] = \int_{-1}^{1} \frac{\mathrm{d}x}{x}\sum_{n=1}^{\infty}\frac{x^n}{n!}\frac{\partial^n}{\partial\xi^n}E(x,\xi,t)$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^{n+1}}{\partial\xi^{n+1}}\left[\int_{-1}^{1}\mathrm{d}x x^n E(x,\xi,t)\right]$$

⇨ Polinomiality of Mellin moments of GPDs

$$\int_{-1}^{1}\mathrm{d}x x^n E(x,\xi,t) = e_0^{(n)}(t) + e_2^{(n)}(t)\xi^2 + \cdots + e_{n+1}^{(n)}(t)\xi^{n+1}$$

$$\int_{-1}^{1} \frac{\mathrm{d}x}{x}[E(x,x+\xi,t)-E(x,\xi,t)] = \sum_{n=0}^{\infty} e_{n+1}^{(n)}(t)$$

⇨ Highest moment generated by Polyakov-Weiss D-Term $e_{n+1}^{(n)} = -\int_{-1}^{1}\mathrm{d}z z^n D(z,t)$

$$\boxed{\Delta(t) = 2\mathcal{P}\int_{-1}^{1}\mathrm{d}x\frac{1}{x}\left[E(x,x,t)-E(x,0,t)\right] = -2\int_{-1}^{1}\mathrm{d}z\frac{D(z,t)}{1-z}}$$

# Subtraction Function: relation with D-term

$$\Delta(t) = -2 \int_{-1}^{1} \mathrm{d}z \, \frac{D(z,t)}{1-z}$$

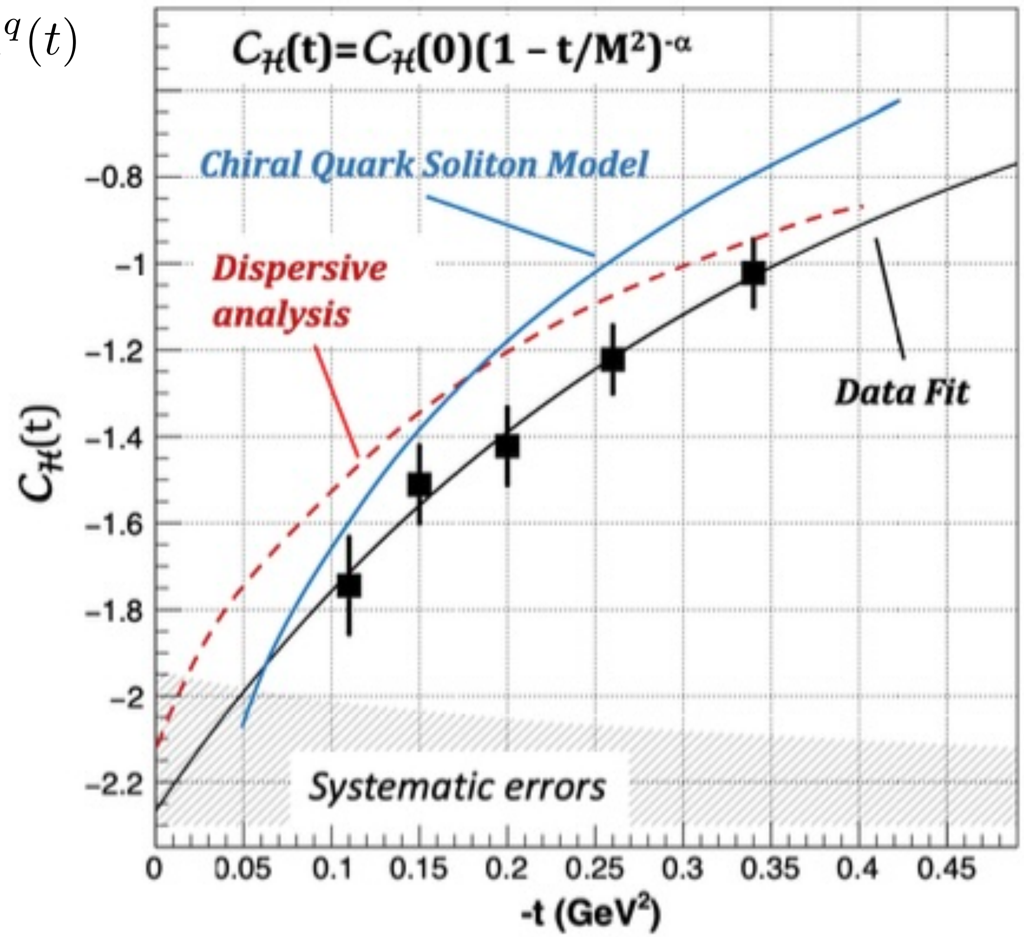
$\Longrightarrow$ Gegenbauer expansion of D-term $D(z,t) = (1-z^2) \sum_{\substack{n=1 \\ n\,odd}}^{\infty} d_n(t) C_n^{(3/2)}(z)$

$$\Delta(t) = -4 \sum_{\{n\,odd\}}^{\infty} d_n(t)$$

$\Longrightarrow$ Relation to EMT for factors $d_1(t) = 5\,C(t) = \frac{5}{4} D(t)$

# Extraction of D-term form factor

$$\mathcal{C}_H(t) = -2\sum_q e_q^2 \Delta^q(t)$$




Extraction from data:
- neglecting gluon contribution
- assuming:

$$\mathcal{C}_H(t) = 8\sum_q e_q^2 \sum_{\substack{n=1\\ n\,odd}}^{\infty} d_n(t) \approx \frac{10}{9} d_1^Q(t)$$

Fit to data: $\mathcal{C}_H(t) = \dfrac{\mathcal{C}_H(0)}{(1-t/M^2)^{\alpha}}$

$$\mathcal{C}_H(0) = -2.27 \pm 0.16 \pm 0.36 \qquad \lambda = 2.76 \pm 0.23 \pm 0.48$$

$$M^2 = 1.02 \pm 0.13 \pm 0.21\,\mathrm{GeV}^2$$

*Girod, Elouadrhiri, Burkert,Nature 557 (2018) 7705; Burkert, et al., Rev. Mod. Phys. 95 (2023) 014002*

# Mechanical properties of the nucleon

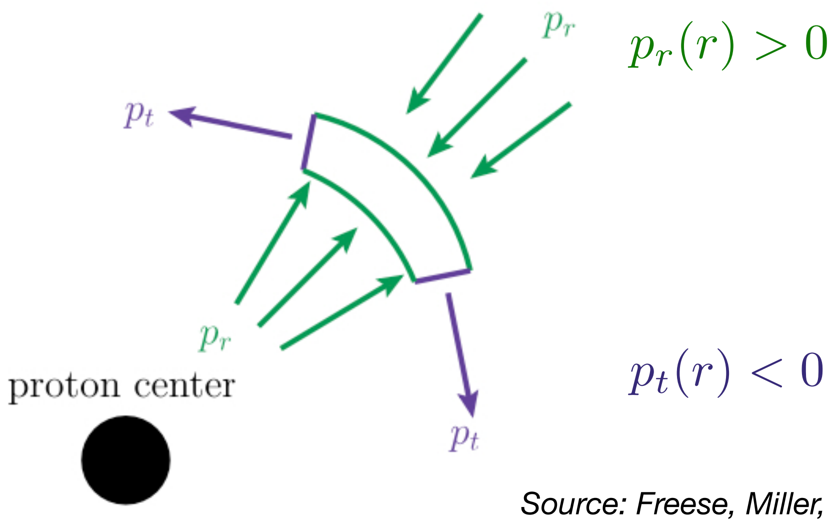
- Fourier transform in coordinate space:

$$T^{ij} = \underbrace{s(\vec{r})}_{\text{shear forces}} \left( \frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij} \right) + \underbrace{p(\vec{r})}_{\text{isotropic pressure}}\delta^{ij}$$

$$p(r) = \frac{1}{6M_N}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}D(r) \qquad s(r) = -\frac{1}{4M_N}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}D(r)$$

radial pressure $p_r(r) = p(r) + \frac{2}{3}s(r)$

tangential pressure $p_t(r) = p(r) - \frac{1}{3}s(r)$

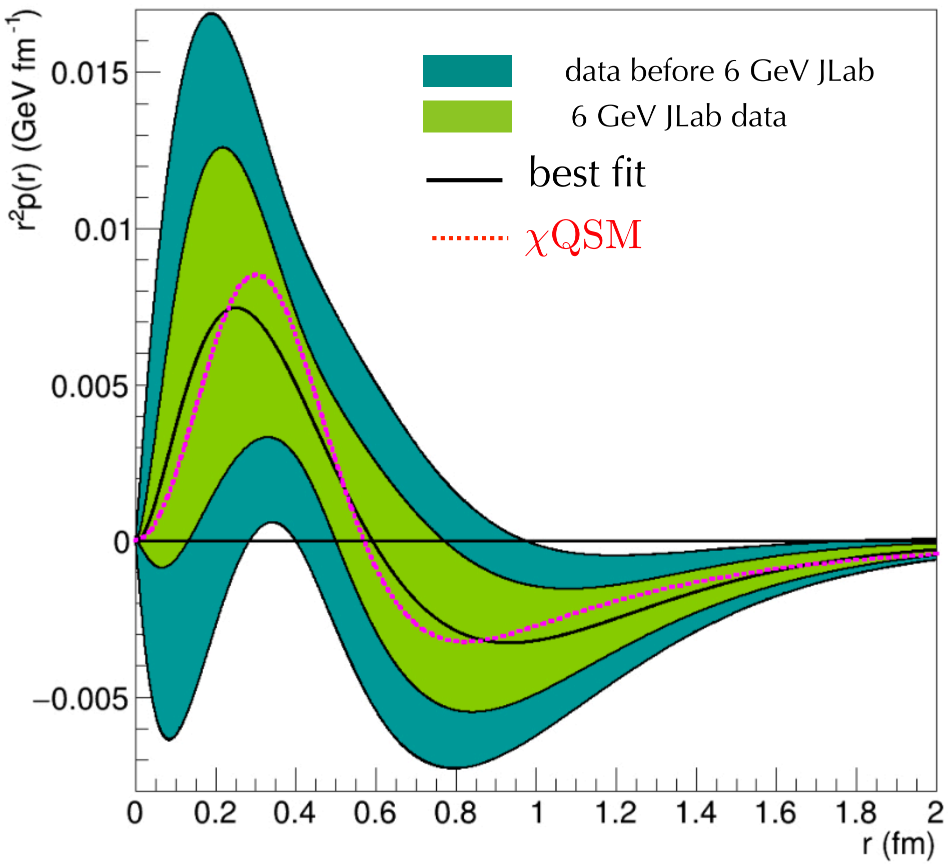



Source: Freese, Miller, PRD104 (2021)

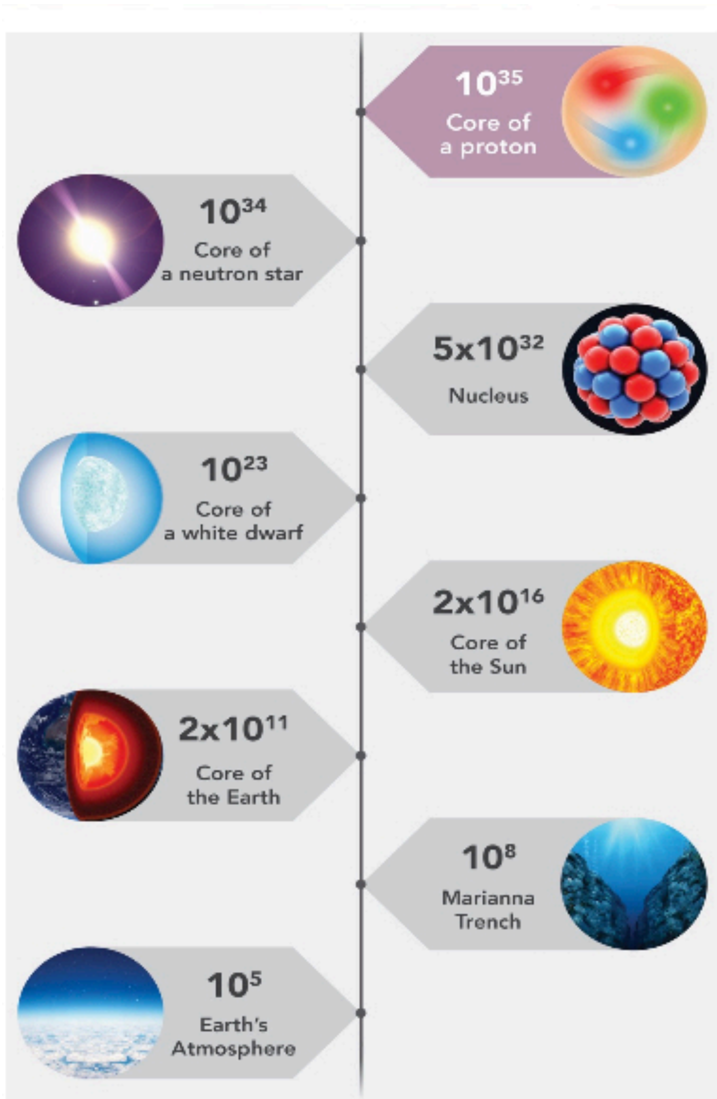
*Polyakov, Schweitzer, IJMA 33 (2018) 1830025;*
*Lorcé, Moutarde, Trawinski, EPJ C79 (2019) 89*

# Pressure distributions from data

$$p(r) = \frac{1}{6M_N}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}D(r)$$




*Girod, Elouadrhiri,Burkert, Nature 557 (2018) 7705*




*Source: L. Elouadrhiri*

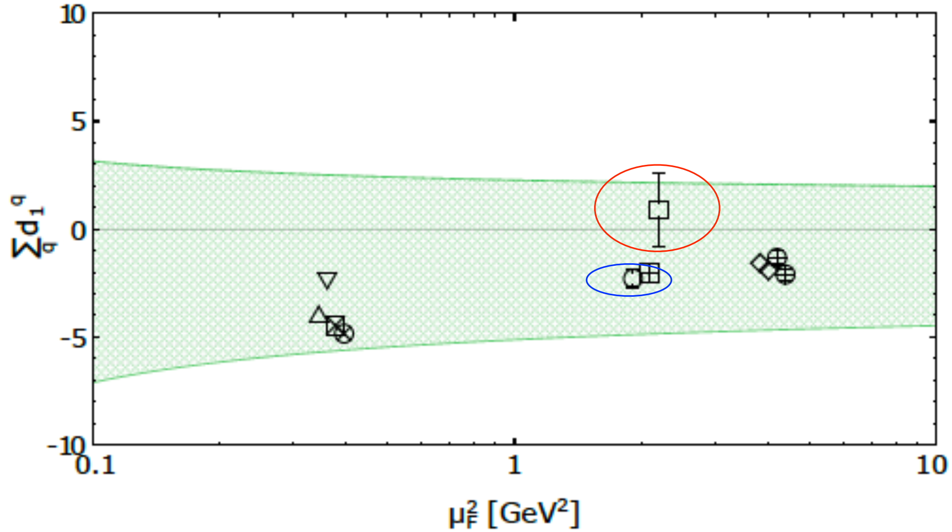
- Gluon contribution neglected
- Systematic errors including model-dependent assumptions in the extraction
- More flexible analyses based on unconstrained artificial neural network techniques give results compatible with zero with large uncertainties *(Kumericki, 2019; Dutrieux et al., 2021)*

Necessary to verify model assumptions in the exp extraction
with more data coming from JLab, COMPASS and the future EIC, EIcC

*Kumericki, Nature 570 (2019) 7759; Dutrieux et al, Eur. Phys. J. C81 (2021) 4*



global fit to DVCS data
with artificial neural networks




CLAS data, with fixed param., Girod et al.

CLAS data, with neural networks Kumericki

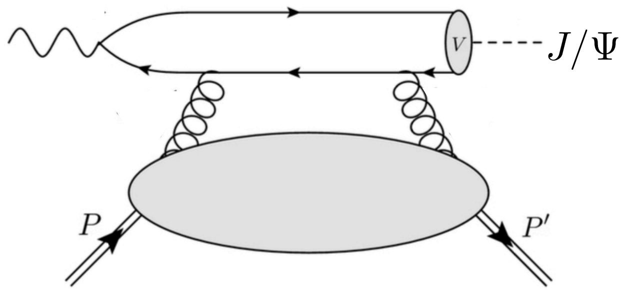
$$\sum_q d_1^q < 0$$

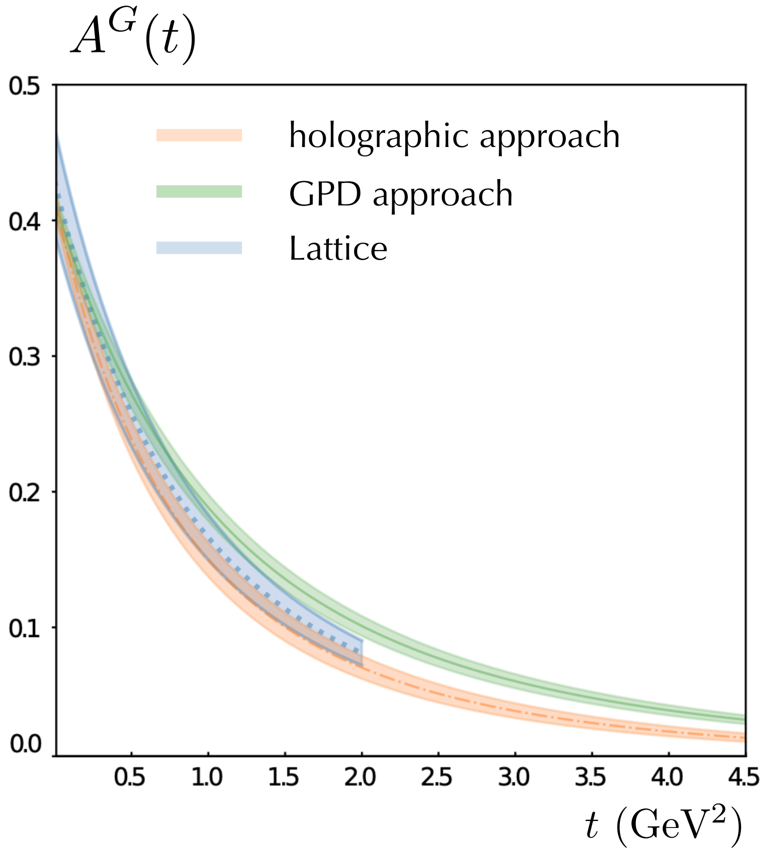
in all model calculations
for a stable proton

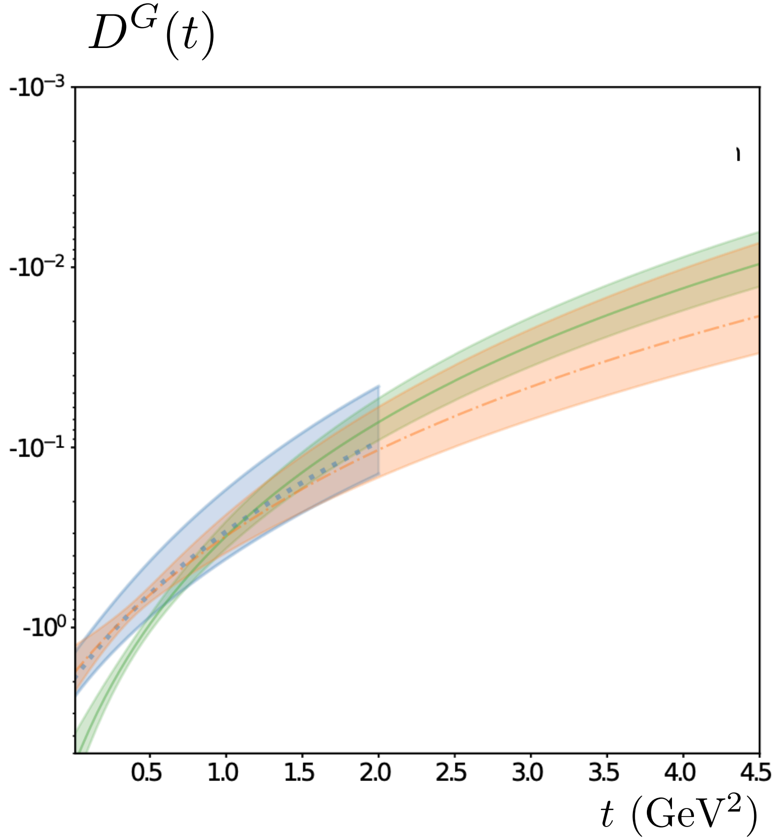
| Marker in Fig. 3 | $\sum_q d_1^q(\mu_F^2)$ | $\mu_F^2$ in GeV$^2$ | # of flavours | Type |
|---|---|---|---|---|
| ○ | $-2.30 \pm 0.16 \pm 0.37$ | 2.0 | 3 | from experimental data |
| □ | $0.88 \pm 1.69$ | 2.2 | 2 | from experimental data |
| ◇ | $-1.59$ | 4 | 2 | $t$-channel saturated model |
| | $-1.92$ | 4 | 2 | $t$-channel saturated model |
| △ | $-4$ | 0.36 | 3 | $\chi$QSM |
| ▽ | $-2.35$ | 0.36 | 2 | $\chi$QSM |
| ⊠ | $-4.48$ | 0.36 | 2 | Skyrme model |
| ⊞ | $-2.02$ | 2 | 3 | LFWF model |
| ⊗ | $-4.85$ | 0.36 | 2 | $\chi$QSM |
| ⊕ | $-1.34 \pm 0.31$ | 4 | 2 | lattice QCD ($\overline{\text{MS}}$) |
| | $-2.11 \pm 0.27$ | 4 | 2 | lattice QCD ($\overline{\text{MS}}$) |

# Gluonic EMT Form Factors

*Exp (Hall C, JLab): Duran et al., Nature 615 (2023) 7954*
*Lattice: Pefkou, Hackett, Shanahan, PRD105 (2022) 054509*

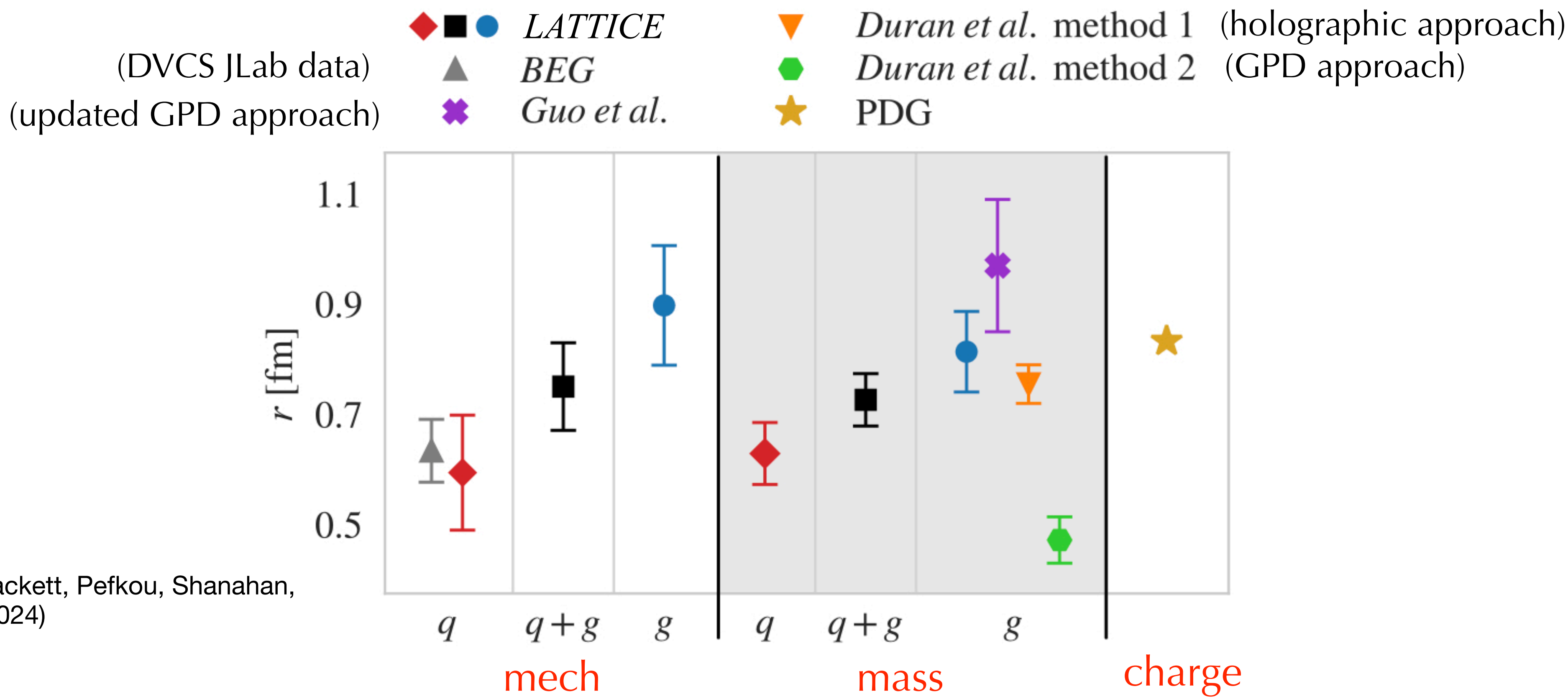



- proof of concept of feasibility to extract gluonic structure
- preliminary new results from GlueX and further measurements planned with SOLID at JLab
- JLab22 crucial for these measurements: high luminosity and leverage in t
- EIC: complementary measurements for $\Upsilon$ photo- and electro-production, but require L=100 fb$^{-1}$

# Proton size




Source: Hackett, Pefkou, Shanahan, PRL132 (2024)

$$\sqrt{\frac{\int d^3r\, r^2\, p_r(r)}{\int d^3r\, p_r(r)}} \qquad \sqrt{\frac{\int d^3r\, r^2\, T^{00}(r)}{\int d^3r\, T^{00}(r)}} \qquad \sqrt{-6\frac{G_E'(0)}{G_E(0)}}$$

- The separate quark and gluon contributions are renormalization scale dependent (results shown at $\mu^2 = 4\,\mathrm{GeV}^2$)
- Proton mechanical quark radius is ~30% smaller than charge radius
- Gluon contributions extend the radial (mass and mech) size of the proton beyond that of the quark contributions