2nd European School on the Physics of the EIC and Related Topics

PHYSICS OF GENERALIZED PARTON DISTRIBUTIONS

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UNIVERSITÀ DI PAVIA

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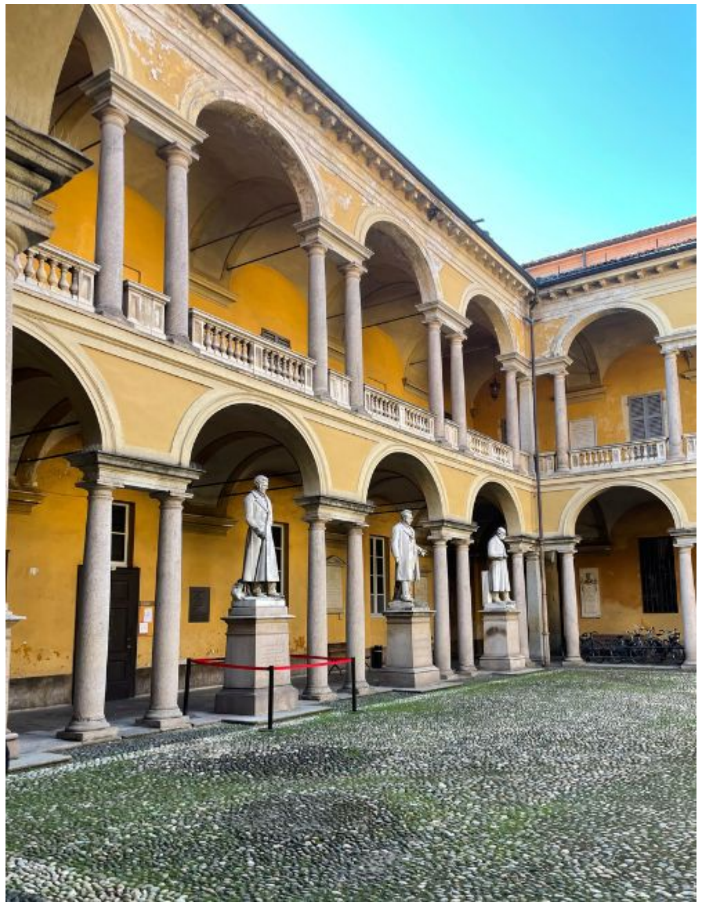
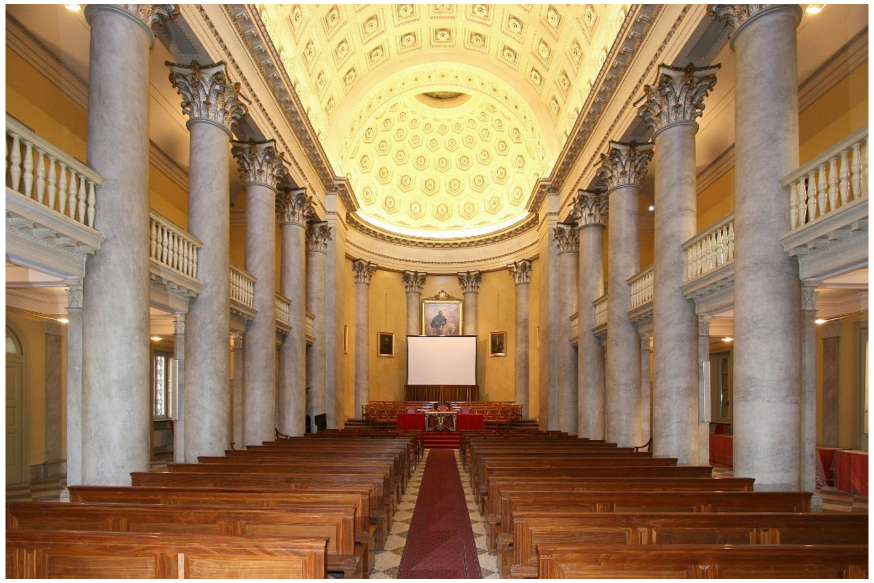
Map of the Universities in Italy

Map of the Universities in Italy

Founded in 1361, it is one of the oldest universities in the world

24000 students, 18 departments,
85 different courses
and 18 colleges





Hadron Physics Group
at the Department of Physics:
3 permanent staff
1 PhD students
3 postdocs
a large network of collaborators

Plan of lectures

0. Brief introduction

1. Partonic interpretation of Generalized Parton Distributions (GPDs)

2. Multidimensional picture of the proton in the 1+2D

3. Sum rule for Angular Momentum

4. Form Factors of Energy Momentum Tensor

5. Observables for GPDs

A few references on GPDs

- M. Diehl, Phys. Rep. 388 (2003) 41
- K. Goeke, M. Polyakov, M. Vanderhaeghen, Prog. Part. Nucl. Phys. 47 (2001) 401
- X. Ji, Ann. Rev. Nucl. Part. Sci. 54 (2004) 413
- A.V. Belitsky, A.V. Radyushkin, Phys. Rept. (2005) 418
- S. Boffi, B. Pasquini, Riv. Nuovo Cim. 30 (2007) 387
- M. Guidal, H. Moutarde, M. Vanderhaeghen, Rept. Prog. Phys. 76 (2013) 066202
- N. D'Hose, S. Niccolai, A. Rostomyan, Eur. Phys. J. A52 (2016) 151
- K. Kumericki, S. Liuti, H. Moutarde, Eur. Phys. J. A52 (2016) 157

Recent Review

Eur. Phys. J A52 (2016)

The European Physical Journal A
All Volumes & Issues

The 3-D Structure of the Nucleon

ISSN: 1434-6001 (Print) 1434-601X (Online)

In this topical collection (17 articles)

Editors: M. Anselmino, M. Guidal, P. Rossi

Pedagogical lecture notes

Cédric Mezrag, “An Introductory Lecture on Generalised Parton Distributions”,
Few. Body Syst. 63 (2022) 62

Pedagogical intro to PDFs and their generalizations

Cédric Lorcé, Andreas Metz, BP, Peter Schweitzer, to appear in the arXiv by the end of July

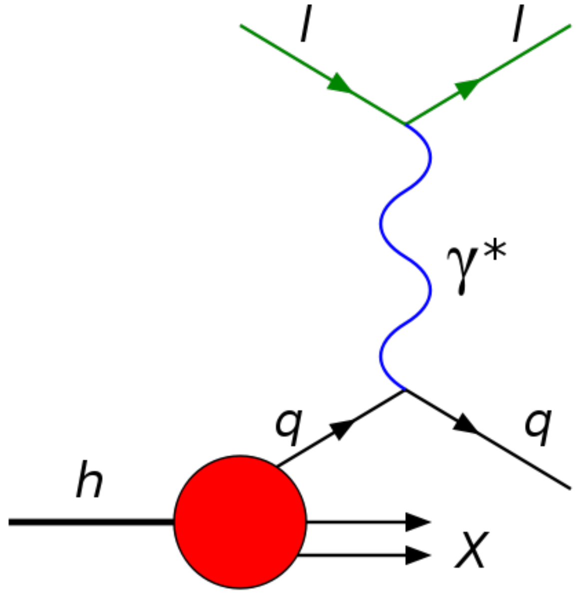
Parton Distribution Functions (PDFs)

$$\boxed{\frac{1}{2}\int\frac{\mathrm{d}z^-}{2\pi}e^{ik^+z^-}\langle p^+,\vec{0}_\perp,\Lambda'|\bar{\psi}(-\frac{z}{2})\Gamma\psi(\frac{z}{2})|p^+,\vec{0}_\perp,\Lambda\rangle_{z^+=0,\vec{z}_\perp=0}}$$

Depend on

Λ, Λ', Γ: nucleon and quark polarizations

$x=\dfrac{k^+}{p^+}$: longitudinal momentum fraction

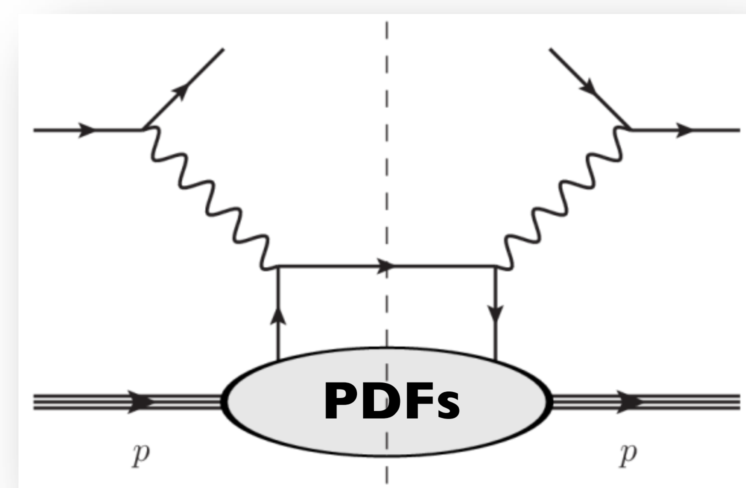


$Q^2 \gg$

$\nu \gg$

$x=\frac{Q^2}{2M\nu}$ finite

Deep Inelastic Scattering



Parton Distribution Functions (PDFs)

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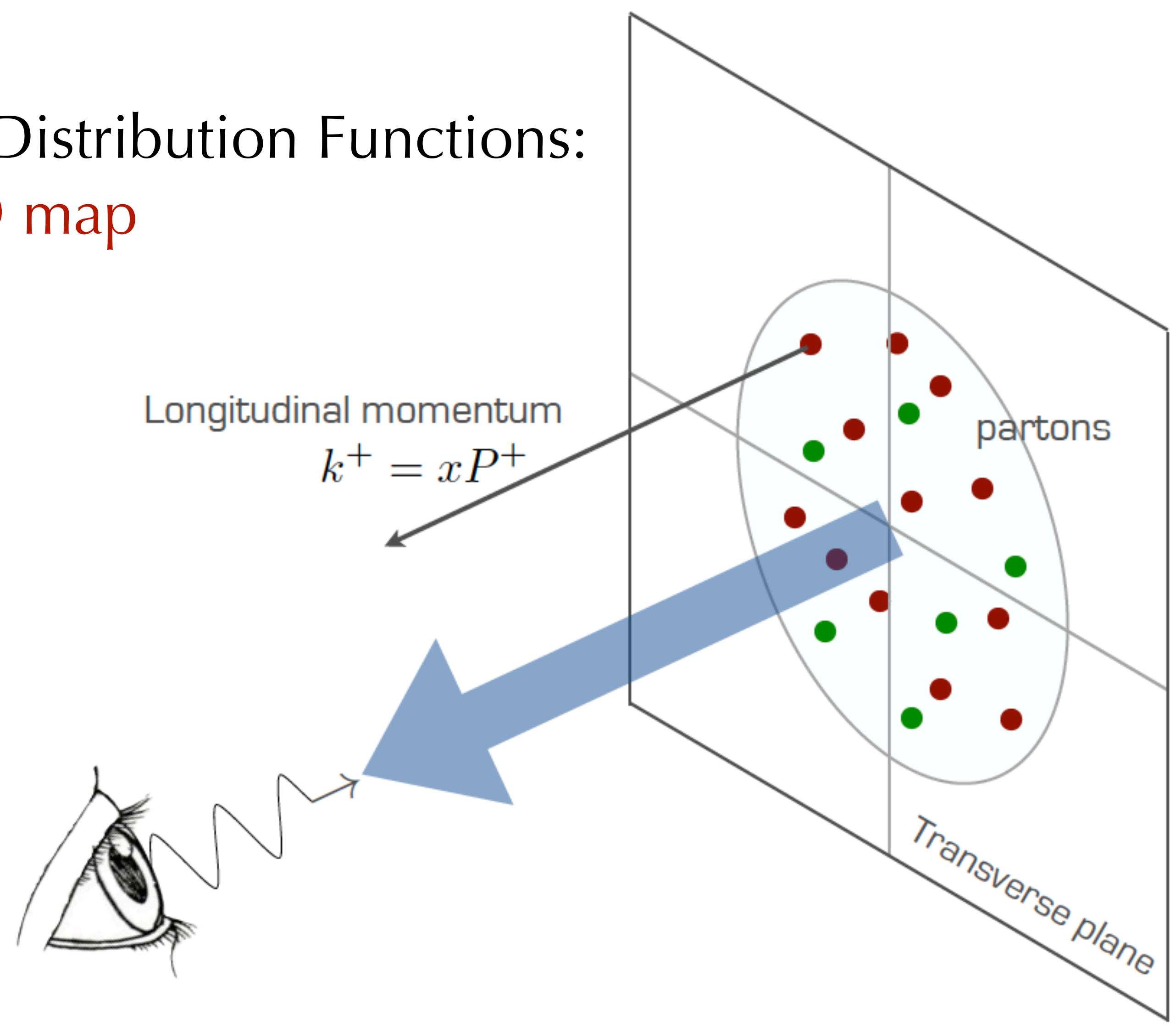
Depend on

Λ, Λ', Γ: nucleon and quark polarizations

$x = \dfrac{k^+}{p^+}$: longitudinal momentum fraction

Parton Distribution Functions:
1D+0D map



Generalized Parton Distributions (GPDs)

$$\boxed{\frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ik^+z^-} \langle p'^+, -\frac{\vec{\Delta}_\perp}{2}, \Lambda' | \bar{\psi}(-\frac{z}{2})\, \Gamma\, \psi(\frac{z}{2}) | p^+, \frac{\vec{\Delta}_\perp}{2}, \Lambda \rangle_{z^+=0, \vec{z}_\perp=0}}$$

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non-diagonal
matrix elements

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non-diagonal matrix elements

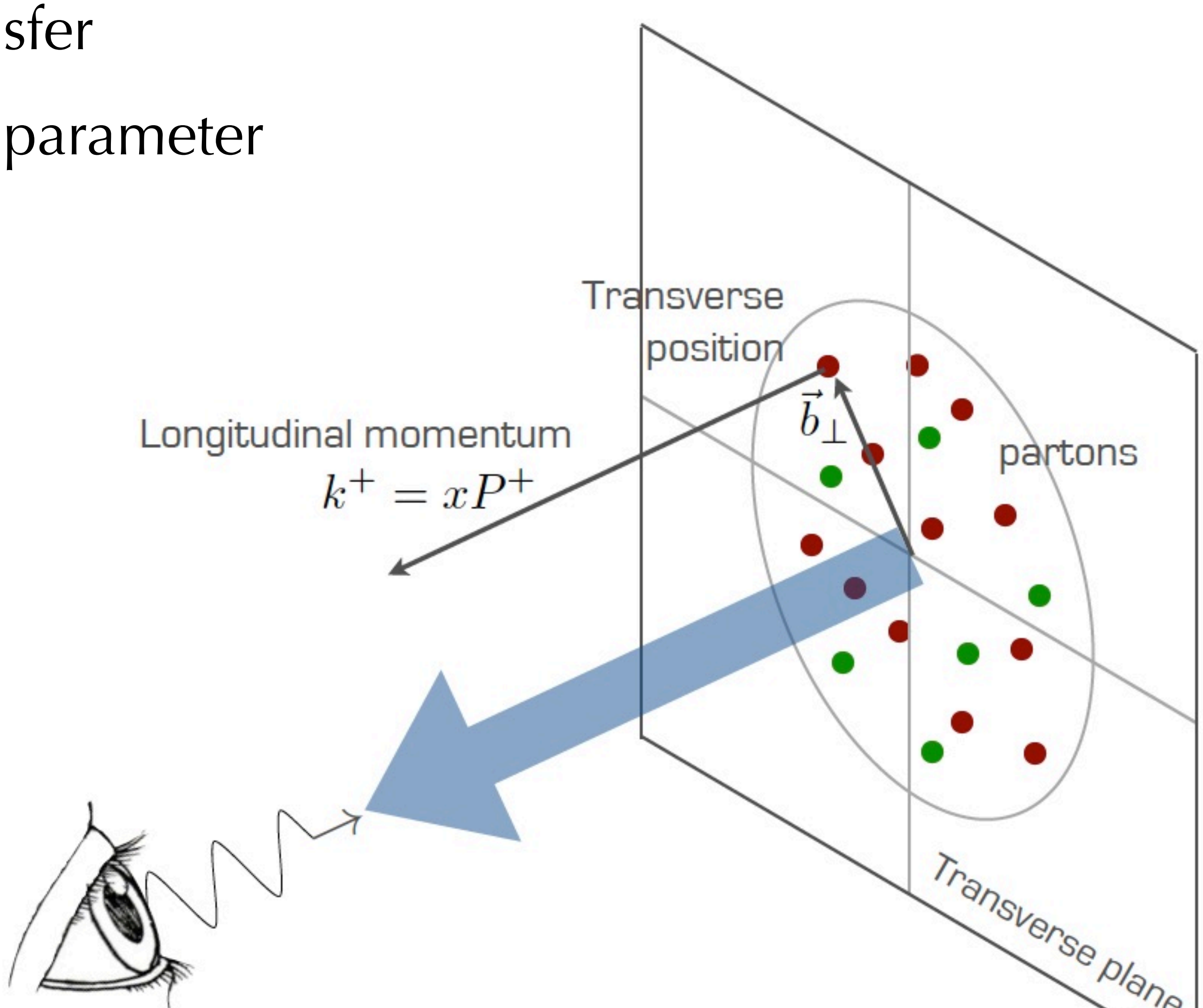
Depend on

Λ, Λ', Γ: nucleon and quark polarizations

$x = \dfrac{k^+}{p^+}$: longitudinal momentum fraction

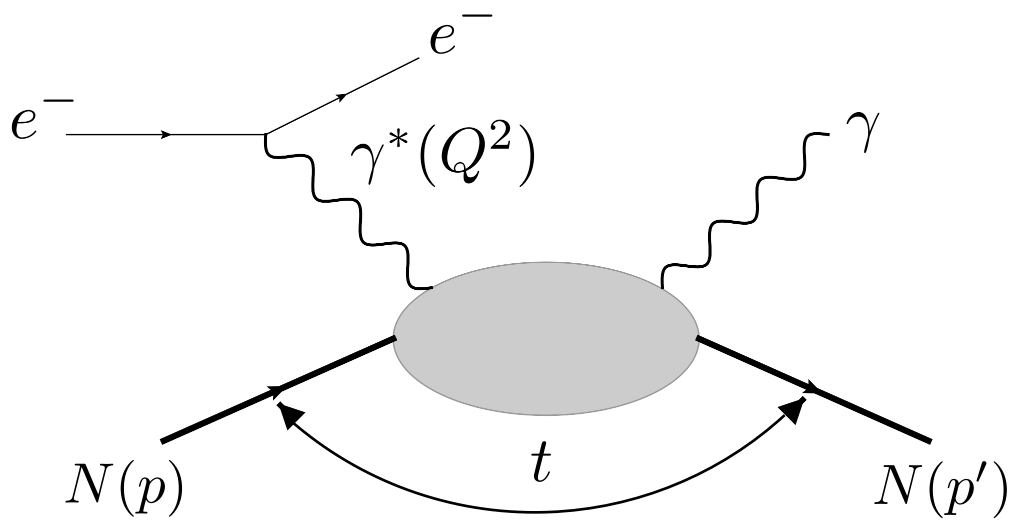
Δ: momentum transfer

$\vec{b}_\perp \xleftrightarrow{\text{FT}} \vec{\Delta}_\perp$: impact parameter

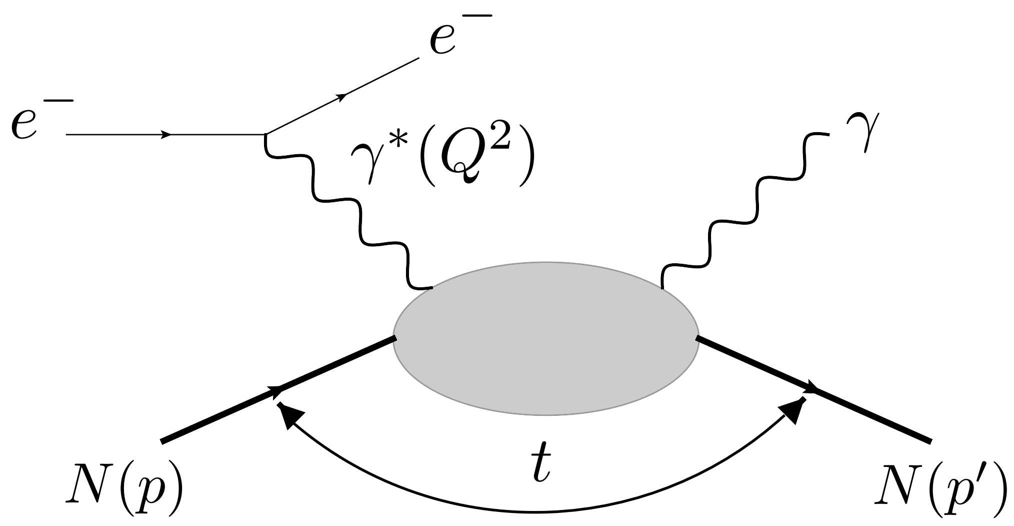


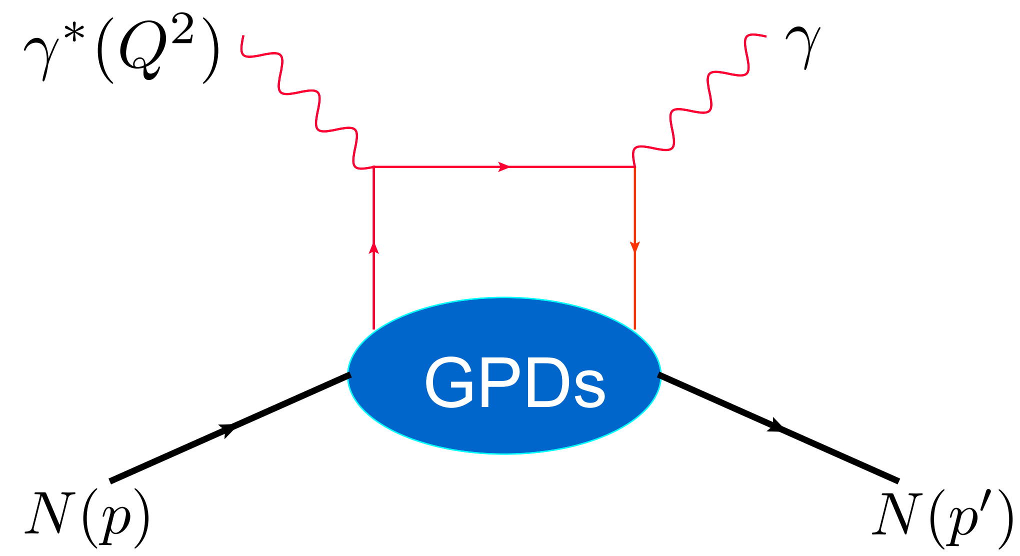
1+2D maps

How to access GPDs



How to access GPDs

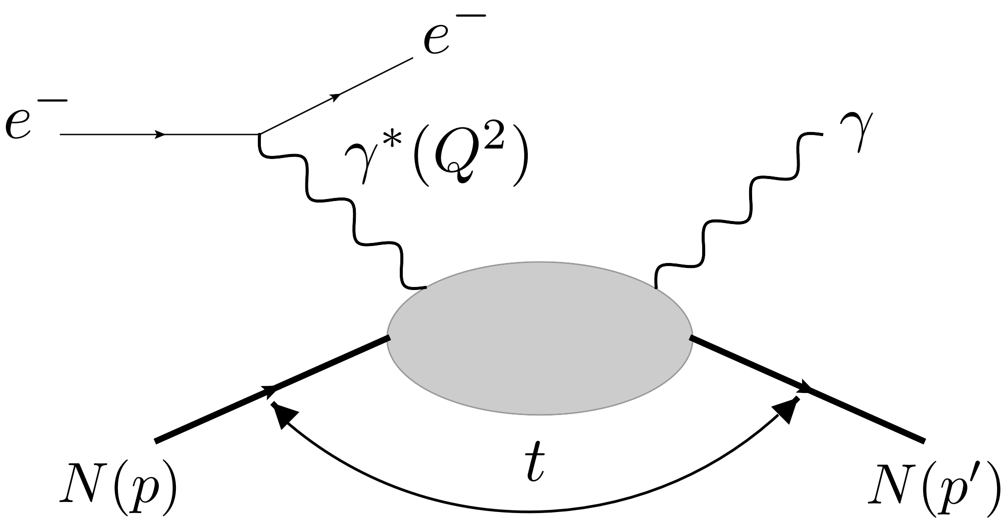


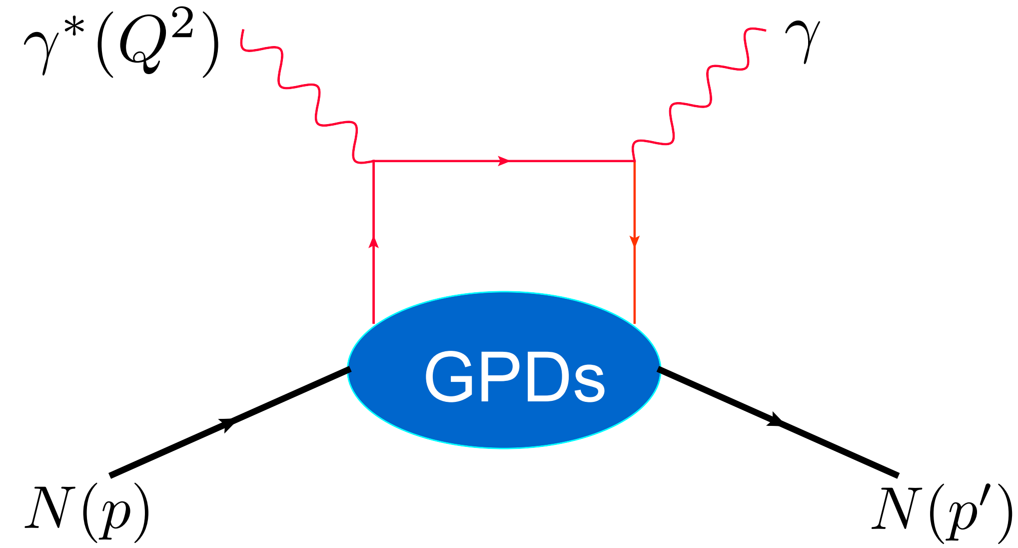


factorization for large Q^2, $|t| << Q^2$, s

$$\mathcal{M} = [\text{parton Ampl.}] \otimes [\text{GPDs}]$$

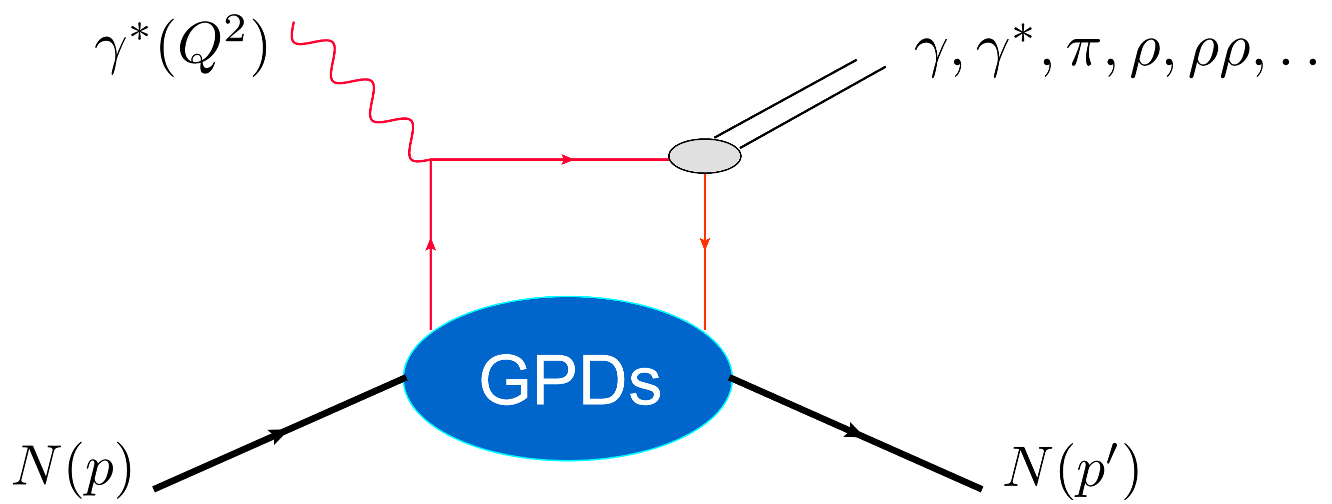
How to access GPDs





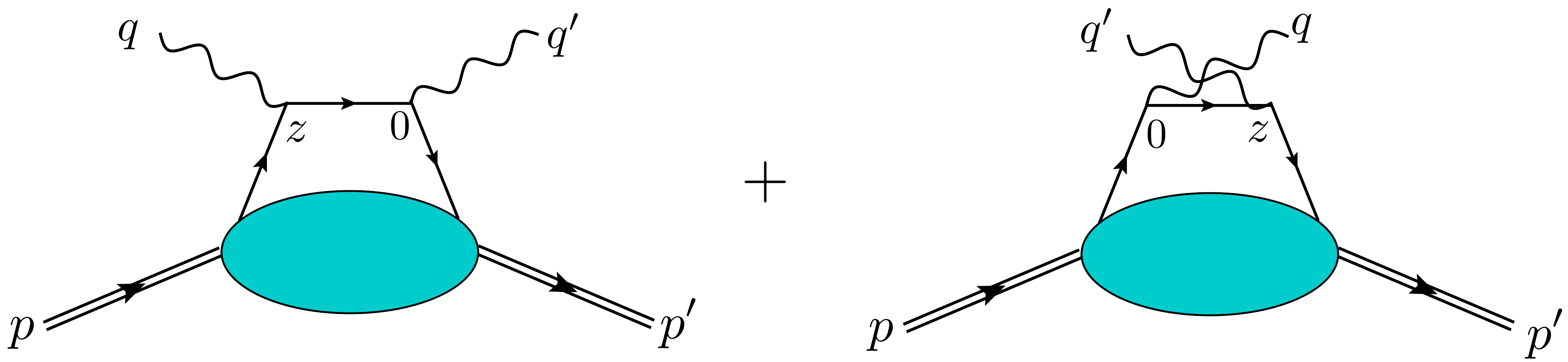
factorization for large Q², |t|<< Q² , s

$\mathcal{M} =$[parton Ampl.] $\otimes$ [GPDs]



universality: the same GPDs enter a variety of exclusive reactions

DVCS at leading-order



$$H^{\mu\nu} = i\int d^4z e^{-iq\cdot z}\langle p'|T[J^\mu(0)J^\nu(z)]|p\rangle = i\int d^4z e^{iq'\cdot z}\langle p'|T[J^\mu(z)J^\nu(0)]|p\rangle$$

In the Bjorken limit and at Leading Order (LO) in 1/Q:

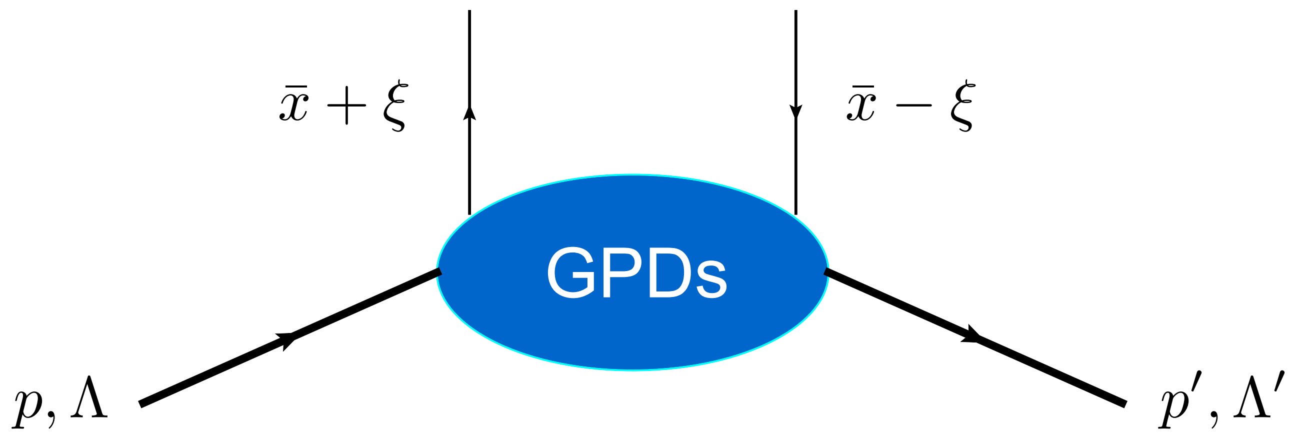
$$H^{\mu\nu}_{LO} \quad = \quad -g_T^{\mu\nu}\mathcal{F} - i\varepsilon_T^{\mu\nu}\widetilde{\mathcal{F}}$$

$$\mathcal{F} = \sum_q e_q^2 \int_{-1}^{+1} d\bar{x}\, C^+(\bar{x},\xi)\, \Phi^{q,\gamma^+}(\bar{x},\xi,t) \qquad \widetilde{\mathcal{F}} = \sum_q e_q^2 \int_{-1}^{+1} d\bar{x}\, C^-(\bar{x},\xi)\, \Phi^{q,\gamma^+\gamma_5}(\bar{x},\xi,t)$$

GPD correlator: $\Phi^{q,[\Gamma]}(\bar{x},\xi,t) = \langle p',\Lambda'|\int \frac{dz^-}{4\pi} e^{i\bar{x}P^+z^-}\bar{\psi}\left(-\frac{z}{2}\right)\Gamma\psi\left(\frac{z}{2}\right)|p,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$

Hard scattering coefficients: $C^{\pm}(\bar{x},\xi) = \frac{1}{\bar{x}-\xi+i\epsilon} \pm \frac{1}{\bar{x}+\xi-i\epsilon}$

Leading-Twist GPDs



$$\Phi^{[\Gamma]}(\bar{x},\xi,t) = \langle p',\Lambda'| \int \frac{dz^-}{4\pi} e^{i\bar{x}P^+z^-} \bar{\psi}\left(-\frac{z}{2}\right) \Gamma \psi\left(\frac{z}{2}\right) |p,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$$

$$\Gamma = \begin{cases} \gamma^+ & H^q,\ E^q & \text{unpol.} \\ \gamma^+\gamma^5 & \tilde{H}^q,\ \tilde{E}^q & \text{long. pol.} \\ i\sigma^{+i}\gamma^5 & H_T^q,\ E_T^q,\ \tilde{H}_T^q,\ \tilde{E}_T^q & \text{transv. pol.} \end{cases}$$

- $p \neq p' \Rightarrow$ GPDs depend on two momentum fractions $\bar{x}$, ξ, and t

$$\bar{x} = \frac{(k+k')^+}{(p+p')^+} = \frac{\bar{k}^+}{P^+}$$

average fraction of the longitudinal momentum carried by partons

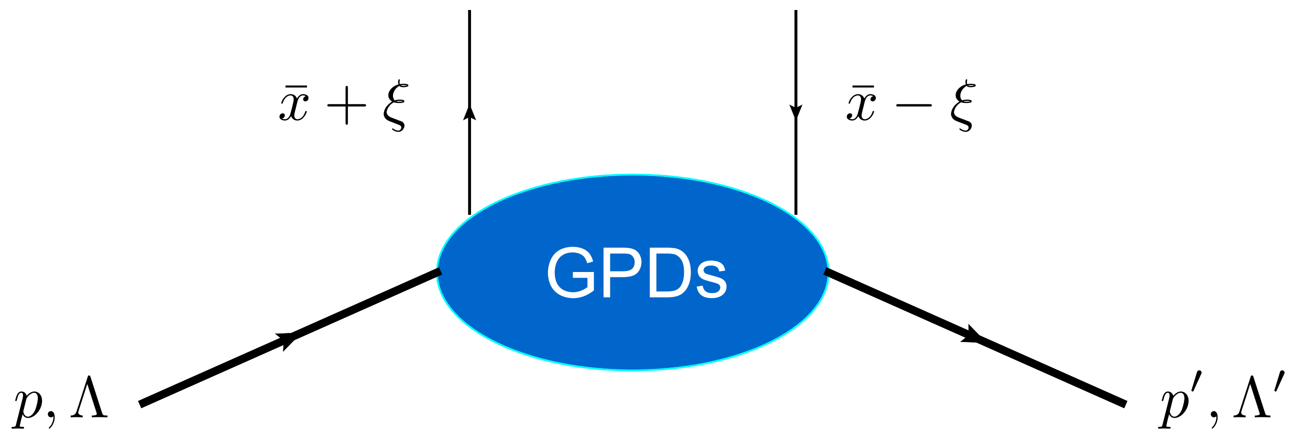
$$\xi = \frac{(p-p')^+}{(p+p')^+} = -\frac{\Delta^+}{2P^+}$$

skewness parameter: fraction of longitudinal momentum transfer

$$t = (p-p')^2 \equiv \Delta^2$$

t-channel momentum transfer squared

Leading-Twist GPDs



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- DVCS amplitudes involve Compton Form Factors:

$$\mathcal{F} = \sum_q e_q^2 \int_{-1}^{+1} d\bar{x} C^+(\bar{x},\xi)\, \Phi^{q,\gamma^+}(\bar{x},\xi,t) \longrightarrow \mathcal{H} = \int_{-1}^{+1} d\bar{x} \frac{H(\bar{x},\xi,t)}{\bar{x}-\xi+i\epsilon}$$

$$\mathrm{Re}\mathcal{H} \overset{\mathrm{LO}}{=} \mathcal{P}\int_{-1}^{1} \mathrm{d}\bar{x} \frac{H(\bar{x},\xi,t)}{\bar{x}-\xi} \qquad \mathrm{Im}\mathcal{H} \overset{\mathrm{LO}}{=} H(\xi,\xi,t)$$

Need of a gauge link

$$\Phi^{[\Gamma]}(\bar{x},\xi,t) = \langle p',\Lambda'| \int \frac{dz^-}{4\pi} e^{i\bar{x}P^+z^-}\, \bar{\psi}\left(-\frac{z}{2}\right)\Gamma\psi\left(\frac{z}{2}\right)|p,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$$

not invariant under $\psi(z) \to e^{i\alpha(z)}\psi(z)$

$$\Downarrow$$

$$\Phi^{[\Gamma]}(\bar{x},\xi,t) = \langle p',\Lambda'| \int \frac{dz^-}{4\pi} e^{i\bar{x}P^+z^-}\, \bar{\psi}\left(-\frac{z}{2}\right)\Gamma\,\mathcal{U}_{[-\frac{z}{2},\frac{z}{2}]}\psi\left(\frac{z}{2}\right)|p,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$$

$$\mathcal{U}(-\tfrac{z}{2},\tfrac{z}{2}) \to e^{i\alpha(-\frac{z}{2})}\,\mathcal{U}(-\tfrac{z}{2},\tfrac{z}{2})\,e^{-i\alpha(\frac{z}{2})}$$

Need of a gauge link

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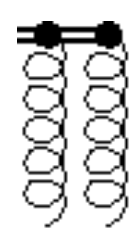
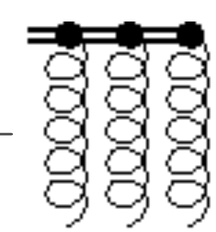
$$\mathcal{U}(-\tfrac{z}{2},\tfrac{z}{2}) \to e^{i\alpha(-\frac{z}{2})}\,\mathcal{U}(-\tfrac{z}{2},\tfrac{z}{2})\,e^{-i\alpha(\frac{z}{2})}$$

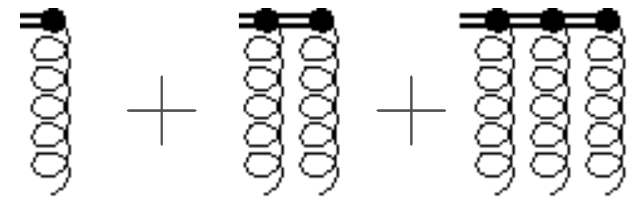
$$\mathcal{U}_{[-\frac{z}{2}\ \frac{z}{2}]} = \mathcal{P}\exp\left[-ig\int_{-\frac{z}{2}}^{\frac{z}{2}} \mathrm{d}\eta^\mu\, A_\mu(\eta)\right] = \sum_{n=0}^{\infty}\frac{1}{n!}(-ig)^n\mathcal{P}\int_{-\frac{z}{2}}^{\frac{z}{2}} \mathrm{d}\eta_n^{\mu_n}\ldots\mathrm{d}\eta_1^{\mu_1}A_{\mu_n}(\eta_n)\ldots A_{\mu_1}(z_1)$$

(1 +  +  +  + ...

Wilson line definition for GPDs

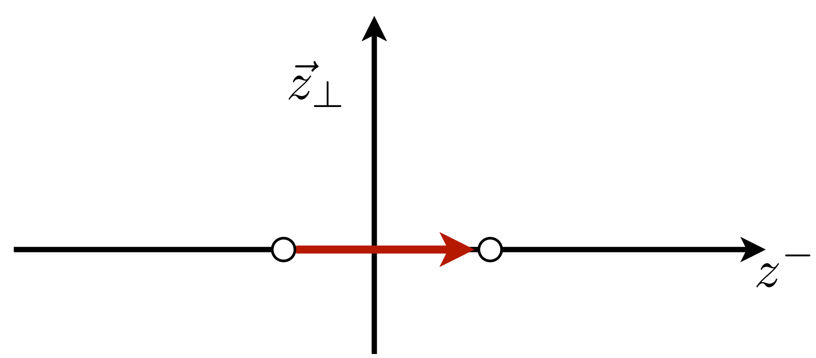
$$\Phi^{[\Gamma]}(\bar{x},\xi,t) = \langle p',\Lambda'| \int \frac{dz^-}{4\pi} e^{i\bar{x}P^+z^-} \bar{\psi}\left(-\frac{z}{2}\right)\Gamma\mathcal{U}_{[-\frac{z}{2},\frac{z}{2}]}\psi\left(\frac{z}{2}\right)|p,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$$

$$U_{[a,b]} = \mathcal{P}\exp\left[-ig\int_a^b d\eta^\mu A_\mu(\eta)\right] = (1 + \ldots$$



exchange of more than 2 partons
between hard scattering process (H) and soft amplitude (A)
is suppressed except for gluons with polarization A^+

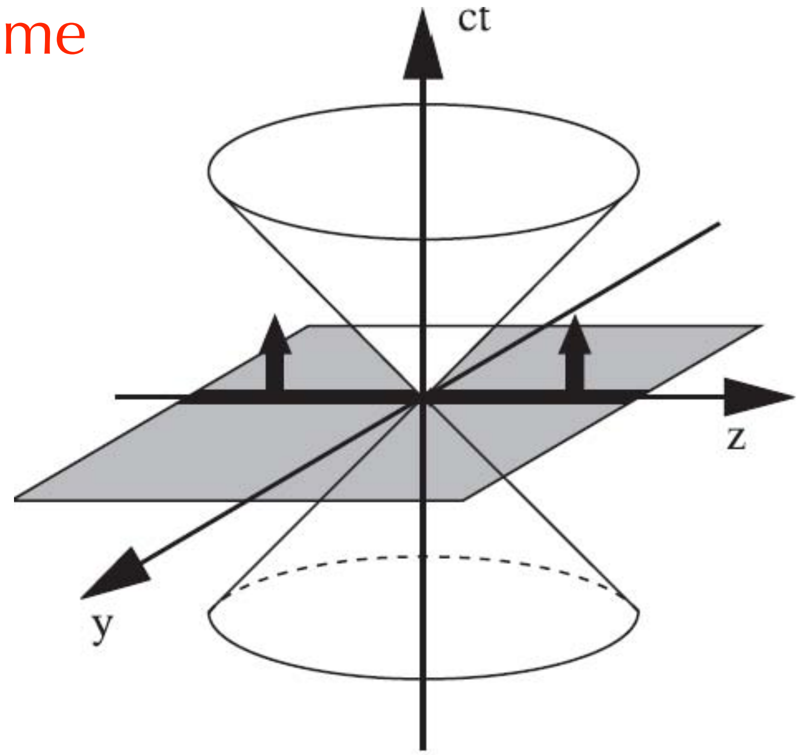
$$\mathcal{U}_{[-\frac{z}{2}\ \frac{z}{2}]} = \mathcal{P}\exp\left[-ig\int_{-\frac{z}{2}}^{\frac{z}{2}} \mathrm{d}\eta^-\ A^+(\eta)\right]_{z^+=0,z_\perp=0}$$



for convenience, choose light-cone gauge: $A^+ = 0$ in which $U = 1$

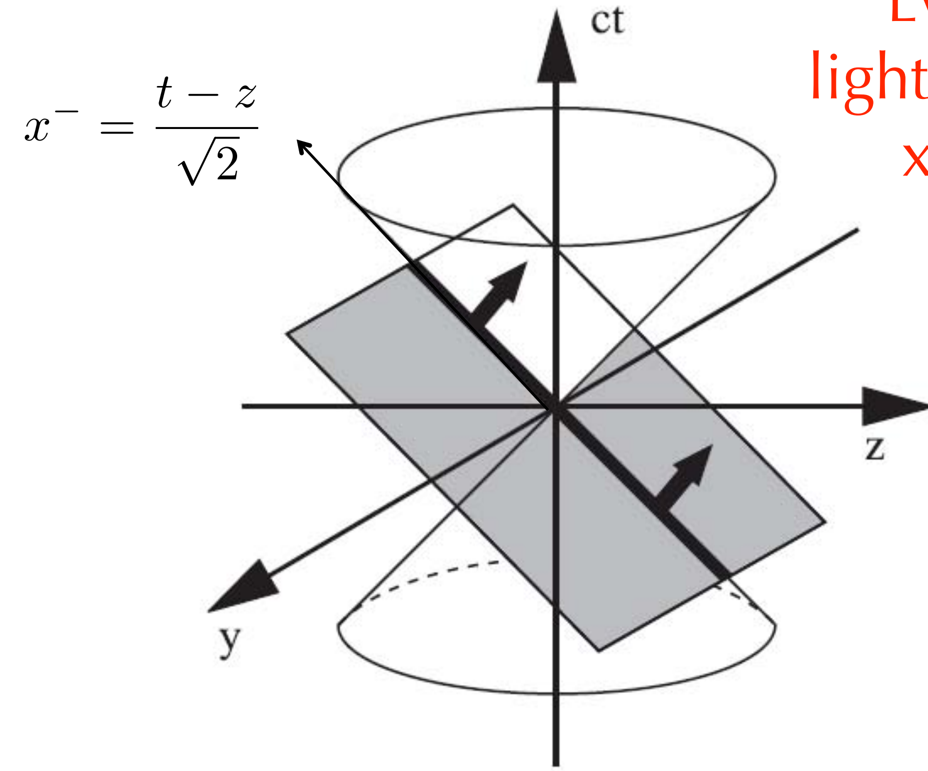
Partonic interpretation

Evolve in
ordinary time



Instant Form

Evolve in
light-front time
$x^+ = t+z$



Light-Front Form

coordinates

x^0	time	$x^+ = \dfrac{x^0 + x^3}{\sqrt{2}}$	time
x^1, x^2, x^3	space	$x^- = \dfrac{x^0 - x^3}{\sqrt{2}}, \; \vec{x}_\perp = (x^1, x^2)$	space

Hamiltonian

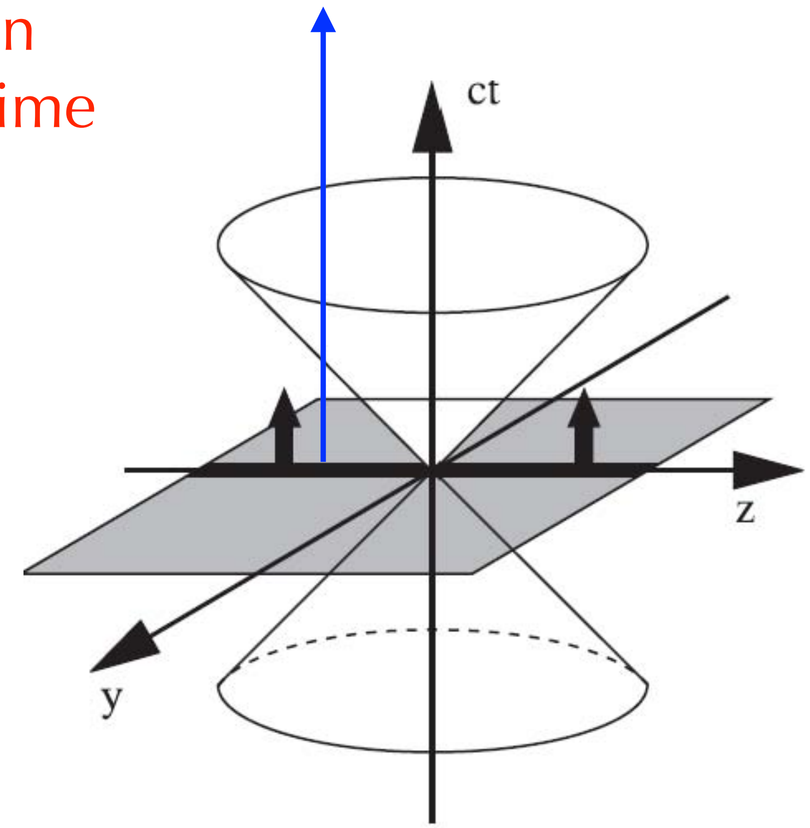
$$H = \sqrt{P^2 + M_0^2} + V \qquad\qquad P^- = \frac{\vec{P}_\perp^2 + M_0^2}{P^+} + V$$

generators of Poincaré group interaction independent

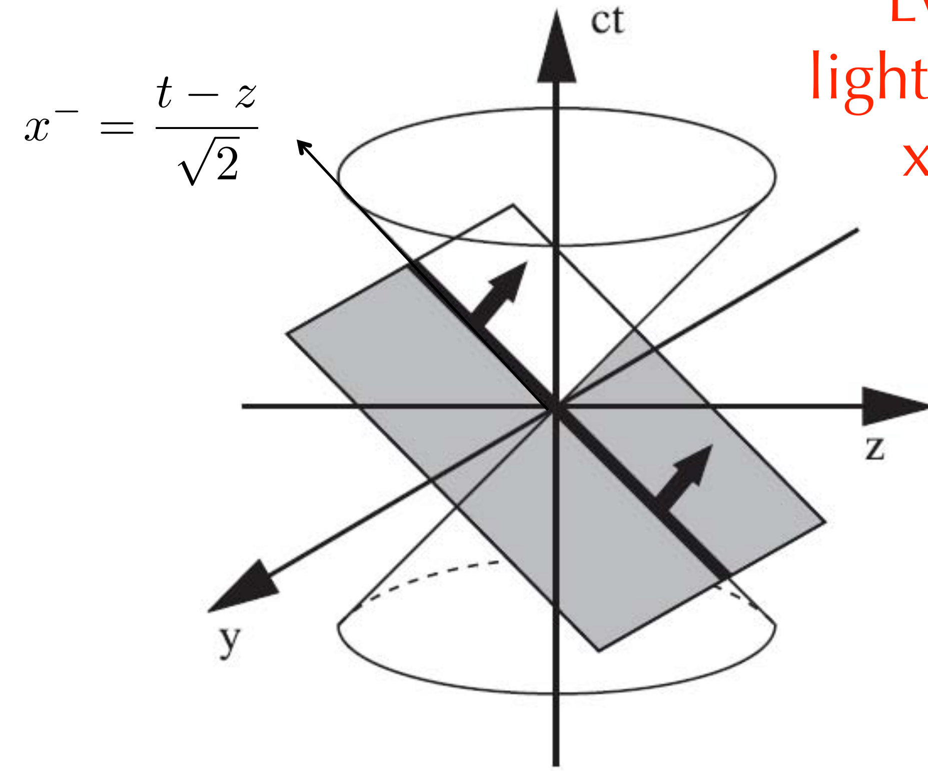
6 7

P.A.M. Dirac, Rev. Mod. Phys. 21, 392 (1949)

Evolve in ordinary time



Evolve in light-front time $x^{+} = t+z$



Instant Form

Light-Front Form

coordinates

Instant Form		Light-Front Form	
x^0	time	$x^{+} = \dfrac{x^0 + x^3}{\sqrt{2}}$	time
x^1, x^2, x^3	space	$x^{-} = \dfrac{x^0 - x^3}{\sqrt{2}}, \; \vec{x}_\perp = (x^1, x^2)$	space

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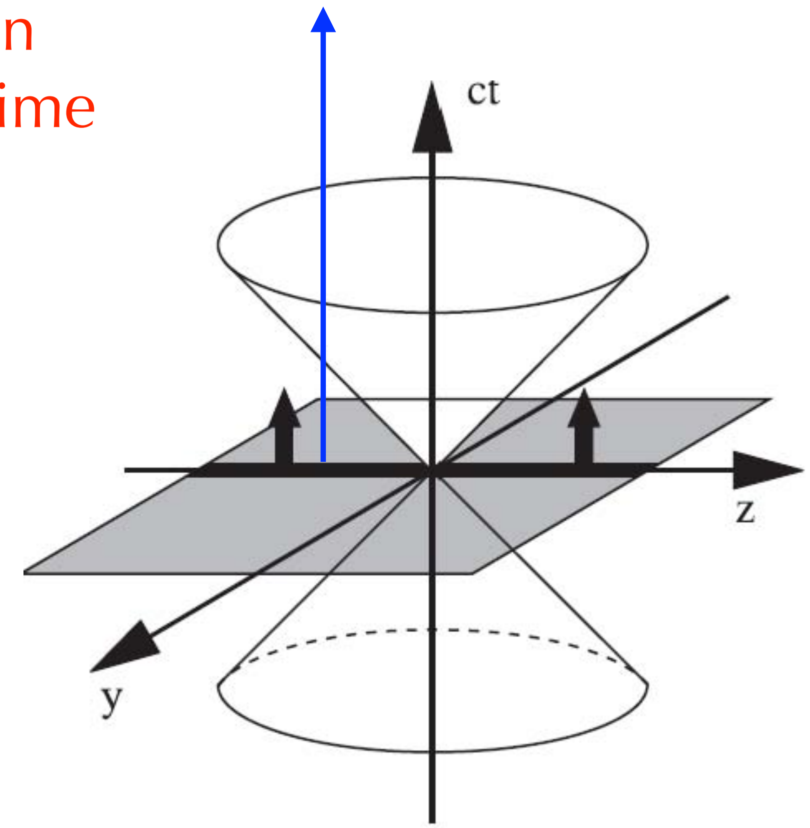
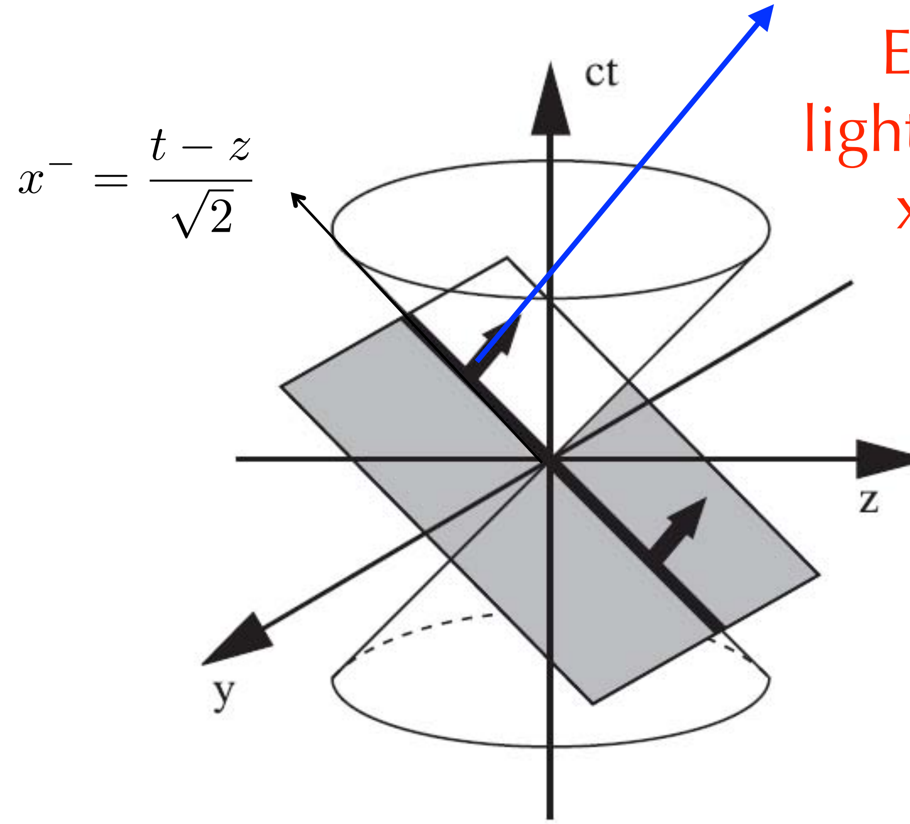


Instant Form

Evolve in
light-front time
x⁺ = t+z



Light-Front Form

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generators of Poincaré group interaction independent

P.A.M. Dirac, Rev. Mod. Phys. 21, 392 (1949)

Good and bad components

- Decompose the four-component fermion field in bad (-) and good (+) components

$$\psi = \psi^+ + \psi^- \qquad \text{with } \psi^+ = P_+\psi \text{ and } \psi^- = P_-\psi$$

- Properties of projector operators: $P_+ = \frac{1}{2}\gamma^-\gamma^+ \quad P_- = \frac{1}{2}\gamma^+\gamma^-$

$$P_+ + P_- = I \qquad (P_+)^2 = P_+ \quad (P_-)^2 = P_- \qquad P_+P_- = P_-P_+ = 0$$

- Projecting the Dirac equation and using the light-cone gauge $A^+ = 0$

$$i\gamma^- \frac{\partial}{\partial x^-}\psi_- = -\vec{\gamma}_\perp \cdot \vec{D}_\perp \psi_+ + m\psi_+ \qquad\qquad i\gamma^+ D_+ \psi_+ = -\vec{\gamma}_\perp \cdot \vec{D}_\perp \psi_- + m\psi_-$$

constrained field — independent dynamical degree of freedom

R. Jaffe, hep-ph/9602236

Partonic interpretation of GPDs

- Unpolarized GPDs

$$\Phi^{[\gamma^+]}(\bar{x},\xi,t) = \langle P',\Lambda'|\int \tfrac{\mathrm{d}z^-}{4\pi} e^{i\bar{x}P^+z^-}\,\bar{\psi}(-\tfrac{z}{2})\,\gamma^+\,\psi(\tfrac{z}{2})|P,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$$

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$$\Longrightarrow\ \bar{\psi}(-\frac{z}{2})\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\gamma^0\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\sqrt{2}P_+\psi(\frac{z}{2})$$

Partonic interpretation of GPDs

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$$\Longrightarrow\ \bar{\psi}(-\frac{z}{2})\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\gamma^0\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\sqrt{2}P_+\psi(\frac{z}{2})$$

$$= \sqrt{2}\psi^\dagger(-\frac{z}{2})P_+P_+\psi(\frac{z}{2}) = \sqrt{2}\psi^\dagger_+(-\frac{z}{2})\psi_+(\frac{z}{2})$$

Partonic interpretation of GPDs

- Unpolarized GPDs

$$\Phi^{[\gamma^+]}(\bar{x},\xi,t) = \langle P',\Lambda'|\int \frac{\mathrm{d}z^-}{4\pi} e^{i\bar{x}P^+z^-}\,\bar{\psi}(-\tfrac{z}{2})\,\gamma^+\,\psi(\tfrac{z}{2})|P,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$$

$$\Longrightarrow\ \bar{\psi}(-\frac{z}{2})\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\gamma^0\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\sqrt{2}P_+\psi(\frac{z}{2})$$

$$= \sqrt{2}\psi^\dagger(-\frac{z}{2})P_+P_+\psi(\frac{z}{2}) = \sqrt{2}\psi_+^\dagger(-\frac{z}{2})\psi_+(\frac{z}{2})$$ good components of the quark fields

Partonic interpretation of GPDs

- Unpolarized GPDs

$$\Phi^{[\gamma^+]}(\bar{x},\xi,t) = \langle P',\Lambda'| \int \frac{\mathrm{d}z^-}{4\pi} e^{i\bar{x}P^+z^-} \bar{\psi}(-\tfrac{z}{2})\, \gamma^+\, \psi(\tfrac{z}{2})|P,\Lambda\rangle|_{z^+=0,\vec{z}_\perp=0}$$

$$\Longrightarrow \bar{\psi}(-\frac{z}{2})\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\gamma^0\gamma^+\psi(\frac{z}{2}) = \psi^\dagger(-\frac{z}{2})\sqrt{2}P_+\psi(\frac{z}{2})$$

$$= \sqrt{2}\psi^\dagger(-\frac{z}{2})P_+P_+\psi(\frac{z}{2}) = \sqrt{2}\psi_+^\dagger(-\frac{z}{2})\psi_+(\frac{z}{2})$$ good components of the quark fields

$$\Longrightarrow \psi_+(z^-,\vec{z}_\perp) = \int \frac{dk^+}{2k^+}\frac{d\vec{k}_\perp}{(2\pi)^3}\theta(k^+)\sum_\mu [b_q(w)u_+(w)\exp[-ik^+z^- + i\vec{k}_\perp\cdot\vec{z}_\perp]$$

$$+d_q^\dagger(w)v_+(w)\exp[ik^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp]$$ with $w = (k^+,\vec{k}_\perp,\mu)$

$b_q, b_q^\dagger$ annihilation and creation operator of quark

$d_q, d_q^\dagger$ annihilation and creation operator of antiquark

❖Homework: derive the operator structure in the different regions using positivity condition $k^{'+}, k^{+} > 0$ and momentum conservation $k^{+} - k^{'+} = p^{+} - p^{'+} = 2\xi P^{+}$

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$b,\ b^{\dagger}$ quarks

$d,\ d^{\dagger}$ antiquarks

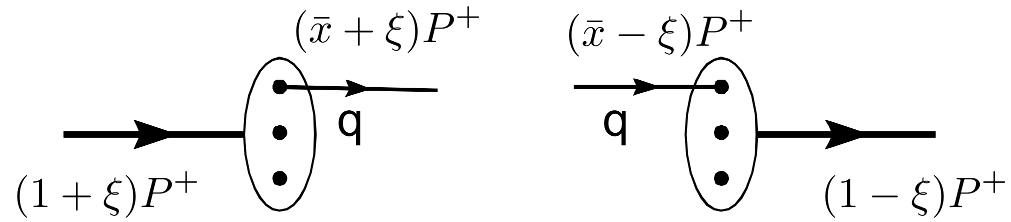
non-diagonal matrix elements of momentum-density matrix

⇓

we loose the probabilistic interpretation of the PDF

we gain information on the quark-momentum correlation

DGLAP region $\xi \leq \bar{x} \leq 1$



$$\langle N, (1-\xi)P^{+} | b^{\dagger}_{\lambda'}[(\bar{x}-\xi)P^{+}] b_{\lambda}[(\bar{x}+\xi)P^{+}] | N, (1+\xi)P^{+} \rangle$$

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⇩

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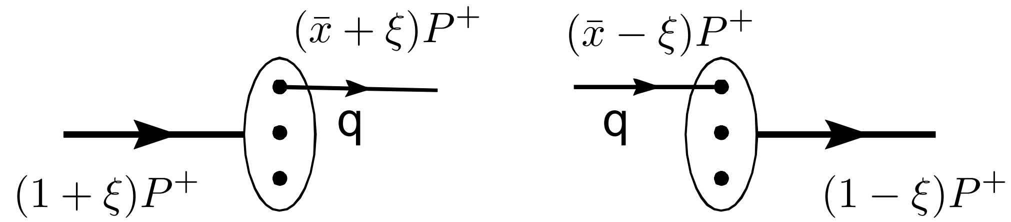
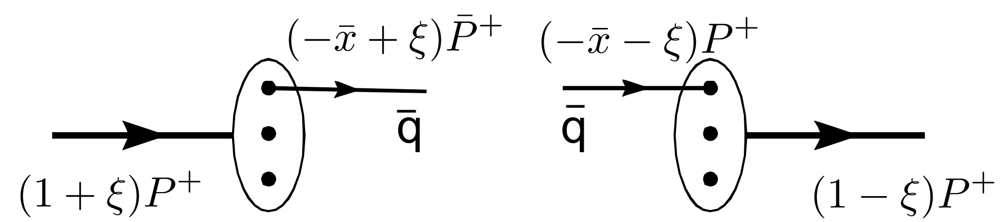
DGLAP region $\xi \leq \bar{x} \leq 1$



$$\langle N, (1-\xi)P^{+} | b^{\dagger}_{\lambda'}[(\bar{x}-\xi)P^{+}] b_{\lambda}[(\bar{x}+\xi)P^{+}] | N, (1+\xi)P^{+} \rangle$$

DGLAP region $-1 \leq \bar{x} \leq -\xi$



$$\langle N, (1-\xi)P^{+} | d^{\dagger}_{\lambda'}[(-\bar{x}-\xi)P^{+}] d_{\lambda}[(-\bar{x}+\xi)P^{+}] | N, (1+\xi)P^{+} \rangle$$

❖Homework: derive the operator structure in the different regions using positivity condition $k^{'+}, k^{+} > 0$ and momentum conservation $k^{+} - k^{'+} = p^{+} - p^{'+} = 2\xi P^{+}$

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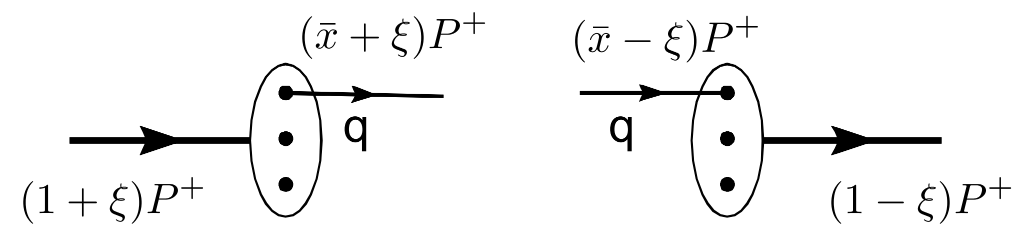
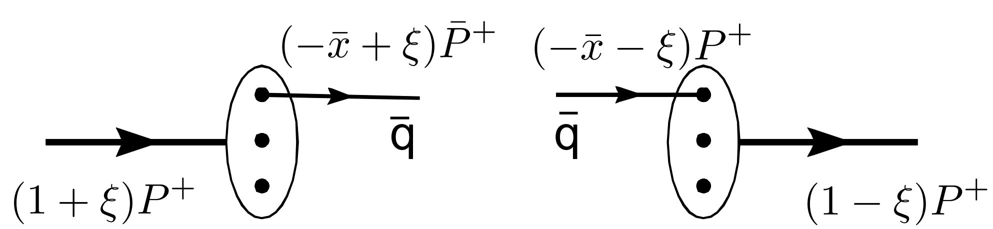
DGLAP region $\xi \le \bar{x} \le 1$



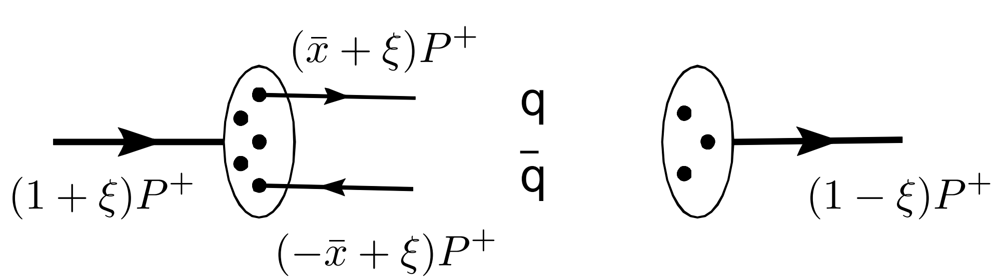
$$\langle N, (1-\xi)P^{+} | b^{\dagger}_{\lambda'}[(\bar{x}-\xi)P^{+}] b_{\lambda}[(\bar{x}+\xi)P^{+}] | N, (1+\xi)P^{+} \rangle$$

DGLAP region $-1 \le \bar{x} \le -\xi$



$$\langle N, (1-\xi)P^{+} | d^{\dagger}_{\lambda'}[(-\bar{x}-\xi)P^{+}] d_{\lambda}[(-\bar{x}+\xi)P^{+}] | N, (1+\xi)P^{+} \rangle$$

ERBL region $-\xi \le \bar{x} \le \xi$



$$\langle N, (1-\xi)P^{+} | b_{\lambda'}[(\bar{x}+\xi)P^{+}] d_{\lambda}[(-\bar{x}+\xi)P^{+}] | N, (1+\xi)P^{+} \rangle$$

❖Homework: derive the operator structure in the different regions using positivity condition $k'^+, k^+ > 0$ and momentum conservation $k^+ - k'^+ = p^+ - p'^+ = 2\xi P^+$

$b,\, b^\dagger$ quarks

$d,\, d^\dagger$ antiquarks

non-diagonal matrix elements of momentum-density matrix

⇓

we loose the probabilistic interpretation of the PDF

we gain information on the quark-momentum correlation

Spin projection

helicity space

$b^\dagger_\uparrow b_\uparrow + b^\dagger_\downarrow b_\downarrow$

$H^q,\, E^q$

$b^\dagger_\uparrow b_\uparrow - b^\dagger_\downarrow b_\downarrow$

$\tilde{H}^q,\, \tilde{E}^q$

transverse-spin space

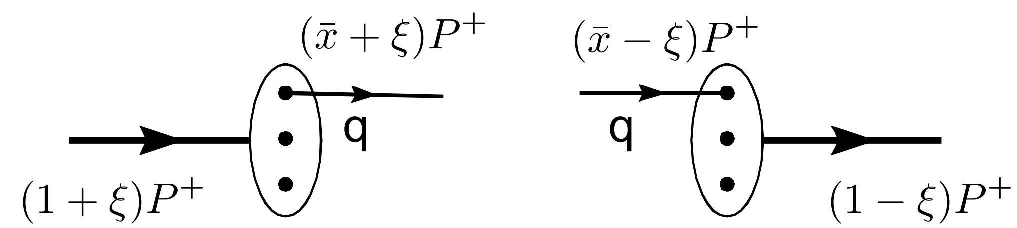
$b^\dagger_\rightarrow b_\rightarrow + b^\dagger_\leftarrow b_\leftarrow$

$H^q_T,\, E^q_T$

$b^\dagger_\rightarrow b_\leftarrow - b^\dagger_\leftarrow b_\rightarrow$

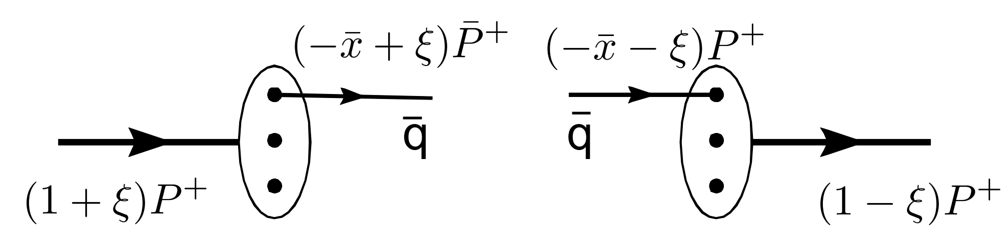
$\tilde{H}^q_T,\, \tilde{E}^q_T$

DGLAP region $\xi \leq \bar{x} \leq 1$



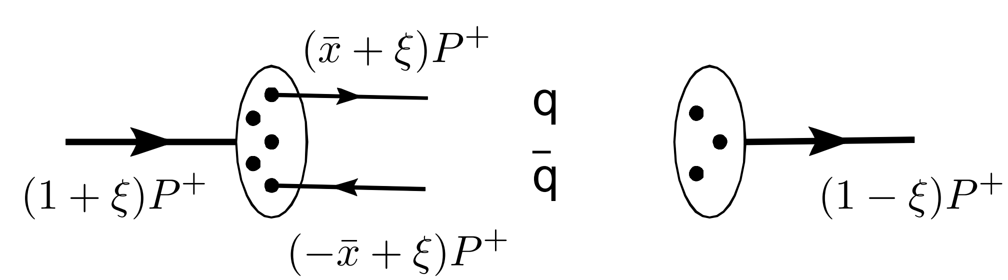
$$\langle N, (1-\xi)P^+ | b^\dagger_{\lambda'}[(\bar{x}-\xi)P^+] b_\lambda[(\bar{x}+\xi)P^+] | N, (1+\xi)P^+\rangle$$

DGLAP region $-1 \leq \bar{x} \leq -\xi$

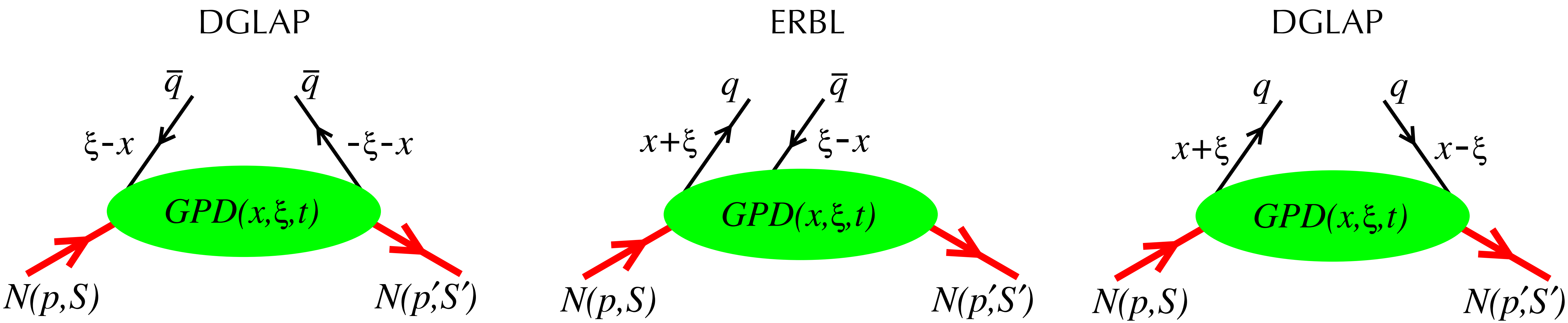


$$\langle N, (1-\xi)P^+ | d^\dagger_{\lambda'}[(-\bar{x}-\xi)P^+] d_\lambda[(-\bar{x}+\xi)P^+] | N, (1+\xi)P^+\rangle$$

ERBL region $-\xi \leq \bar{x} \leq \xi$



$$\langle N, (1-\xi)P^+ | b_{\lambda'}[(\bar{x}+\xi)P^+] d_\lambda[(-\bar{x}+\xi)P^+] | N, (1+\xi)P^+\rangle$$



$-1 \leq x \leq -\xi$ $\qquad$ $-\xi \leq x \leq \xi$ $\qquad$ $\xi \leq x \leq 1$

Time reversal invariance: $H(x,\xi,t) = H(x,-\xi,t)$ (similarly for E, $\widetilde{H}$, $\widetilde{E}$, H_T, E_T, $\widetilde{H}_T$)

$$\widetilde{E}_T(x,\xi,t) = -\widetilde{E}_T(x-\xi,t)$$

Hermiticity: $[H(x,\xi,t)]^* = H(x,-\xi,t)$ (similarly for E, $\widetilde{H}$, $\widetilde{E}$, H_T, E_T, $\widetilde{H}_T$)

$$[\widetilde{E}_T(x,\xi,t)]^* = -\widetilde{E}_T(x-\xi,t)$$

$\Downarrow$

GPDs are real valued functions

Quark polarization

Nucleon pol.

	U	T_x	T_y	L
U	$\mathcal{H}$	$i\,\frac{\Delta_y}{2M}\,\mathcal{E}_T$	$-i\,\frac{\Delta_x}{2M}\,\mathcal{E}_T$	
T_x	$i\,\frac{\Delta_y}{2M}\,\mathcal{E}$	$\mathcal{H}_T + \frac{\Delta_x^2-\Delta_y^2}{2M^2}\,\tilde{\mathcal{H}}_T$	$\frac{\Delta_x\Delta_y}{M^2}\,\tilde{\mathcal{H}}_T$	$\frac{\Delta_x}{2M}\tilde{\mathcal{E}}$
T_y	$-i\,\frac{\Delta_x}{2M}\,\mathcal{E}$	$\frac{\Delta_x\Delta_y}{M^2}\,\tilde{\mathcal{H}}_T$	$\mathcal{H}_T - \frac{\Delta_x^2-\Delta_y^2}{2M^2}\,\tilde{\mathcal{H}}_T$	$\frac{\Delta_y}{2M}\tilde{\mathcal{E}}$
L		$\frac{\Delta_x}{2M}\tilde{\mathcal{E}}_T$	$\frac{\Delta_y}{2M}\tilde{\mathcal{E}}_T$	$\tilde{\mathcal{H}}$

ξ-odd

$$\mathcal{H} = \sqrt{1-\xi^2}\left(H - \frac{\xi^2}{1-\xi^2}\,E\right) \qquad \mathcal{H}_T = \sqrt{1-\xi^2}\left(H_T - \frac{\vec{\Delta}_\perp^2}{2M^2}\,\frac{\tilde{\mathcal{H}}_T}{\sqrt{1-\xi^2}} + \frac{\xi\,\tilde{\mathcal{E}}_T}{\sqrt{1-\xi^2}}\right)$$

$$\mathcal{E} = \frac{E}{\sqrt{1-\xi^2}} \qquad \mathcal{E}_T = \frac{2\tilde{H}_T + E_T - \xi\,\tilde{E}_T}{\sqrt{1-\xi^2}}$$

$$\tilde{\mathcal{H}} = \sqrt{1-\xi^2}\left(\tilde{H} - \frac{\xi^2}{1-\xi^2}\,\tilde{E}\right) \qquad \tilde{\mathcal{H}}_T = -\frac{\tilde{H}_T}{2\sqrt{1-\xi^2}}$$

$$\tilde{\mathcal{E}} = \frac{\xi\tilde{E}}{\sqrt{1-\xi^2}} \qquad \tilde{\mathcal{E}}_T = \frac{\tilde{E}_T - \xi\,E_T}{\sqrt{1-\xi^2}}$$

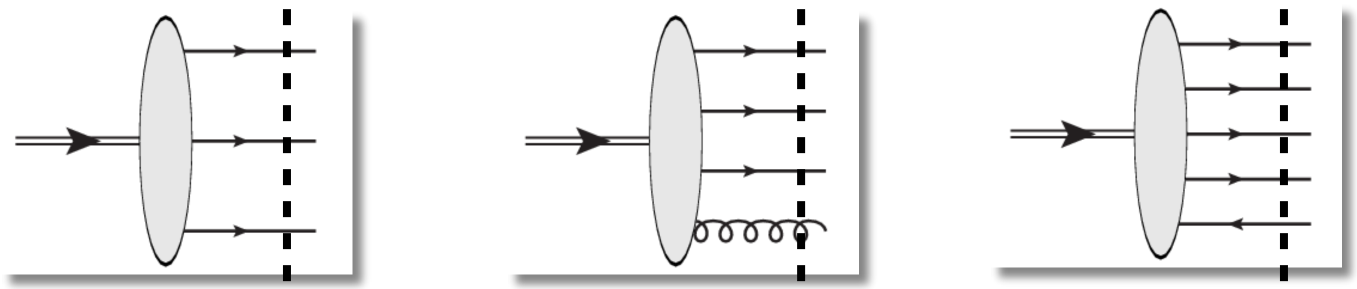
✦ Helicity structure: nucleon (Λ', Λ) quark (λ', λ)

$$U = (++) + (--) \qquad T_x = (-+) + (+-) \qquad T_y = i[(-+) - (+-)] \qquad L = (++) - (--)$$

Light-front Wave Function Overlap Representation of GPDs

Light-cone Fock expansion

$$|P\rangle = \Psi_{qqq}\,|qqq\rangle + \Psi_{qqqg}\,|qqqg\rangle + \Psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle + \ldots$$

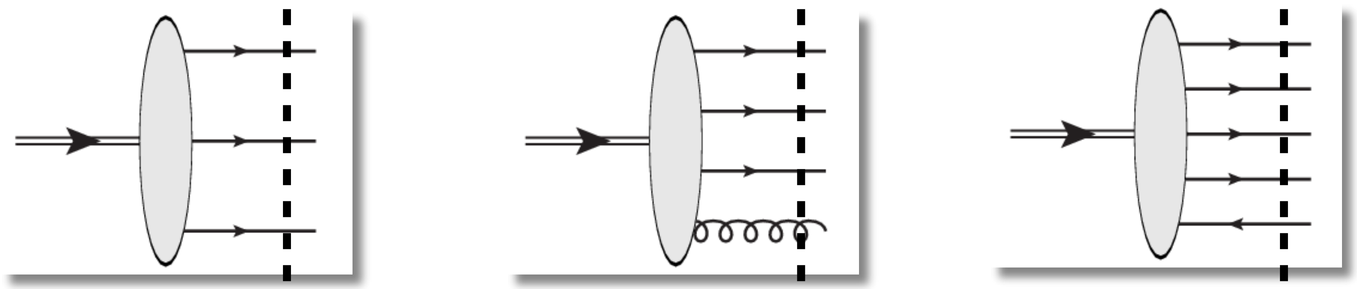


Fixed light-cone time $x^{+}=0$

Brodsky, Pauli, Phys. Rept. 301 (1998) 299

Light-cone Fock expansion

$$|P\rangle \;=\; \Psi_{qqq}\,|qqq\rangle \;+\; \Psi_{qqqg}\,|qqqg\rangle \;+\; \Psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle \;+\; \ldots$$



Fixed light-cone time $x^+=0$

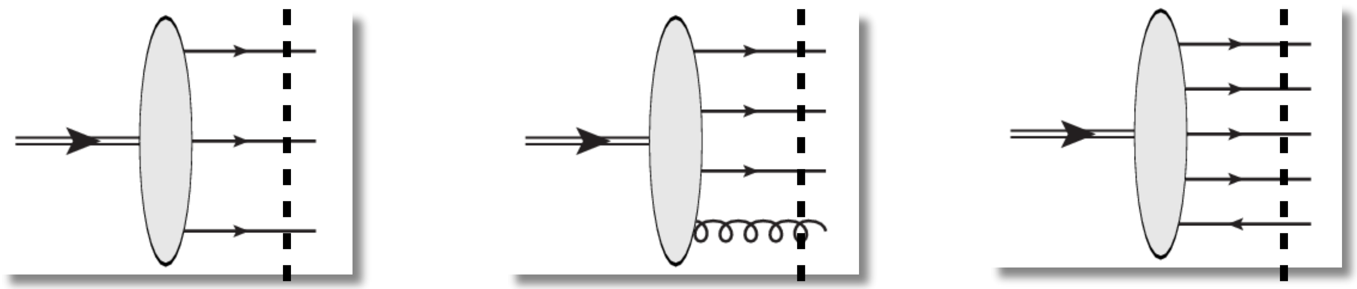
Light-front wave functions: Probability Amplitude for the N,β Fock state

$$|(P^+, \vec{P}_\perp), \Lambda\rangle \;=\; \sum_{N,\beta} [dx]_N [d\vec{k}_\perp]_N \Psi^{\Lambda}_{N,\beta}(x_i, \vec{k}_{\perp\, i}) |N, \beta; (x_i P^+, x_i \vec{P}_\perp + \vec{k}_{\perp\, i}), \lambda_i\rangle$$

Brodsky, Pauli, Phys. Rept. 301 (1998) 299

Light-cone Fock expansion

$$|P\rangle \ = \ \Psi_{qqq}\,|qqq\rangle \ + \ \Psi_{qqqg}\,|qqqg\rangle \ + \ \Psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle \ + \ \ldots$$



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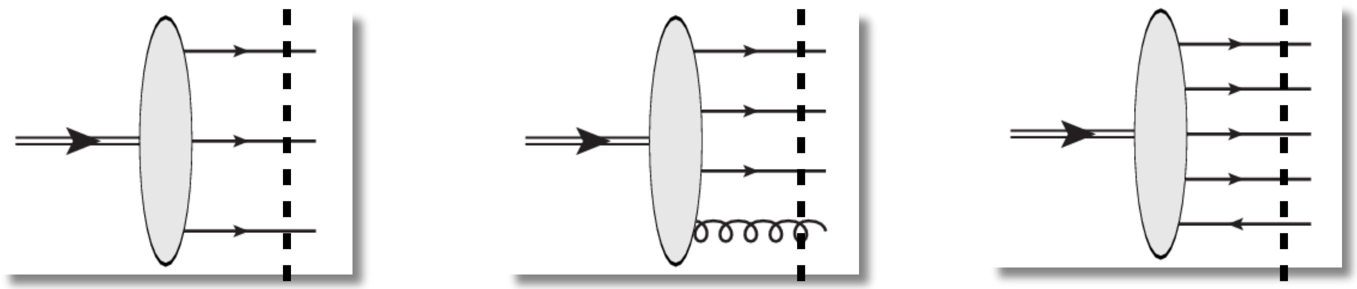
Internal variables: $x_i = \dfrac{p_i^+}{P^+}$ $\qquad \sum_{i=1}^{N} x_i = 1 \qquad \sum_{i=1}^{N} \vec{k}_{i\perp} = \vec{0}_\perp$

Frame Independent

Brodsky, Pauli, Phys. Rept. 301 (1998) 299

Light-cone Fock expansion

$$|P\rangle \;=\; \Psi_{qqq}\,|qqq\rangle \;+\; \Psi_{qqqg}\,|qqqg\rangle \;+\; \Psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle \;+\; \ldots$$



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Internal variables: $x_i = \dfrac{p_i^+}{P^+}$ $\qquad \displaystyle\sum_{i=1}^{N} x_i = 1 \qquad \sum_{i=1}^{N}\vec{k}_{i\perp} = \vec{0}_\perp$

Frame Independent

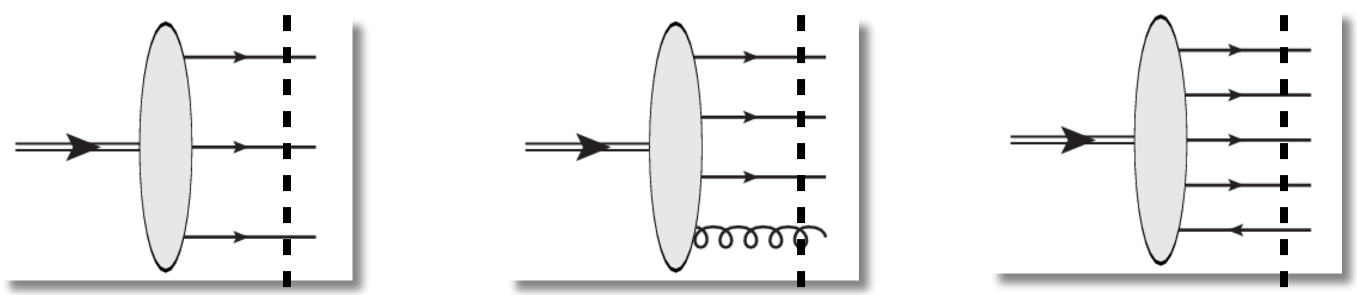
Eigenstates of parton light-front helicity

$$\hat{S}_{iz}\Psi^\Lambda_{\lambda_1\ldots\lambda_N} = \lambda_i\Psi^\Lambda_{\lambda_1\ldots\lambda_N}$$

Brodsky, Pauli, Phys. Rept. 301 (1998) 299

Light-cone Fock expansion

$$|P\rangle \;=\; \Psi_{qqq}\,|qqq\rangle \;+\; \Psi_{qqqg}\,|qqqg\rangle \;+\; \Psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle \;+\;\ldots$$



Fixed light-cone time $x^+=0$

Light-front wave functions: Probability Amplitude for the N,β Fock state

$$|(P^+,\vec{P}_\perp),\Lambda\rangle \;=\; \sum_{N,\beta}[dx]_N[d\vec{k}_\perp]_N\Psi^{\Lambda}_{N,\beta}(x_i,\vec{k}_{\perp\,i})|N,\beta;(x_iP^+,x_i\vec{P}_\perp+\vec{k}_{\perp\,i}),\lambda_i\rangle$$

Internal variables: $x_i=\dfrac{p_i^+}{P^+}$ $\quad\sum_{i=1}^{N}x_i=1\quad$ $\sum_{i=1}^{N}\vec{k}_{i\perp}=\vec{0}_\perp$

Frame Independent

Eigenstates of parton light-front helicity

$$\hat{S}_{iz}\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N}=\lambda_i\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N}$$

Eigenstates of total OAM

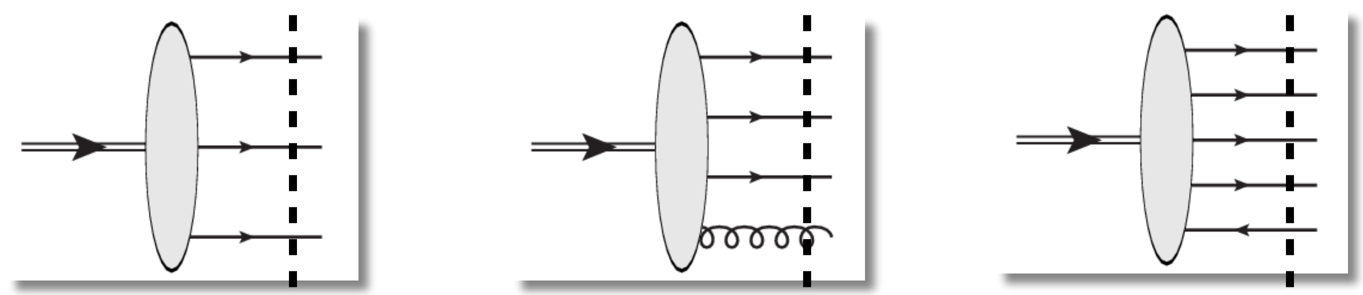
$$\hat{L}_z\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N}=\ell_z\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N}$$

⚠ $A^+=0$ **gauge**

Brodsky, Pauli, Phys. Rept. 301 (1998) 299

Light-cone Fock expansion

$$|P\rangle \;=\; \Psi_{qqq}\,|qqq\rangle \;+\; \Psi_{qqqg}\,|qqqg\rangle \;+\; \Psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle \;+\; \ldots$$



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Internal variables: $x_i = \dfrac{p_i^+}{P^+}$ $\quad\sum_{i=1}^{N} x_i = 1$ $\quad\sum_{i=1}^{N}\vec{k}_{i\perp} = \vec{0}_\perp$

Frame Independent

Eigenstates of parton light-front helicity

$$\hat{S}_{iz}\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N} = \lambda_i\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N}$$

$$\boxed{\Lambda = \sum_{i=1}^{N}\lambda_i + \ell_z}$$

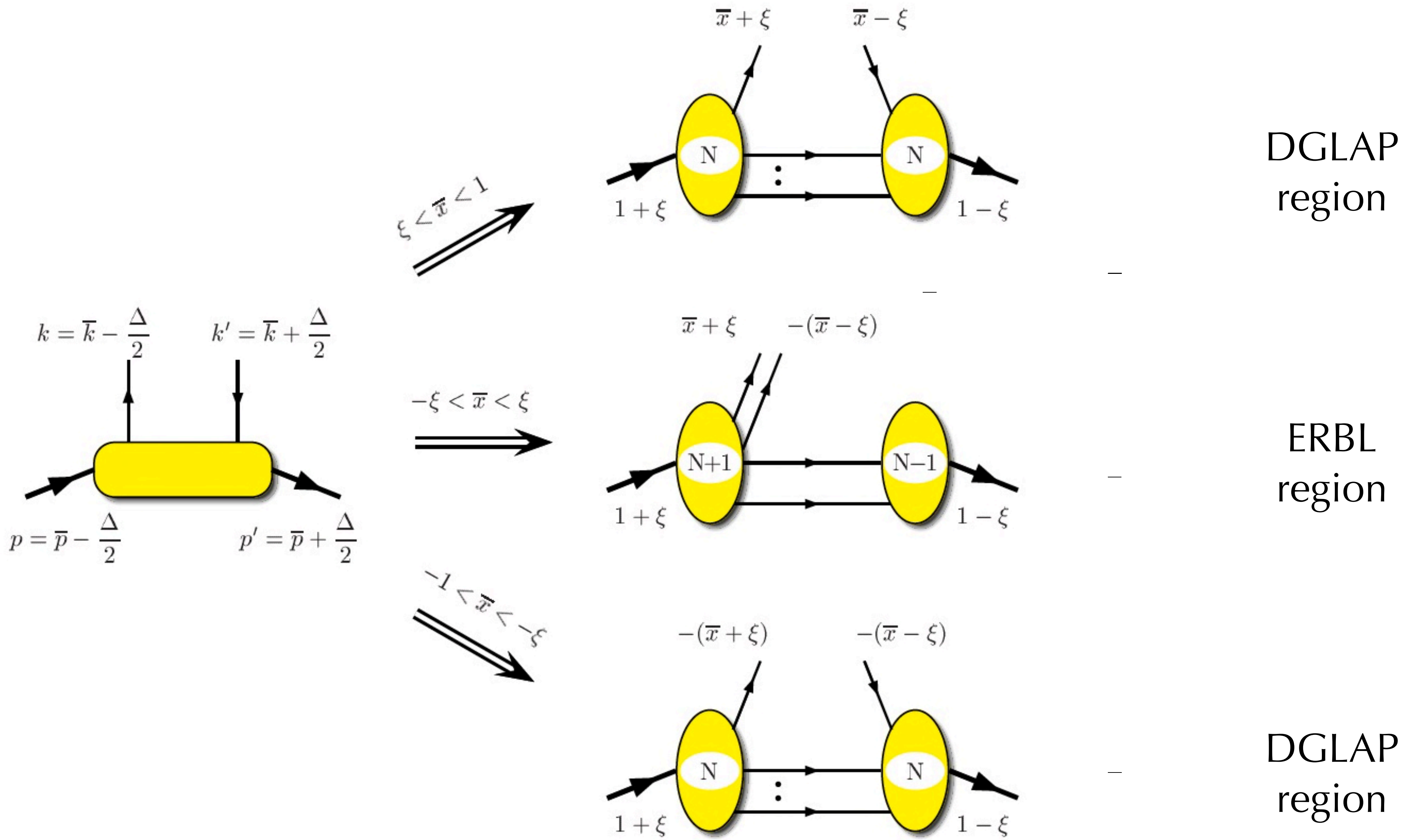
Eigenstates of total OAM

$$\hat{L}_z\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N} = \ell_z\Psi^{\Lambda}_{\lambda_1\ldots\lambda_N}$$

$A^+ = 0$ **gauge**

Brodsky, Pauli, Phys. Rept. 301 (1998) 299

Light-Front Wave Function Overlap Representation



$\text{GPDs} \sim \sum_N \int [d^3k]_N \, \Psi_N^*(k'_N)\Psi_N(k_N)\delta(\dots)$ interference of probability amplitudes

$\text{PDFs} \sim \sum_N \int [d^3k]_N \, |\Psi_N(k_N)|^2\delta(\dots)$ probability density

Diehl, Feldmann, Jakob, Kroll, NPB596, 2001
Diehl, Hwang, Brodsky, NPB596, 2001
Boffi, Pasquini, NPB649, 2003

Properties of GPDs

- Forward limit: ordinary parton distributions

$H^q(x, \xi = 0, t = 0) = q(x)$ unpolarized quark distributions

$\widetilde{H}^q(x, \xi = 0, t = 0) = \Delta q(x)$ long. polarized quark distributions

$H^q_T(x, \xi = 0, t = 0) = h_1(x)$ transv. polarized quark distributions

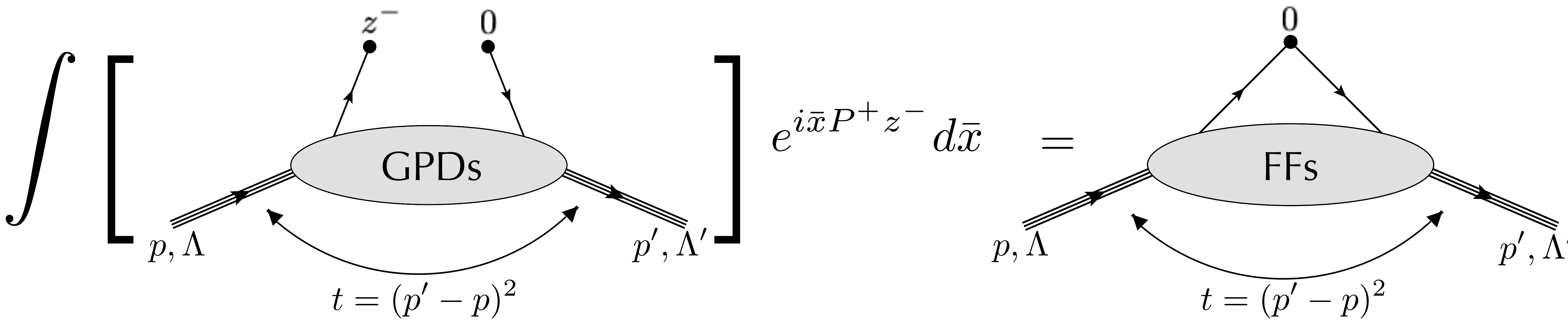
$x \in [-1, 1]$: $x > 0$: quarks $x < 0$: antiquarks

analogous relations for gluons, except for transversity distribution

- all the other GPDs do NOT appear in inclusive DIS $\Longrightarrow$ new information

- They all depend on the renormalisation scale ($\mu^2 = Q^2$) with different evolution equations in the DGLAP and ERBL regions

Properties of GPDs



$$\int_{-1}^{1} d\bar{x}\, H^q(\bar{x}, \xi, t) = F_1^q(t)$$ Dirac Form Factor

$$\int_{-1}^{1} d\bar{x}\, E^q(\bar{x}, \xi, t) = F_2^q(t)$$ Pauli Form Factor

$$\int_{-1}^{1} d\bar{x}\, \widetilde{H}^q(\bar{x}, \xi, t) = G_A^q(t)$$ Axial Form Factor

$$\int_{-1}^{1} d\bar{x}\, \widetilde{E}^q(\bar{x}, \xi, t) = G_P^q(t)$$ Pseudoscalar Form Factor

- matrix elements of local operators

$\longrightarrow$ can be calculated on the lattice

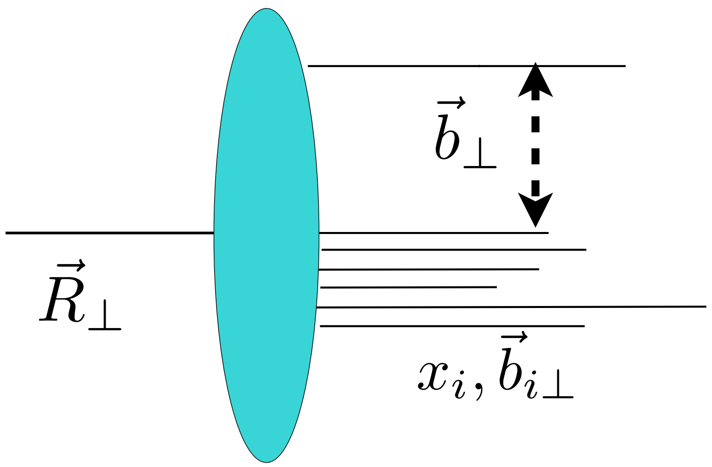
- renormalisation scale independent

- Lorentz invariance

$\longrightarrow$ ξ independent

GPDs
in Impact Parameter Space

Impact Parameter Space



- average transverse position of the partons

$$\vec{R}_\perp = \frac{\sum_i p_i^+ \vec{b}_{\perp i}}{\sum_i p_i^+} \qquad (i = q, \bar{q}, g)$$

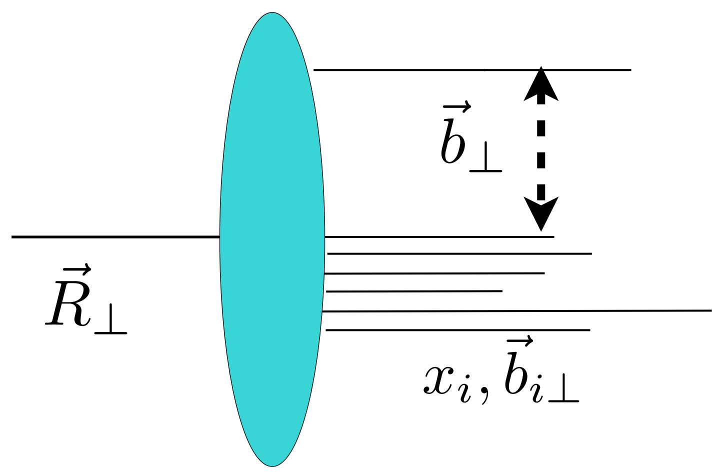
- $\mathrm{b}_\perp$: transverse distance between the struck parton and the centre of momentum of the hadron

[Burkardt, 2003]

Isomorphism between Galilei and subgroup of Light-Front operators

Galilei transformation:	Transverse boost:
$m_i \to m_i \qquad \vec{p}_i \to \vec{p}_i - m_i \vec{v}$	$p_i^+ \to p_i^+ \qquad \vec{p}_{\perp i} \to \vec{p}_{\perp i} - p_i^+ \vec{v}$
Center of mass:	Center of plus momentum:
$\vec{r}_* = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$	$\vec{R}_\perp = \frac{\sum_i p_i^+ \vec{b}_{\perp i}}{\sum_i p_i^+}$

Impact Parameter Space



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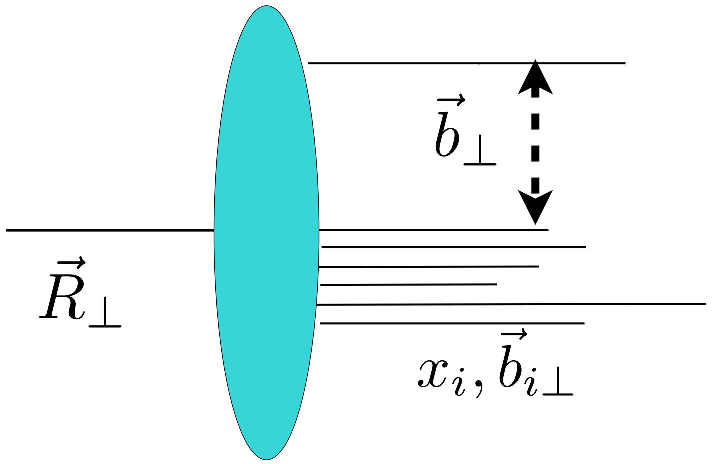


- average transverse position of the partons

$$\vec{R}_\perp = \frac{\sum_i p_i^+ \vec{b}_{\perp i}}{\sum_i p_i^+} \quad (i = q, \bar{q}, g)$$

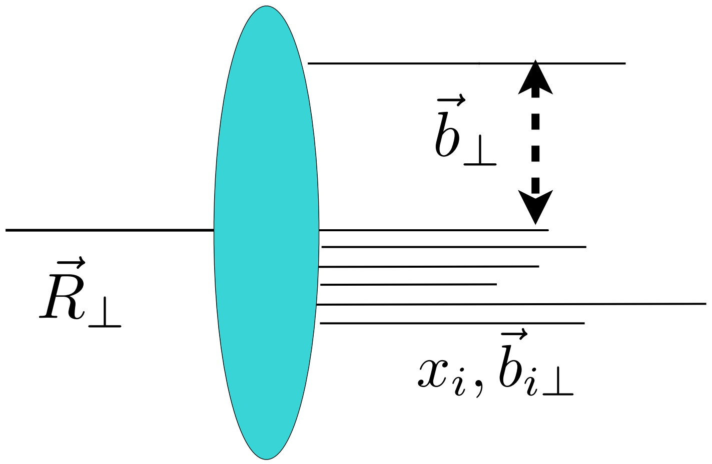
- $b_\perp$: transverse distance between the struck parton and the centre of momentum of the hadron

[Burkardt, 2003]

- Localized wave packet in the transverse plane

$$|p^+, \vec{R}_\perp = \vec{0}_\perp, \lambda\rangle = \mathcal{N} \int \frac{d^2\vec{p}_\perp}{(2\pi)^2} |p^+, \vec{p}_\perp, \lambda\rangle \quad \text{with} \quad |\mathcal{N}|^2 \int \frac{d^2\vec{p}_\perp}{(2\pi)^2} = 1$$

Impact Parameter Space



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- Impact parameter dependent GPD (IPD)

$$H(x, 0, \vec{b}_\perp) = \langle p^+, \vec{R}_\perp = 0, \Lambda| \int \frac{dz^-}{4\pi} e^{ixp^+z^-} \bar{\psi}(-\frac{z^-}{2}, \vec{b}_\perp)\gamma^+\psi(\frac{z^-}{2}, \vec{b}_\perp)|p^+, \vec{R}_\perp = 0, \Lambda\rangle$$

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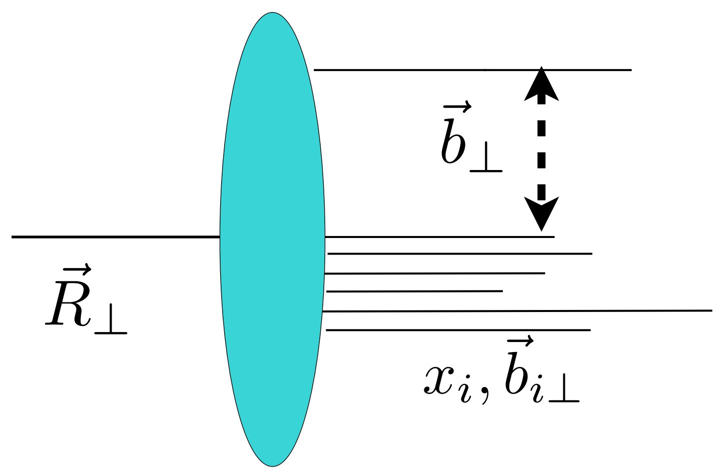

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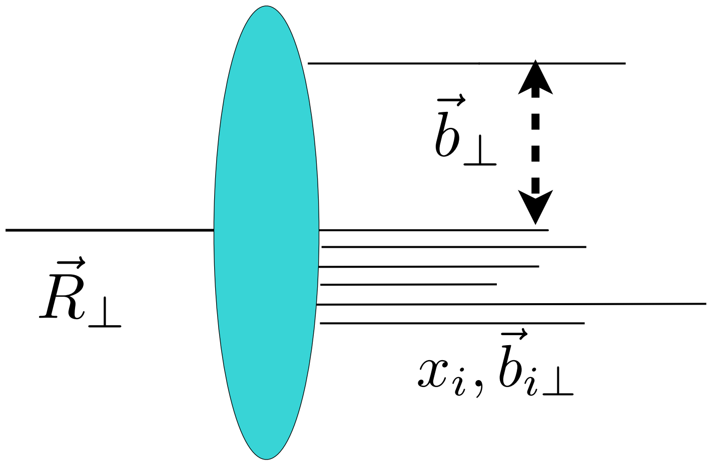
x>0

$$H(x, 0, \vec{b}_\perp) \sim \sum_\lambda \left| b_\lambda(xp^+, \vec{b}_\perp)|p^+, \vec{R}_\perp = 0, \Lambda\rangle \right|^2 \geq 0$$

x<0

$$H(x, 0, \vec{b}_\perp) \sim -\sum_\lambda \left| d_\lambda^\dagger(xp^+, \vec{b}_\perp)|p^+, \vec{R}_\perp = 0, \Lambda\rangle \right|^2 \leq 0$$

Impact Parameter Space



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- IPDs=Fourier transform of GPDs (homework)

$$H^q(x, 0, \vec{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H^q(x, 0, t) e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} \qquad (t = -\Delta_\perp^2)$$

Homework

$$H(x,0,\vec{b}_\perp) = \langle p^+, \vec{R}_\perp = 0, \Lambda | \hat{O}_q(x,\vec{b}_\perp) | p^+, \vec{R}_\perp = 0, \Lambda \rangle$$

where $\hat{O}_q \ (x,\vec{b}_\perp) = \int \frac{dz^-}{4\pi} e^{ixp^+z^-} \bar{\psi}(-\frac{z^-}{2},\vec{b}_\perp)\gamma^+\psi(\frac{z^-}{2},\vec{b}_\perp)$

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- Insert the expression for the localized wave packet in the transverse plane

$$|p^+, \vec{R}_\perp = \vec{0}_\perp, \lambda\rangle = \mathcal{N} \int \frac{d^2\vec{p}_\perp}{(2\pi)^2} |p^+, \vec{p}_\perp, \lambda\rangle \quad \text{with} \quad |\mathcal{N}|^2 \int \frac{d^2\vec{p}_\perp}{(2\pi)^2} = 1$$

$$H(x,0,\vec{b}_\perp) = |\mathcal{N}|^2 \int \frac{d^2 p_\perp}{(2\pi)^2} \int \frac{d^2 p'_\perp}{(2\pi)^2} \langle p^+, \vec{p}'_\perp, \Lambda | \hat{O}_q(x,\vec{b}_\perp) | p^+, \vec{p}_\perp, \Lambda \rangle$$

Homework

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- Use translation invariance

$$\langle p^+, \vec{p}'_\perp, \Lambda | \hat{O}_q(x,\vec{b}_\perp) | p^+, \vec{p}_\perp, \Lambda \rangle = e^{i\vec{b}_\perp \cdot (\vec{p}_\perp - \vec{p}_\perp)} \langle p^+, \vec{p}'_\perp, \Lambda | \hat{O}_q(x,\vec{0}_\perp) | p^+, \vec{p}_\perp, \Lambda \rangle$$

Homework

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- Change variables of integration:

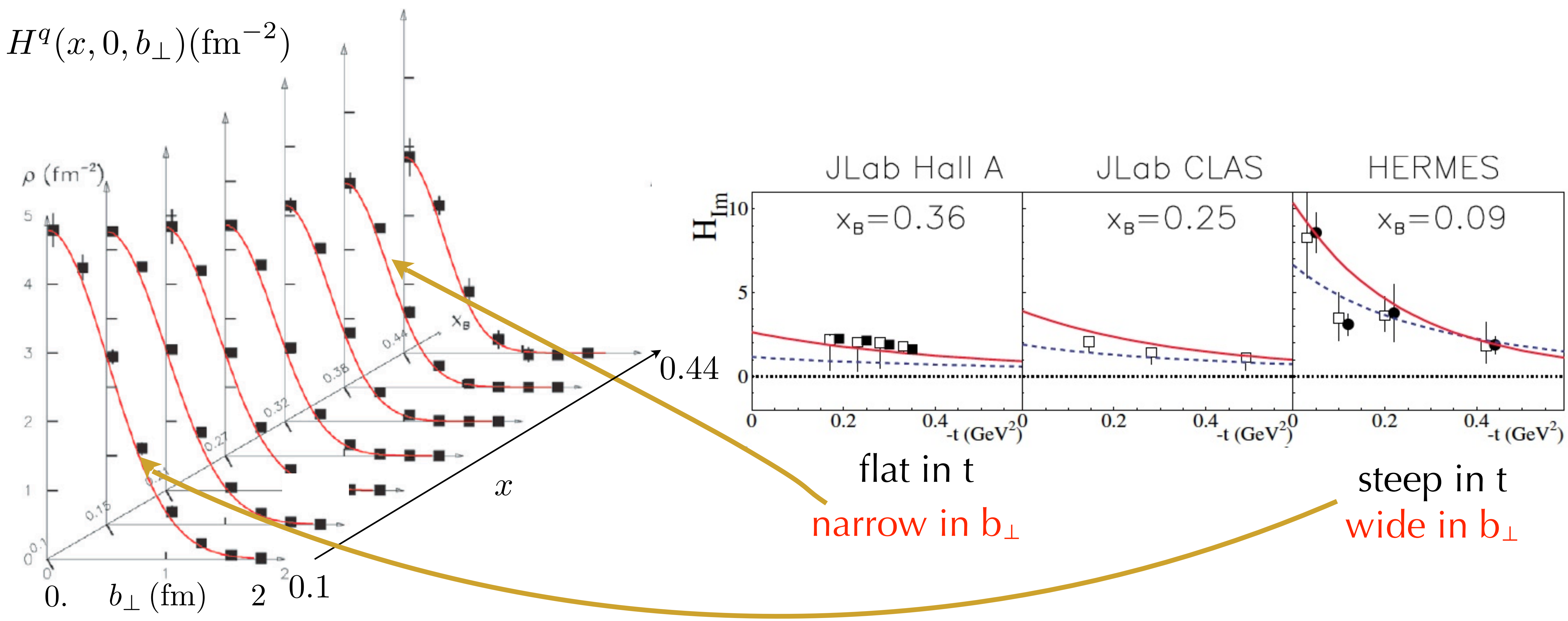
$$\int d^2p_\perp \int d^2p'_\perp = \int d^2\Delta_\perp \int d^2P_\perp \quad \text{where} \quad \vec{\Delta}_\perp = \vec{p}'_\perp - \vec{p}_\perp \ \ \vec{P}_\perp = \vec{p}'_\perp + \vec{p}_\perp$$

$$H(x,0,\vec{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} \langle p^+, \vec{p}'_\perp, \Lambda | \hat{O}_q(x,\vec{0}_\perp) | p^+, \vec{p}_\perp, \Lambda \rangle$$

The unpolarized GPD H

$$H^q(x, 0, \vec{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H^q(x, 0, t) e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} \qquad (t = -\vec{\Delta}_\perp^2)$$

extrapolation from data



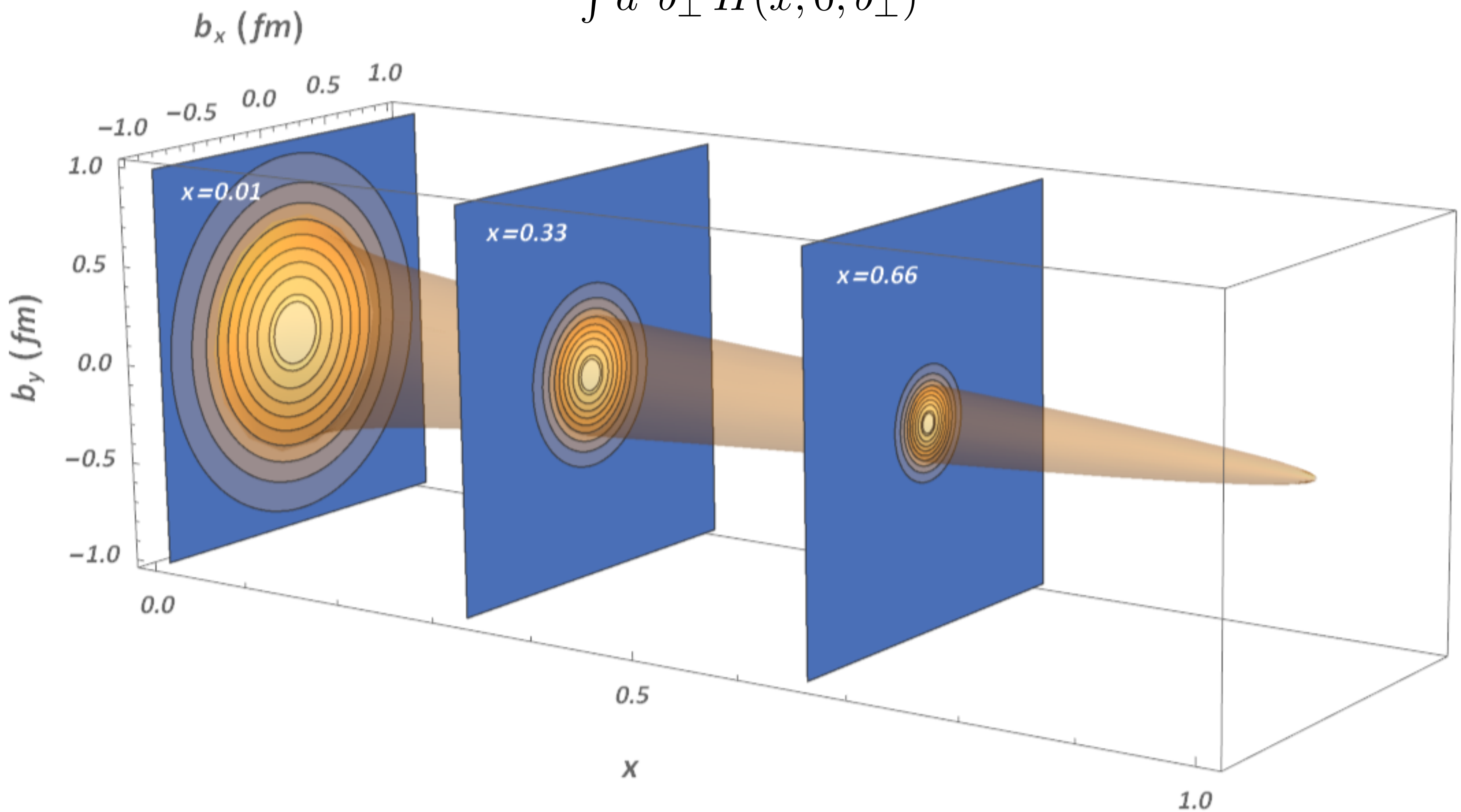
$$\longrightarrow \int dx\, H^q(x, 0, \vec{b}_\perp) = F_1(\vec{b}_\perp)$$

$H^q(x, 0, \vec{b}_\perp)$:decomposition in x of e.m. form factor

Guidal et al., Rep. Prog. Phys. ***76*** *(2013) 066202*

x-dependent Transverse Charge Radius

$$\langle b_\perp^2(x)\rangle = \frac{\int d^2\vec{b}_\perp\, \vec{b}_\perp^2\, H(x,0,b_\perp)}{\int d^2\vec{b}_\perp\, H(x,0,b_\perp)}$$



As x ⟶1, the active parton carries all the momentum and represents the centre of momentum

$$\langle b_\perp^2\rangle = \int dx\, \langle b_\perp^2(x)\rangle = \frac{2}{3}\langle r^2_{e.m.}\rangle = 0.43 \pm 0.01\,\mathrm{fm}^2$$

input from FF data

Dupré et al., PRD95(2017)011501

Assumptions in tomographic pictures

- Interpretation of exclusive processes in terms of GPDs (factorization) is valid for $-t \ll Q^2$

- One needs to integrate t up to infinity to perform the Fourier transform

  model-dependent extrapolation

- Experimental data are accessible only at non vanishing ξ

  model-dependent extrapolation at $\xi = 0$

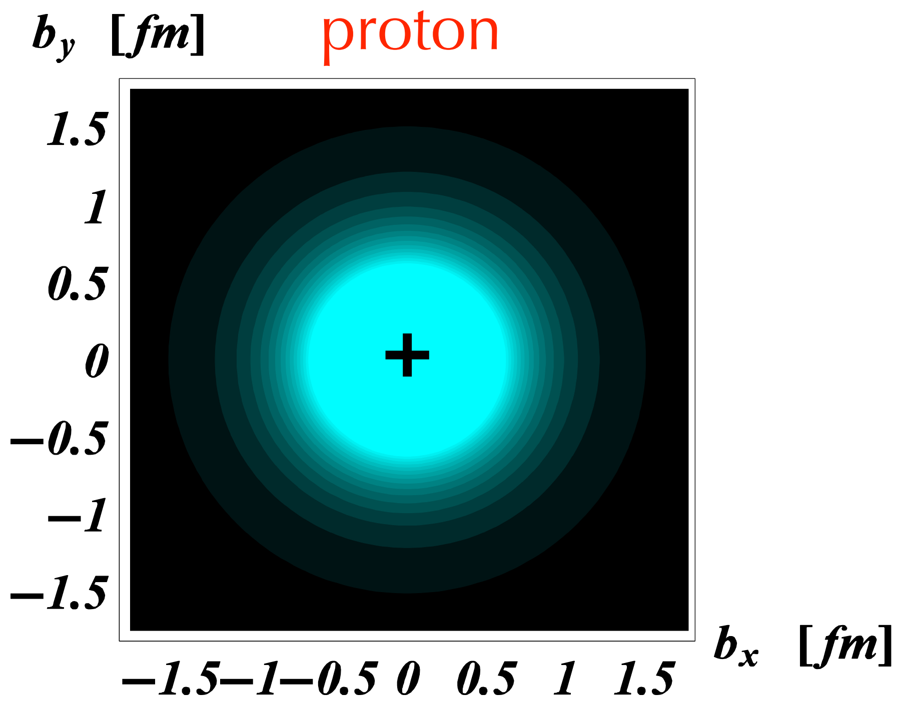
Need of controlled model dependence

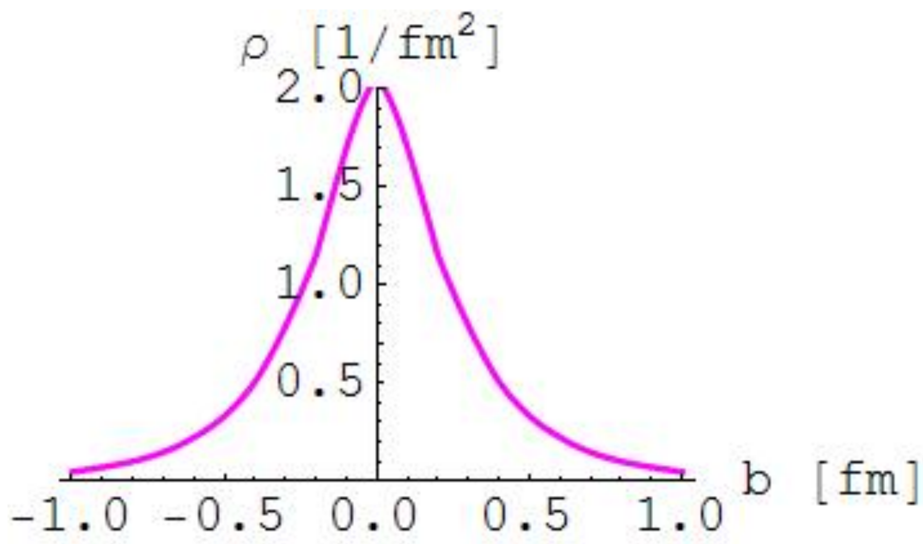
Charge density of partons in the transverse plane

$$\rho^q(b_\perp) = e_q \int d^2\Delta_\perp e^{i\Delta_\perp \cdot b_\perp} \int dx H^q(x, 0, \Delta_\perp^2) = \int d^2\Delta_\perp e^{i\Delta_\perp \cdot b_\perp} F_1^q(\Delta_\perp^2)$$

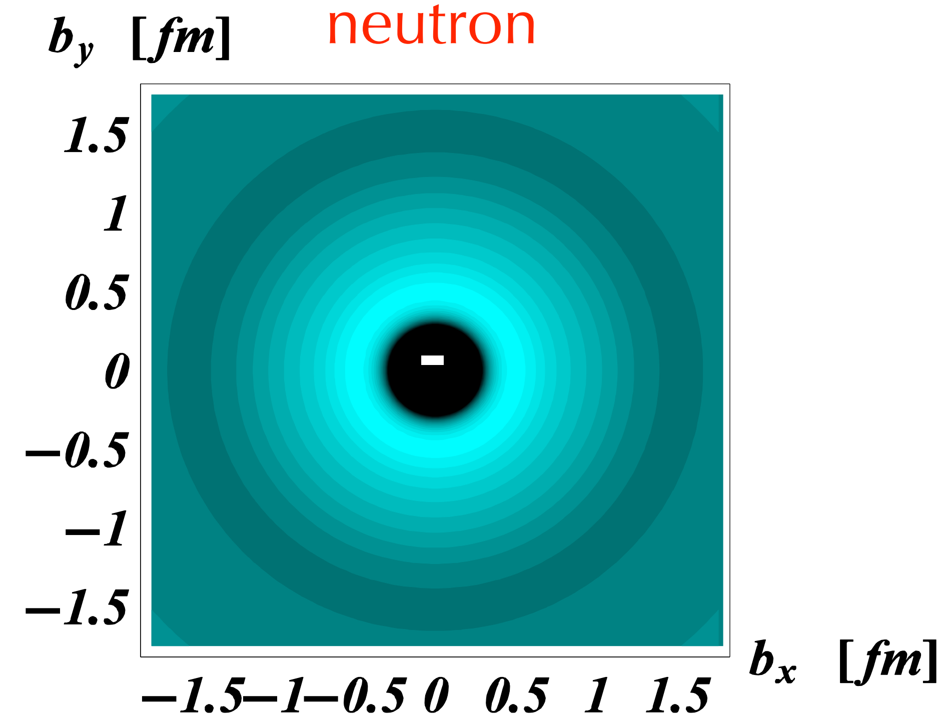
Infinite-Momentum-Frame Parton charge density in the transverse plane

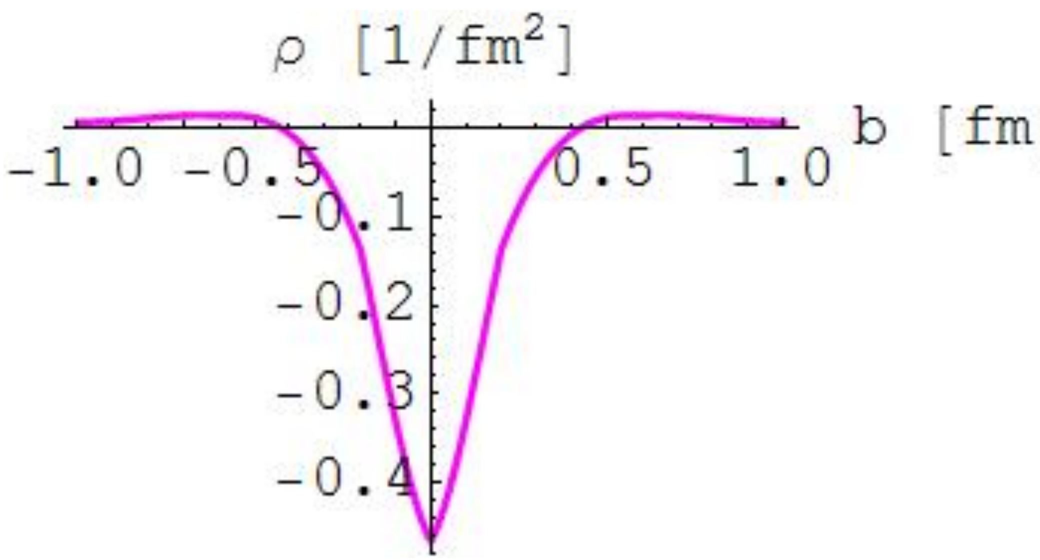
proton





neutron





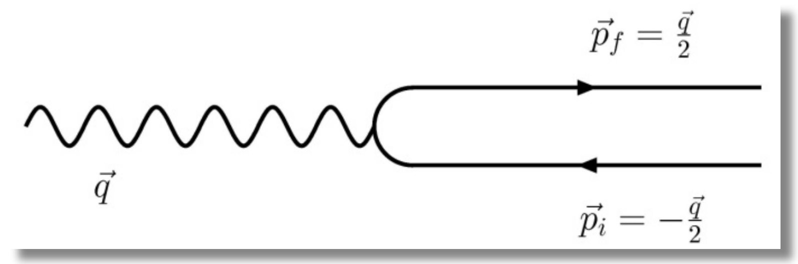
Miller (2007); Burkardt (2007)

Textbook interpretation

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2)$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

Breit frame



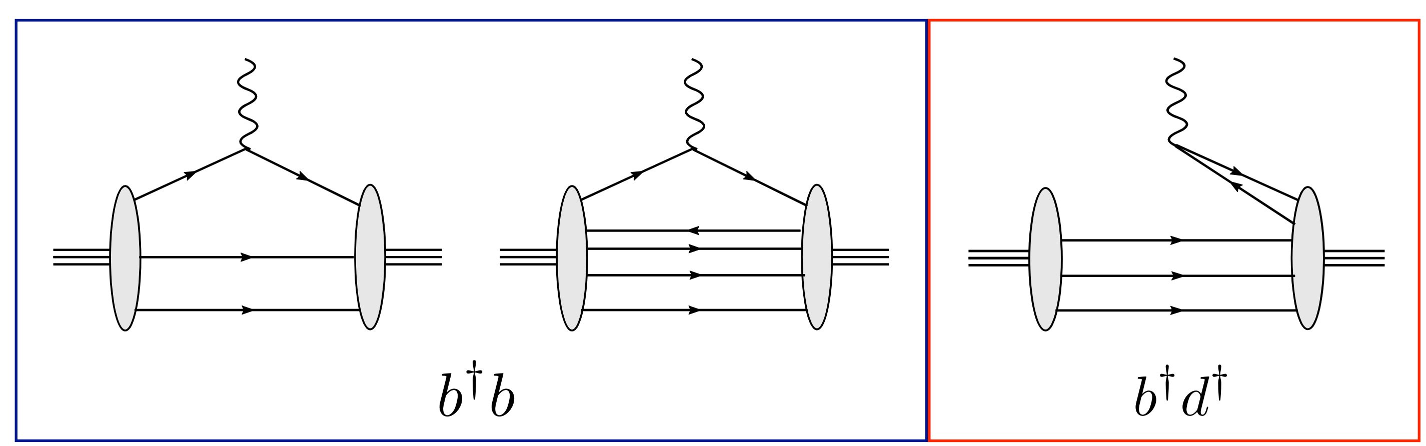
Spatial charge density

$$\rho(\vec{r}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}} G_E(Q^2)$$

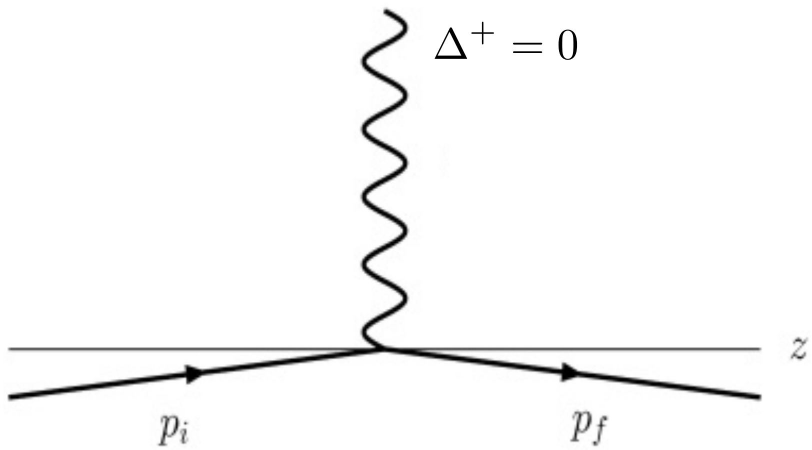
[Ernst, Sachs, Wali (1960)]
[Sachs (1962)]

No probabilistic/charge interpretation

Creation/annihilation of pairs



Drell-Yan-West frame



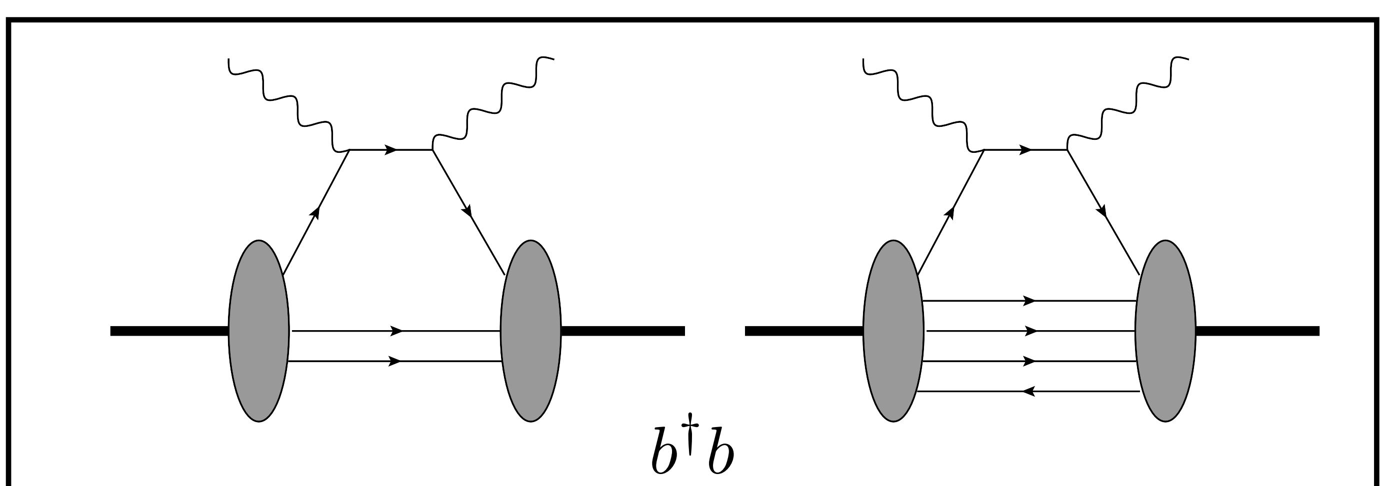
✓ $\Delta^+ = 0$ ⟶ no sensitivity to longitudinal Lorentz contraction

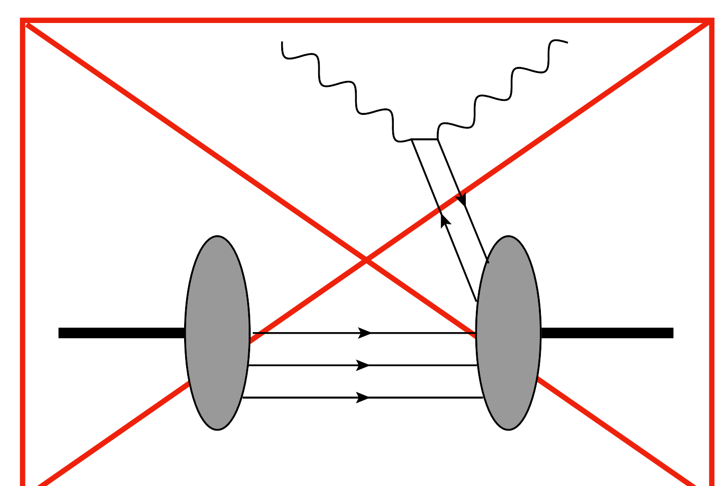
✓ $\vec{\Delta}_\perp \neq 0$: Transverse boosts ⟶ no transverse Lorentz contraction

✓ Particle number is conserved in Drell-Yan frame $\Delta^+ = 0$

$$\rho(\vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \frac{e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp}}{2P^+} J^+(\vec{\Delta}_\perp)$$

probabilistic/charge interpretation





Operator	Forward matrix element	Non-forward matrix element	Position-space interpretation
$e_q \bar{\psi}_q(0)\gamma^+\psi_q(0)$	Q	$F_1(t)$	$\rho(b_\perp^2)$
$\int \frac{dz^-}{4\pi} e^{ixp^+z^-} \bar{\psi}(-\frac{z^-}{2})\gamma^+\psi(\frac{z^-}{2})$	$q(x)$	$H_q(x,0,t)$	$q(x,b_\perp^2)$

$\rho(b_\perp^2)$ 2-dim distribution of charge in the transverse plane

$q(x,b_\perp^2)$ 2-dim. "distribution of the PDF" in the transverse plane

M. Burkardt, Int. J. Mod. Phys. A18 (2003) 173

Take-home message

Parton distribution functions $\longleftrightarrow$ Generalized Parton distributions

enter at cross section level
in fully inclusive reactions

squared wave functions

$\longrightarrow$ probability density

enter at amplitude level
in fully exclusive reactions

correlate wave function with different parton configurations

$\longrightarrow$ quantum-mechanical interference terms

provide a decomposition in x of form factors

at $\xi = 0$ and in impact parameter space ($\vec{b}_\perp \leftrightarrow \vec{\Delta}_\perp$) we recover the probabilistic interpretation for the position of partons in the transverse plane with longitudinal momentum x