



CLASS II



Ignazio Scimemi



PID2022-136510NB-C31,
"Europa Excelencia" No.
EUR2023-143460,



2020-T1/TIC-20204,
2022-T1/TIC-24024



Marie Skłodowska-
Curie Actions – Staff
Exchanges,
HORIZON-
MSCA-2023-
SE-01-101182937-
Hel

RECOVERING THE BACKGROUND FIELD METHOD

.....

Nucl. Phys. B 185 (1981) 189

THE BACKGROUND FIELD METHOD BEYOND ONE LOOP

Vol. B13 (1982)

ACTA PHYSICA POLONICA

No 1-2

L.F. Abbott*

CERN--Geneva

ABSTRACT

The background field approach to multi-loop calculations in gauge field theories is presented. A relation between the gauge-invariant effective action computed using this method and the effective action of the conventional functional approach is derived. Feynman rules are given and renormalization is discussed. It is shown that the renormalization programme can be carried out without any reference to fields appearing inside loops. Finally, as an explicit example, the two-loop contribution to the β function of pure Yang-Mills theory is calculated using the background field method.

INTRODUCTION TO THE BACKGROUND FIELD METHOD*

BY L. F. ABBOTT**

CERN, Geneva

(Received July 20, 1981)

The background field approach to calculations in gauge field theories is presented. Conventional functional techniques are reviewed and the background field method is introduced. Feynman rules and renormalization are discussed and, as an example, the Yang-Mills β function is computed.

PACS numbers: 11.10.Np, 11.10.Gh

BACKGROUND FIELD METHOD IN HISTORY

Recall that the generating functional for disconnected graphs was

$$Z[J] = \int \delta Q \exp i[S[Q] + J \cdot Q]. \quad (2.2.1)$$

Let us define an analogous quantity in which we write the classical action S as a function of the field Q plus an arbitrary background field ϕ . Thus

$$\tilde{Z}[J, \phi] = \int \delta Q \exp i[S[Q + \phi] + J \cdot Q], \quad (2.2.2)$$

The generator of connected graphs is:

$$W[J] = -i \ln[Z[J]] \longrightarrow \tilde{W}[J, \phi] = -i \ln[\tilde{Z}[J, \phi]]$$

$$\text{Classical fields: } \bar{Q} = \frac{\delta W[J]}{\delta J} \longrightarrow \tilde{Q} = \frac{\delta \tilde{W}[J, \phi]}{\delta J}$$

Effective action or generator of 1PI:

$$\Gamma[\bar{Q}] = W[J] - J \cdot \bar{Q} \longrightarrow \tilde{\Gamma}[\tilde{Q}, \phi] = \tilde{W}[J, \phi] - J \cdot \tilde{Q}$$

MÉTODO DE CAMPO DE FONDO EN LA HISTORIA

We re-write previous definitions

$$\tilde{Z}[J, \phi] = Z[J]e^{-iJ \cdot \phi}$$

$$\tilde{Q} = \bar{Q} - \phi$$

$$\tilde{W}[J, \phi] = W[J] - J \cdot \phi \quad \longrightarrow \quad \tilde{\Gamma}[\tilde{Q}, \phi] = (W[J] - J \cdot \phi) - J \cdot (\bar{Q} - \phi) \\ = \Gamma[\bar{Q}] = \Gamma[\tilde{Q} + \phi]$$

$$\Gamma[\phi] = \tilde{\Gamma}[0, \phi]$$

The effective action expressed in terms of the background field is the same as the effective action of the dynamic field and the background field, when the former vanishes: the dynamic fields are only virtual.

GAUGE IN THE BACKGROUND FIELD METHOD (BACKGROUND FIELD GAUGE)

$$\tilde{\Gamma}[0,A] = \Gamma[A]$$

Gauge fixing: If $\tilde{\Gamma}[0,A]$ is calculated in the gauge $\tilde{G}^a = \tilde{G}^a(Q, A)$ then the result is the same as $\Gamma[\bar{Q}]$ with $G^a = \tilde{G}^a(Q - A, A)$ and $\bar{Q} = A$.
Green functions may differ, but cross section is the same

Explicit gauge invariance: there is a choice of gauge fixing term $\tilde{G}^a$ such that $\tilde{\Gamma}[0,A]$ is an invariant functional of A:

$$\tilde{G}^a = \partial^\mu Q_\mu^a + g f^{abc} A^{b,\mu} Q_\mu^c$$

With this choice $\tilde{\Gamma}[\tilde{Q}, A]$ is invariant for $\delta A_\mu^a = -f^{abc} \omega^b A_\mu^c + \frac{1}{g} \partial_\mu \omega^a,$

... and $\tilde{\Gamma}[0,A]$ keeps A gauge invariance $\delta \tilde{Q}_\mu^a = -f^{abc} \omega^b \tilde{Q}_\mu^c.$

BACKGROUND FIELD EXPANSION METHOD

For each fermion field we have two copies of QCD fields: causal (+, also negative frequency) and anticausal (-, also positive frequency).

$$W_{\text{DY}}^{\mu\nu} = \int \frac{d^4 y}{(2\pi)^4} e^{-i(yq)} \int [D\bar{q}^{(+)} Dq^{(+)} DA^{(+)}] \int [D\bar{q}^{(-)} Dq^{(-)} DA^{(-)}] \\ \times \Psi_{p_1}^{*(-)} \Psi_{p_2}^{*(-)} e^{iS_{\text{QCD}}^{(+)} - iS_{\text{QCD}}^{(-)}} J_{\mu}^{\dagger(-)}(y) J_{\nu}^{(+)}(0) \Psi_{p_1}^{(+)} \Psi_{p_2}^{(+)}$$

The hadronic tensors we consider have two causally independent sectors that exchange real emissions.

MODES

Hadrons are formed only from collinear fields.

$$\Psi_{p_1} = \Psi_{p_1}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}], \quad \Psi_{p_2} = \Psi_{p_2}[\bar{q}_n, q_n, A_n]$$

$$k^\mu = k^+ \bar{n}^\mu + k^- n^\mu + k_T^\mu \text{ con } n^2 = \bar{n}^2 = 0, n \cdot \bar{n} = 1$$

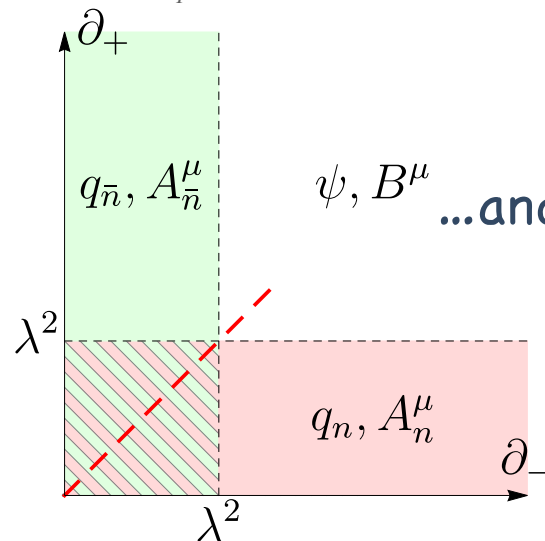
There is a scaling of the momentum of the fields...

$$\{\partial_+, \partial_-, \partial_T\} q_{\bar{n}} \lesssim Q\{1, \lambda^2, \lambda\} q_{\bar{n}},$$

$$\{\partial_+, \partial_-, \partial_T\} A_{\bar{n}}^\mu \lesssim Q\{1, \lambda^2, \lambda\} A_{\bar{n}}^\mu$$

ψ, B^μ

...and we can introduce dynamic and background fields...



Dynamical field

$$q^{(\pm)}(x) = \psi^{(\pm)}(x) + q_n^{(\pm)}(x) + q_{\bar{n}}^{(\pm)}(x)$$

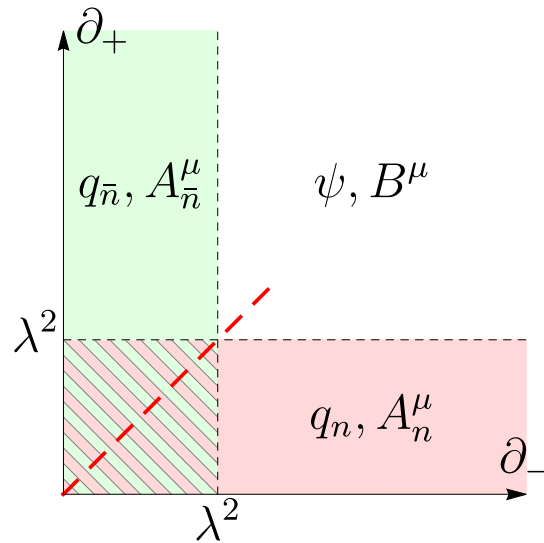
$$A_\mu^{(\pm)}(x) = B_\mu^{(\pm)}(x) + A_{n\mu}^{(\pm)}(x) + A_{\bar{n}\mu}^{(\pm)}(x)$$

Dynamical field

Gauge fixing

$$[\partial_\mu \delta^{AC} + g f^{ABC} (A_{\bar{n}\mu}^{(\pm)B} + A_{n\mu}^{(\pm)B})] B^{(\pm)\mu C} = 0$$

MODES



Each collinear field has “good” and “bad” components selected by standard projectors ($\sim$ SCET)

$$q_{\bar{n}}(x) = \xi_{\bar{n}}(x) + \eta_{\bar{n}}(x) \quad \xi_{\bar{n}}(x) = \frac{\not{n}\not{\bar{n}}}{2}q_{\bar{n}}(x), \quad \eta_{\bar{n}}(x) = \frac{\not{n}\not{\bar{n}}}{2}q_{\bar{n}}(x)$$

$$\begin{aligned} \gamma^+ D_-[A_{\bar{n}}]\xi_{\bar{n}} &= -\mathcal{D}_T[A_{\bar{n}}]\eta_{\bar{n}} \\ \gamma^- D_+[A_{\bar{n}}]\eta_{\bar{n}} &= -\mathcal{D}_T[A_{\bar{n}}]\xi_{\bar{n}} \end{aligned} \quad \Rightarrow \quad \eta_{\bar{n}/n} \sim \lambda \xi_{\bar{n}/n}$$

Finally..

$$\begin{aligned} A_{\bar{n}}^+ &\sim 1, & A_{\bar{n}}^{\mu_T} &\sim \lambda, & A_{\bar{n}}^- &\sim \lambda^2, \\ \xi_{\bar{n}/n} &\sim \lambda, & \eta_{\bar{n}/n} &\sim \lambda^2 \\ A_n^+ &\sim \lambda^2, & A_n^{\mu_T} &\sim \lambda, & A_n^- &\sim 1 \end{aligned}$$

FORMAL RESULT

$$\begin{aligned}
 W_{\text{DY(unsub.)}}^{\mu\nu} &= \int \frac{d^4 y}{(2\pi)^4} e^{-i(yq)} \\
 &\times \int [D\bar{q}_{\bar{n}}^{(+)} Dq_{\bar{n}}^{(+)} DA_{\bar{n}}^{(+)}][D\bar{q}_{\bar{n}}^{(-)} Dq_{\bar{n}}^{(-)} DA_{\bar{n}}^{(-)}] e^{iS_{\text{QCD}}^{(+)}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] - iS_{\text{QCD}}^{(-)}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}]} \\
 &\times \int [D\bar{q}_n^{(+)} Dq_n^{(+)} DA_n^{(+)}][D\bar{q}_n^{(-)} Dq_n^{(-)} DA_n^{(-)}] e^{iS_{\text{QCD}}^{(+)}[\bar{q}_n, q_n, A_n] - iS_{\text{QCD}}^{(-)}[\bar{q}_n, q_n, A_n]} \\
 &\times \Psi_{p_1}^{*(-)}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] \Psi_{p_2}^{*(-)}[\bar{q}_n, q_n, A_n] \mathcal{W}_{\text{eff}}^{\mu\nu}[\bar{q}_{\bar{n}}, \bar{q}_n, \dots](y) \Psi_{p_1}^{+}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] \Psi_{p_2}^{+}[\bar{q}_n, q_n, A_n]
 \end{aligned}$$

FORMAL RESULT

The dynamic (hard) degrees of freedom are integrated, obtaining

$$\mathcal{W}_{eff}^{\mu\nu}[\bar{q}_{\bar{n}}, \bar{q}_n, \dots](y) = \int [D\bar{\psi}^{(+)} D\psi^{(+)} DB^{(+)}][D\bar{\psi}^{(-)} D\psi^{(-)} DB^{(-)}] e^{iS_{QCD}^{(+)}[\bar{\psi}, \psi, B] - iS_{QCD}^{(-)}[\bar{\psi}, \psi, B]} e^{iS_{int}^{(+)} - iS_{int}^{(-)}} \\ \times J_{\mu}^{\dagger(-)}[\bar{\psi} + \bar{q}_{\bar{n}} + \bar{q}_n, \dots](y) J_{\nu}^{(+)}[\bar{\psi} + \bar{q}_{\bar{n}} + \bar{q}_n, \dots](0)$$

And we expand in a series of operators $\mathcal{J}_{N,k}^{\mu\nu} \sim \lambda^{N+4}$

$$\mathcal{W}_{eff}^{\mu\nu}[\bar{q}_{\bar{n}}, \bar{q}_n, \dots](y) = \sum_{N=0}^{\infty} \sum_k \mathcal{W}_{N,k}^{\mu\nu}[\bar{q}_{\bar{n}}, \bar{q}_n, \dots](y)$$

Where N is the power counting and $k=twist$ or lists the operators

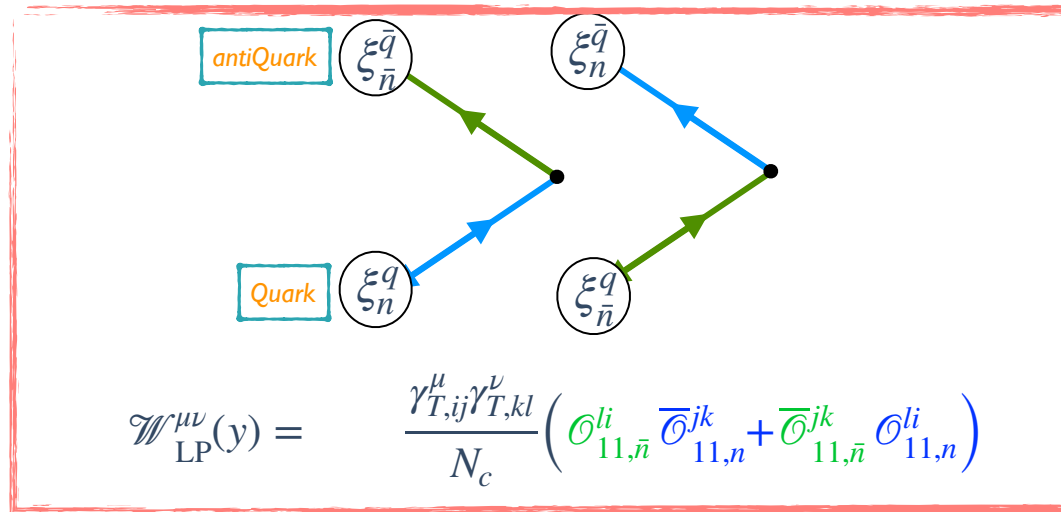
$$\mathcal{W}_{N,(a,b)}^{\mu\nu}[\bar{q}_{\bar{n}}, \bar{q}_n, \dots](y) = C_{N,(a+b)}^{\mu\nu}(y) \otimes \mathcal{O}_a[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] \otimes \mathcal{O}_b[\bar{q}_n, q_n, A_n]$$

hard coefficients

Operators products

LEADING POWER AND LEADING ORDER HADRONIC TENSOR

$\bar{n}$ y n sectors can now be separated



En el gauge del cono de luz:

$$\mathcal{O}_{11,n}^{ji}(\{y^+,0\},y_T) = \bar{\xi}_{n,i}^{(-)}(y^+\bar{n} + y_T) \xi_{n,j}^{(+)}(0),$$

$$\overline{\mathcal{O}}_{11,n}^{ji}(\{y^+,0\},y_T) = \xi_{n,j}^{(-)}(y^+\bar{n} + y_T) \bar{\xi}_{n,i}^{(+)}(0).$$

$$\mathcal{O}_{11,\bar{n}}^{ji}(\{y^-,0\},y_T) = \bar{\xi}_{\bar{n},i}^{(-)}(y^-n + y_T) \xi_{\bar{n},j}^{(+)}(0),$$

$$\overline{\mathcal{O}}_{11,\bar{n}}^{ji}(\{y^-,0\},y_T) = \xi_{\bar{n},j}^{(-)}(y^-n + y_T) \bar{\xi}_{\bar{n},i}^{(+)}(0).$$

WILSON LINES AND GAUGE INVARIANT BLOCKS

To recover gauge invariance, the fields are redefined with Wilson lines (which “dress” the fields with gluons) n -colineal or $\bar{n}$ -colineal)

$$W(x) = [x + \bar{L}\bar{n}] = \mathcal{P} \left\{ \exp \left[ig \int_{\bar{L}}^0 ds (n \cdot A(x + s\bar{n})) \right] \right\} = W_n \quad \chi = W^\dagger \xi$$

$$g\mathcal{A}^\mu = W^\dagger(iD^\mu W) = W^\dagger[i\partial^\mu W + gA^\mu W]$$

$$W(x) = [x + Ln] = \mathcal{P} \left\{ \exp \left[ig \int_{\bar{L}}^0 ds (\bar{n} \cdot A(x + sn)) \right] \right\} = W_{\bar{n}} \quad \chi = W^\dagger \xi$$

$$g\mathcal{A}^\mu = W^\dagger(iD^\mu W) = W^\dagger[i\partial^\mu W + gA^\mu W]$$

WILSON LINES AND SOFT GAUGE INVARIANT BLOCKS

More Wilson lines are needed to solve superposition problems
(zero-bin problem)

$$S_{\bar{n}}(x) = [x + \bar{L}\bar{n}] = \mathcal{P} \left\{ \exp \left[ig \int_{\bar{L}}^0 ds (n \cdot A(x + s\bar{n})) \right] \right\} \quad \Psi = S_{\bar{n}}^\dagger \psi$$

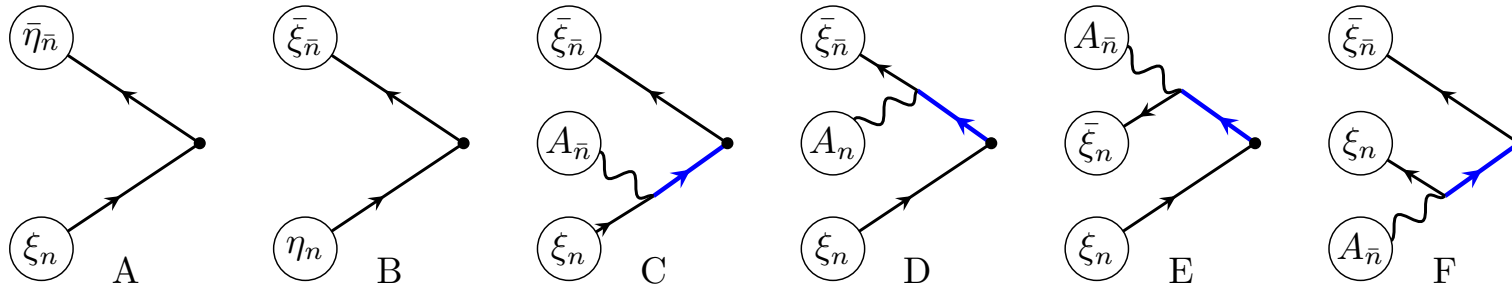
$$g\mathcal{A}_{\bar{n}}^\mu = S_{\bar{n}}^\dagger (iD^\mu S_{\bar{n}}) = S_{\bar{n}}^\dagger [i\partial^\mu S_{\bar{n}} + gA^\mu S_{\bar{n}}]$$

$$S_n(x) = [x + Ln] = \mathcal{P} \left\{ \exp \left[ig \int_{\bar{L}}^0 ds (\bar{n} \cdot A(x + sn)) \right] \right\} \quad \Psi = S_n^\dagger \psi$$

$$g\mathcal{A}_n^\mu = S_n^\dagger (iD^\mu S_n) = S_n^\dagger [i\partial^\mu S_n + gA^\mu S_n]$$

NEXT-TO-LEADING POWER

The main purpose of introducing BFM is to write a proper expansion at NLP and start the study of power corrections

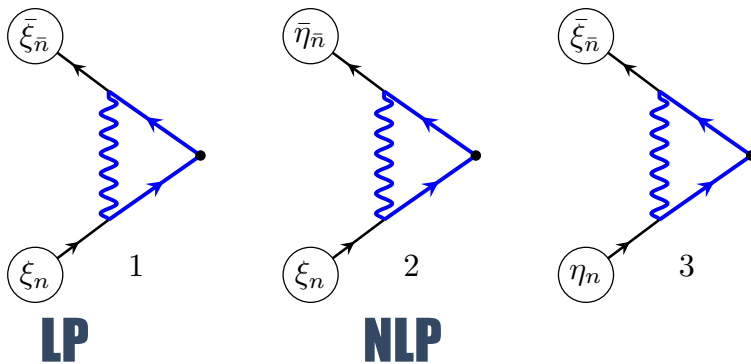


“Bad” component
Contributions

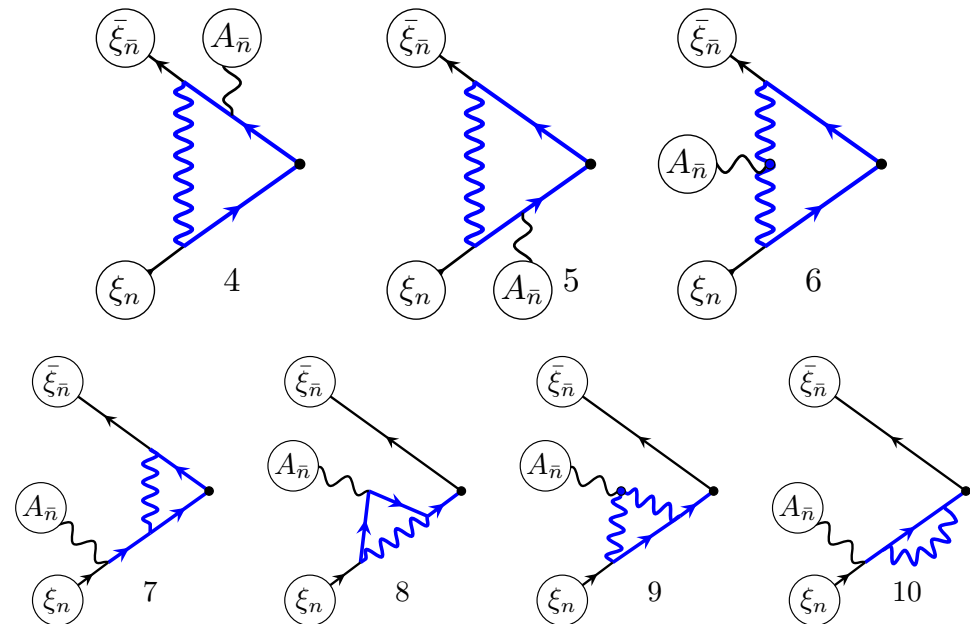
(Hard) background contributions

RECENT DEVELOPMENTS: NEXT-TO-LEADING POWER

“Bad” component
Contributions

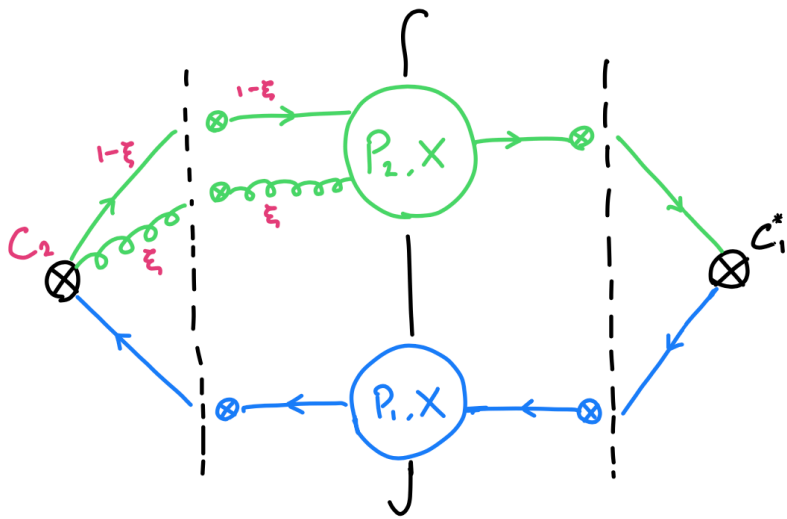


Hard background contributions



A. Vladimirov, V. Moos, I.S. *JHEP* 01 (2022) 110

RECENT DEVELOPMENTS: NEW OBSERVABLES TO NLP



Jets de higher twist

Production of di-jet in e^+e^- with hemisphere dependent algorithms:
Asymmetries at NLP

$$\begin{aligned}
 W^{\mu\nu} = & -N_c g_T^{\mu\nu} H_1(Q^2) \int_0^\infty d\mathbf{b}^2 \frac{J_0(|\mathbf{b}||\mathbf{q}|)}{2\pi} J_{11}(\mathbf{b}^2) J_{11}(\mathbf{b}^2) \\
 & + N_c \left[\frac{n^\mu q_T^\nu}{q^+} + \frac{n^\nu q_T^\mu}{q^+} \right] H_1(Q^2) \int_0^\infty d\mathbf{b}^2 \frac{J_1(|\mathbf{b}||\mathbf{q}|)}{2\pi|\mathbf{b}||\mathbf{q}|} J'_{11}(\mathbf{b}^2) J_{11}(\mathbf{b}^2) \\
 & + N_c \left[\frac{\bar{n}^\mu q_T^\nu}{q^-} - \frac{n^\mu q_T^\nu}{q^+} \right] \int_0^1 d\xi H_2(\xi, Q^2) \int_0^\infty d\mathbf{b}^2 \frac{J_1(|\mathbf{b}||\mathbf{q}|)}{2\pi|\mathbf{b}||\mathbf{q}|} \\
 & \times \left\{ J_{21}(\xi, \mathbf{b}^2) J_{11}(\mathbf{b}^2) - J_{11}(\mathbf{b}^2) J_{21}(\xi, \mathbf{b}^2) \right\}
 \end{aligned}$$



Pictures from M. Jaarsma talk at SCET2024,

R. F. Del Castillo, M. Jaarsma, I. Scimemi., W. Waalewijn, JHEP 02 (2024) 074

SUMMARY OF DEFINITIONS AT LP

$$\xi = (0, \xi^-, \xi_T)$$

Light cone Wilson lines

$$\text{bare TMDPDF} \quad O_q^{bare}(x, b_T) = \frac{1}{2} \sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+ \xi^-} \left\{ T \left[\bar{q}_i \tilde{W}_n^T \right]_a \left(\frac{\xi}{2} \right) |X\rangle \gamma_{ij}^+ \langle X| \bar{T} \left[\tilde{W}_n^{T\dagger} q_j \right]_a \left(-\frac{\xi}{2} \right) \right\}$$

$$\text{bare TMDFF} \quad \mathbb{O}_q^{bare}(z, b_T) = \frac{1}{4zN_c} \sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+ \xi^- / z} \langle 0 | T \left[\tilde{W}_n^{T\dagger} q_j \right]_a \left(\frac{\xi}{2} \right) |X, \frac{\delta}{\delta J}\rangle \gamma_{ij}^+ \langle X, \frac{\delta}{\delta J} | \bar{T} \left[\bar{q}_i \tilde{W}_n^T \right]_a \left(-\frac{\xi}{2} \right) |0\rangle$$

Formal definition of TMD operator

Applying these operators to the hadron states we obtain **unsubtracted** TMDs

$$\Phi_{q \leftarrow h}(x, b_T) = \langle h | O_q^{bare}(x, b_T) | h \rangle$$

$$\Delta_{q \rightarrow h}(z, b_T) = \langle h | \mathbb{O}_q^{bare}(z, b_T) | h \rangle$$

To define individual TMD we have to take into account rapidity divergences, UV divergences and overlap regions

$$F_{q \leftarrow h}(x, b_T; \zeta, \mu) = \sqrt{S(b_T; \zeta)} \langle h | Z_q(\mu) O_q^{bare}(x, b_T) | h \rangle \Big|_{zero-bin}$$

$$D_{q \rightarrow h}(x, b_T; \zeta, \mu) = \sqrt{S(b_T; \zeta)} \langle h | Z_q(\mu) \mathbb{O}_q^{bare}(x, b_T) | h \rangle \Big|_{zero-bin}$$

- μ is scale of UV renormalization.
- ζ is scale of rapidity-divergences separation.

TMD EVOLUTION

COUPLED EVOLUTION OF TMD ...

TMD (standard) anomalous dimension

$$\begin{aligned}\mu^2 \frac{d}{d\mu^2} F_{f \leftarrow h}(x, b; \mu, \zeta) &= \frac{\gamma_F(\mu, \zeta)}{2} F_{f \leftarrow h}(x, b; \mu, \zeta) \\ \zeta \frac{d}{d\zeta} F_{f \leftarrow h}(x, b; \mu, \zeta) &= -\mathcal{D}(\mu, b) F_{f \leftarrow h}(x, b; \mu, \zeta)\end{aligned}$$

TMD rapidity anomalous dimension

Collinear overlap

$$\begin{aligned}\zeta \frac{d}{d\zeta} \gamma_F(\mu, \zeta) &= -\Gamma(\mu) \\ \mu \frac{d}{d\mu} \mathcal{D}(\mu, b) &= \Gamma(\mu)\end{aligned}$$

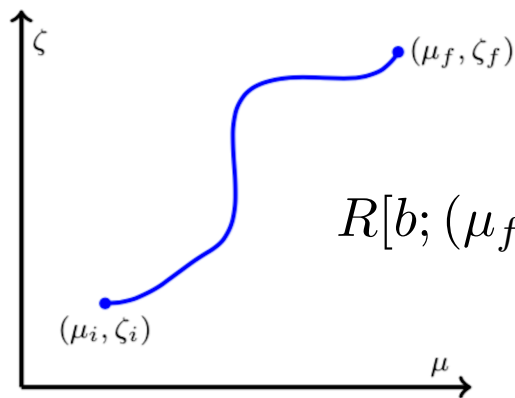
AMBIGUITY IN THE TMD EVOLUTION

COUPLED EVOLUTION OF TMD AND TRUNCATION OF THE PERTURBATIVE SERIES

Integrability Condition...

$$\zeta \frac{d}{d\zeta} \gamma_F(\mu, \zeta) = -\mu \frac{d}{d\mu} \mathcal{D}(\mu, b)$$

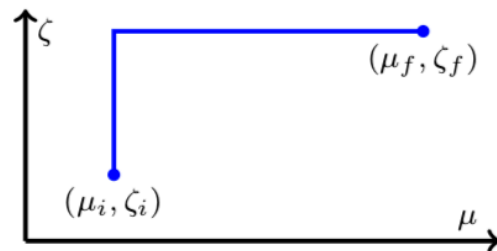
...ensures the path independence of the evolution factor...



$$R[b; (\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i)] = \exp \left[\int_P \left(\gamma_F(\mu, \zeta) \frac{d\mu}{\mu} - \mathcal{D}(\mu, b) \frac{d\zeta}{\zeta} \right) \right]$$

AMBIGUITY IN THE TMD EVOLUTION

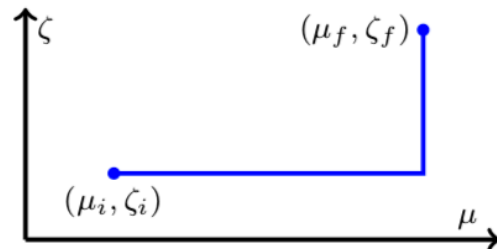
COUPLED EVOLUTION OF TMD AND TRUNCATION OF THE PERTURBATIVE SERIES



Solution 1

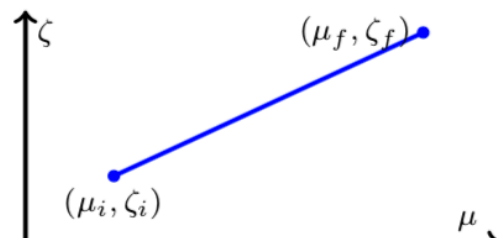
$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_f) - \mathcal{D}(\mu_i, b) \ln \left(\frac{\zeta_f}{\zeta_i} \right)$$

given in [Collins' textbook]



Solution 2

$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_i) - \mathcal{D}(\mu_f, b) \ln \left(\frac{\zeta_f}{\zeta_i} \right)$$



Solution 3

$$\ln R = \int_0^1 \left(\gamma_F(\mu(t), \zeta(t)) \frac{\mu_f - \mu_i}{(\mu_f - \mu_i)t + \mu_i} - \mathcal{D}(\mu(t), b) \frac{\zeta_f - \zeta_i}{(\zeta_f - \zeta_i)t + \zeta_i} \right) dt$$

AMBIGUITY IN THE TMD EVOLUTION

COUPLED EVOLUTION OF TMD AND TRUNCATION OF THE PERTURBATIVE SERIES

In practice due to the truncation of the perturbative series:
Transitivity and reversibility of evolution is lost

$$\zeta \frac{d}{d\zeta} \gamma_F(\mu, \zeta) \neq -\mu \frac{d}{d\mu} \mathcal{D}(\mu, b)$$

**For $Q=M_Z$ the solution path dependence enormous:
at $b=0.5$ an error of about 18% at N³LO on the evolution factor R**

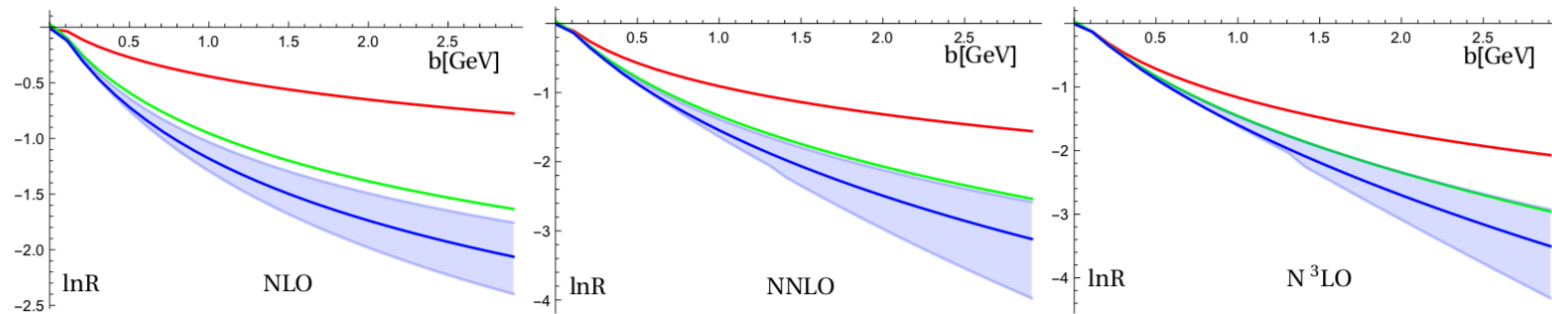


FIG. 3: Comparisons of different solution for $\ln R((M_Z, M_Z^2) \rightarrow (\mu_b, \mu_b^2))$ where $\mu_b = C_0/b + 2$. The blue line is the solution 1. The red line is the solution 2. The green line is the solution 3. The error band is obtained from the improved $\mathcal{D}$ solution at $\mu_0 = \mu_i$ by variation of $\mu_0 \in (0.5, 2)\mu_i$. The blue line with error-band corresponds to the solution used in [18].

EVOLUTION FIELD: NOTATION AND IDEAL CASE

The evolution scales
are treated equally

$$\vec{\nu} = \left(\ln \frac{\mu^2}{1 \text{ GeV}^2}, \ln \frac{\zeta}{1 \text{ GeV}^2} \right)$$

Differentiation

$$\vec{\nabla} = \frac{d}{d\vec{\nu}} = \left(\mu^2 \frac{d}{d\mu^2}, \zeta \frac{d}{d\zeta} \right), \quad \mathbf{curl} = \left(-\zeta \frac{d}{d\zeta}, \mu^2 \frac{d}{d\mu^2} \right)$$

Evolution field

$$\mathbf{E}(\vec{\nu}, b) = \left(\frac{\gamma_F(\vec{\nu})}{2}, -\mathcal{D}(\vec{\nu}, b) \right)$$

TMD Evolution

$$\vec{\nabla} F(x, b; \vec{\nu}) = \mathbf{E}(\vec{\nu}, b) F(x, b; \vec{\nu})$$

Integrability Condition
and Scalar Potential

$$\vec{\nabla} \times \mathbf{E} = 0 \Rightarrow \mathbf{E}(\vec{\nu}, b) = \vec{\nabla} U(\vec{\nu}, b)$$

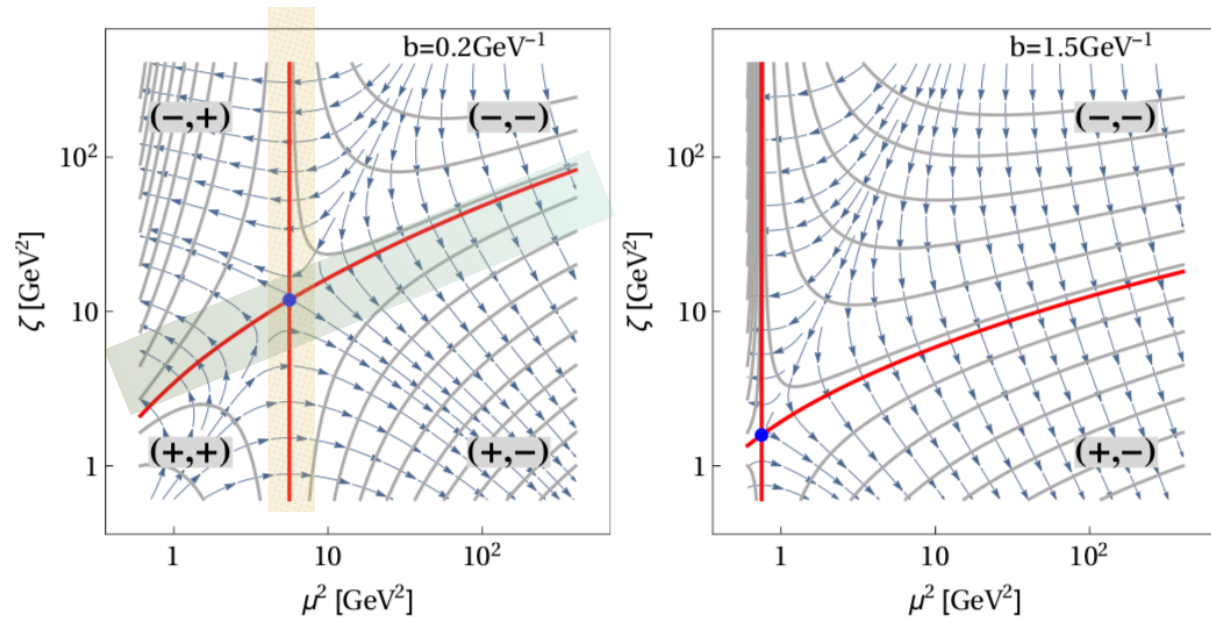
Evolution kernel

$$\ln R[b; \vec{\nu}_f \rightarrow \vec{\nu}_i] = U(\vec{\nu}_f, b) - U(\vec{\nu}_i, b)$$

with

$$U(\vec{\nu}, b) = \int^{\nu_1} \frac{\Gamma(s)s - \gamma_V(s)}{2} ds - \mathcal{D}(\vec{\nu}, b)\nu_2 + \text{const}(b)$$

2D EVOLUTION FIELD: NOTATION AND IDEAL CASE



● =saddle point

$\mathbf{E}(\vec{\nu}_{\text{saddle}}, b) = \vec{0}$
In the saddle
point all
anomalous
dimensions are 0
(No evolution)

Singularities: Landau pole (on the left, not shown)

Equipotential/null-evolution curves:

Special null-evolution curves:

$$\vec{\omega}(t, \vec{\nu}_B, b) = (t, \omega(t, \vec{\nu}_B, b)) \rightarrow \frac{d\vec{\omega}}{dt} \cdot \vec{\nabla} U(\vec{\omega}, b) = 0$$

$$\mu = \mu_{\text{saddle}} \text{ and } \vec{\nu}_B = \vec{\nu}_{\text{saddle}}$$

TRUNCATION OF THE PERTURBATIVE SERIES

The truncation introduces a path difference

$$\delta\Gamma(\mu, b) = \Gamma(\mu) - \mu \frac{d\mathcal{D}(\mu, b)}{d\mu},$$

$$\delta\Gamma^{(N)} = 2 \sum_{n=1}^N \sum_{k=0}^n n \bar{\beta}_{n-1}(a_s) a_s^{n-1} d^{(n,k)} \mathbf{L}_\mu^k$$

$$\text{with } \bar{\beta}_n(a_s) = \beta(a_s) - \sum_{k=0}^{n-1} \beta_k a_s^{k+2}$$

$$\delta\Gamma^{(N)} \sim \mathcal{O}(a_s^{N+1} \mathbf{L}_\mu^N) \text{ with perturbative } D$$

$$\delta\Gamma^{(N)} \sim \mathcal{O}(a_s^{N+1} \mathbf{L}_\mu) \text{ with resummed } D$$

$$\mathbf{L}_\mu = \ln \left(\frac{X^2 b^2}{4e^{-2\gamma_E}} \right)$$

$$\ln \frac{R[b; \{\mu_1, \zeta_1\} \xrightarrow{P_1} \{\mu_2, \zeta_2\}]}{R[b; \{\mu_1, \zeta_1\} \xrightarrow{P_2} \{\mu_2, \zeta_2\}]} = \frac{1}{2} \int_{\Omega(P_1 \cup P_2)} d^2 \nu \delta\Gamma(\vec{\nu}, b) = \int_{\mu_2}^{\mu_1} \frac{d\mu}{\mu} \delta\Gamma(\mu, b) \ln \left(\frac{\zeta_1(\mu)}{\zeta_2(\mu)} \right)$$

The path dependence is enhanced by the difference in rapidity scale

At large value of impact parameter the breaking of integrability condition becomes crucial

RECOVERING PATH INDEPENDENCE

Helmholtz decomposition
of evolution fields

$$\mathbf{E}(\vec{\nu}, b) = \tilde{\mathbf{E}}(\vec{\nu}, b) + \mathbf{\Theta}(\vec{\nu}, b)$$

Basic properties
of evolution fields

$$\text{curl} \tilde{\mathbf{E}} = 0, \quad \vec{\nabla} \cdot \mathbf{\Theta} = 0, \quad \tilde{\mathbf{E}} \cdot \mathbf{\Theta} = 0.$$

Scalar potentials

$$\tilde{\mathbf{E}}(\vec{\nu}, b) = \vec{\nabla} \tilde{U}(\vec{\nu}, b) \quad \mathbf{\Theta}(\vec{\nu}, b) = \text{curl } V(\vec{\nu}, b)$$

Ideally one could
repair the truncation
using decomposition of
the evolution field

$$\text{curl} \mathbf{E} = \text{curl} \mathbf{\Theta} = \frac{\delta \Gamma(\vec{\nu}, b)}{2} \neq 0$$

THE INTEGRABILITY CONDITION IS RE-ESTABLISHED DEFINING THE EVOLUTION KERNEL AS

$$\ln R[b; \vec{\nu}_f \rightarrow \vec{\nu}_i] = \tilde{U}(\vec{\nu}_f, b) - \tilde{U}(\vec{\nu}_i, b)$$

$$\nabla^2 \tilde{U}(\vec{\nu}, b) = \frac{1}{2} \frac{d\gamma_F(\vec{\nu})}{d\nu_1}$$

However in order to fix
completely the evolution potential
one needs boundary condition for
the evolution field:
at the moment no theoretically solid
non-perturbative input is known

RECOVERING PATH INDEPENDENCE

We modify anomalous dimensions such that integrability restored

$$\mu \frac{d\mathcal{D}(\mu, b)}{d\mu} = -\zeta \frac{d\gamma_F(\mu, \zeta)}{d\zeta}$$

It can be done from both sides of the equation.

Improved $\mathcal{D}$

Facilitate

$$\mu \frac{d\mathcal{D}}{d\mu} = \Gamma.$$

by

$$\mathcal{D}(\mu, b) = \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \Gamma(\mu) + \mathcal{D}(\mu_0, b)$$

- In the spirit of [Collins' text book].
- Already used in many studies
- However, it is not the best way

Improved γ

We set

$$\zeta \frac{d\gamma_F}{d\zeta} \equiv -\mu \frac{d\mathcal{D}}{d\mu} = \delta\Gamma - \Gamma$$

Or

$$\gamma_F(\mu, \zeta) \rightarrow \gamma_M(\mu, \zeta, b)$$

$$\gamma_M = (\Gamma - \delta\Gamma) \ln\left(\frac{\mu^2}{\zeta}\right) - \gamma_V$$

- Completely self consistent
- Very natural

IMPROVED-D SCENARIO

$$\mathcal{D}(\mu, b) = \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma(\mu') + \mathcal{D}(\mu_0, b) \longrightarrow \tilde{U}(\vec{\nu}, b; \mu_0) = \int_{\ln \mu_0^2}^{\nu_1} \frac{\Gamma(s)(s - \nu_2) - \gamma_V(s)}{2} ds - \mathcal{D}(\mu_0, b)\nu_2 + \text{const}(b)$$

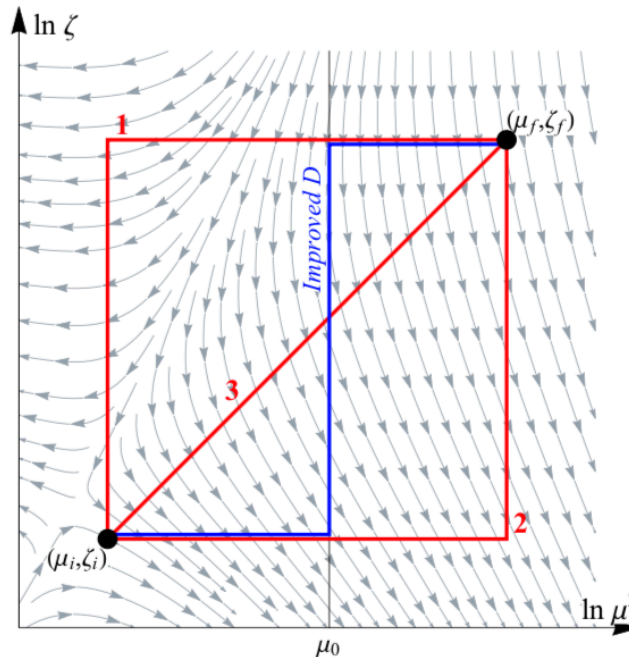
The truncation effects should be minimized by the choice of μ_0

$$\ln R[b; (\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i); \mu_0] = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \left(\Gamma(\mu) \ln \left(\frac{\mu^2}{\zeta_f} \right) - \gamma_V(\mu) \right) - \int_{\mu_0}^{\mu_i} \frac{d\mu}{\mu} \Gamma(\mu) \ln \left(\frac{\zeta_f}{\zeta_i} \right) - \mathcal{D}(\mu_0, b) \ln \left(\frac{\zeta_f}{\zeta_i} \right).$$

This is a mixture of solution 1 and 2.

The **solution dependence** is parameterized by μ_0

In order to compare fits one should agree on a conventional μ_0 scale



The minimization occurs only when one finds a μ_0 such that

$$\delta \Gamma(\mu_0, b) = 0$$

IMPROVED- γ SCENARIO

$$\gamma_M(\mu, \zeta, b) = (\Gamma(\mu) - \delta\Gamma(\mu, b))\mathbf{l}_\zeta - \gamma_V(\mu)$$



$$\tilde{U}(\vec{\nu}, b) = \int^{\nu_1} \frac{(\Gamma(s) - \delta\Gamma(s, b))s - \gamma_V(s)}{2} ds - \mathcal{D}(\vec{\nu}, b)\nu_2 + \text{const}(b)$$

$$\ln R[b; (\mu_f, \zeta_f) \rightarrow (\mu_i, \zeta_i)] = - \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} (2\mathcal{D}(\mu, b) + \gamma_V(\mu)) + \mathcal{D}(\mu_f, b) \ln \left(\frac{\mu_f^2}{\zeta_f} \right) - \mathcal{D}(\mu_i, b) \ln \left(\frac{\mu_i^2}{\zeta_i} \right)$$

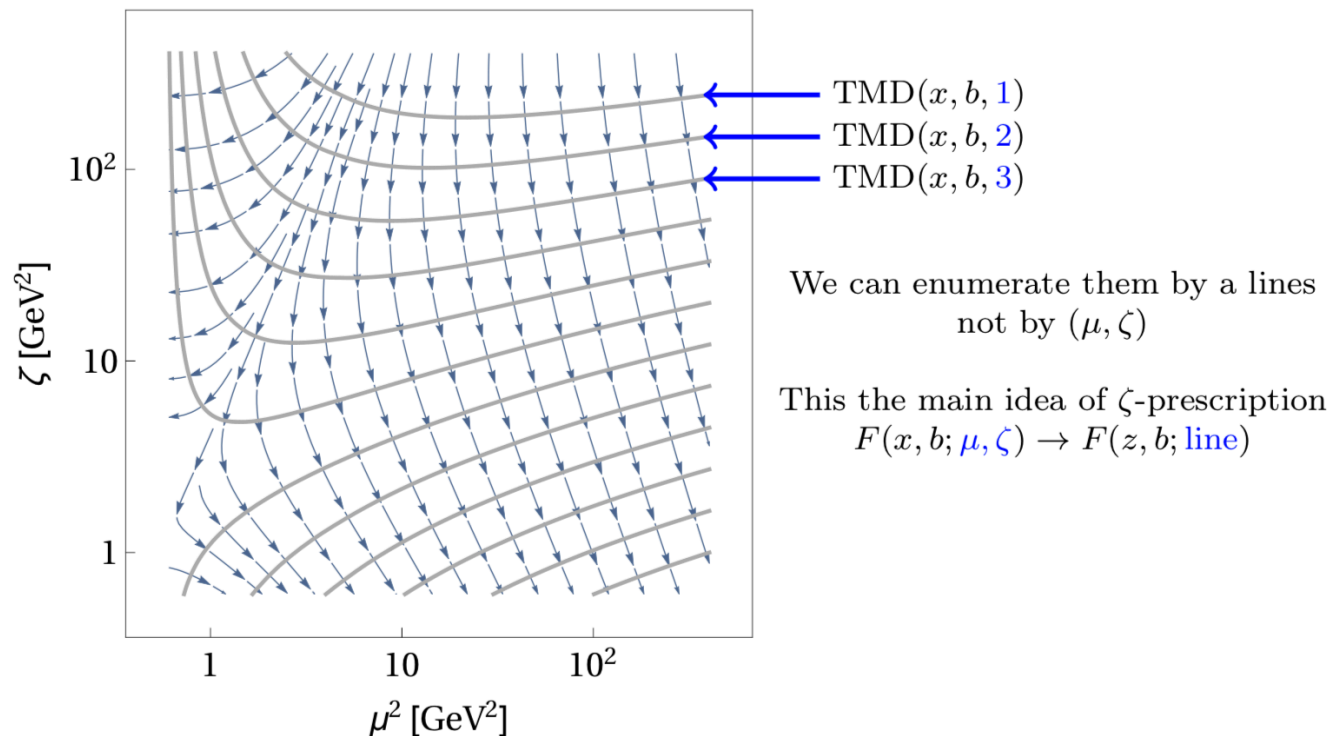
CLEAR ADVANTAGES:

- ★ NO MORE THE INTERMEDIATE SCALE μ_0
- ★ PATH INDEPENDENCE
- ★ SIMPLICITY
- ★ WE ACHIEVE A CLEAR SEPARATION OF EVOLUTION AND NON-PERTURBATIVE PART OF THE TMD

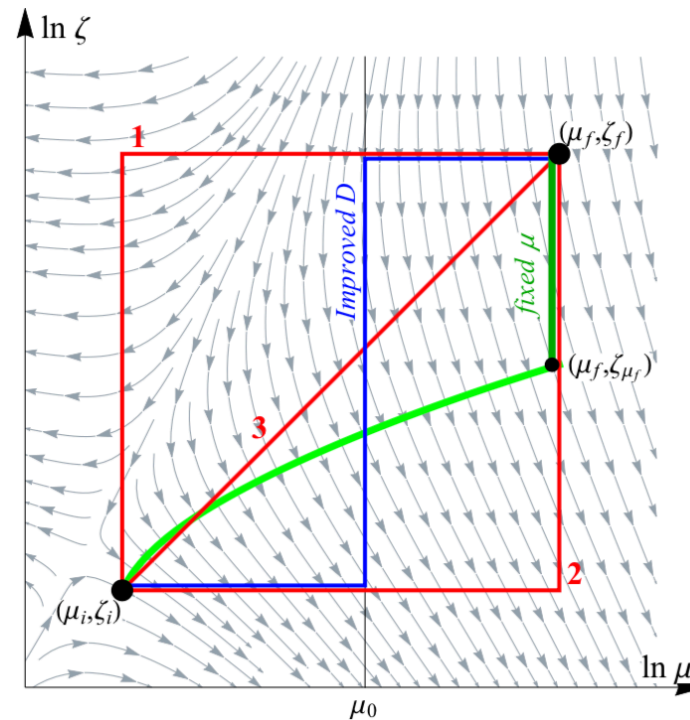
EQUIVALENT TMDs: EQUIPOTENTIAL LINES

The 2-D evolution just connects TMDs on different equipotential lines

TMD distributions on the same equipotential line are equivalent.



EQUIVALENT TMDs: EQUIPOTENTIAL LINES



Field lines orthogonal to equipotential lines

The green line evolves first on an *equipotential line* and then on a vertical line.

TMD ON EQUIPOTENTIAL LINES

The TMDs on equipotential lines are not evolved
so one can define a TMD by a single parameter line

$$F(x, b; \vec{\nu}_B) = F(x, b; \vec{\nu}'_B), \quad \vec{\nu}'_B \in \vec{\omega}(\vec{\nu}_B, b).$$

ONE CAN HAVE AN EVOLUTION ONLY WHEN MOVING BETWEEN
DIFFERENT LINES

$$F(x, b; \vec{\nu}_B) = R[b; \vec{\nu}_B \rightarrow \vec{\nu}'_B] F(x, b; \vec{\nu}'_B)$$

Outcome: the modeling of the non-perturbative part of the TMD does not depend anymore on the relation between renormalization scale and coordinate parameter.

Question: Is there a preferred line?

ζ -prescription

We have just to evolve between different equipotential / null-evolution line

$$F(x, b; \mu_f, \zeta_f) = R[b; (\mu_f, \zeta_f) \rightarrow (\mu_f, \zeta_{\mu_f}(\vec{\nu}_B, b))] F(x, b; \vec{\nu}_B)$$

This is realized choosing $\zeta_\mu(b)$ such that

$$\Gamma_{\text{cusp}}(\mu) \ln \left(\frac{\mu^2}{\zeta_\mu(b)} \right) - \gamma_V(\mu) = 2\mathcal{D}(\mu, b) \frac{d \ln \zeta_\mu(b)}{d \ln \mu^2},$$

With boundary condition

$$\begin{aligned} \mathcal{D}(\mu_{\text{saddle}}, b) &= 0 \\ \gamma_F(\mu_{\text{saddle}}, \zeta_{\text{saddle}}) &= 0 \end{aligned}$$

The TMD on the saddle point has no evolution (Optimal TMD).

We write it as $F(x, b)$ (NO scale!!)

$$\gamma_F(\mu, \zeta) = \Gamma_{\text{cusp}}(\mu) \ln \left(\frac{\mu^2}{\zeta} \right) - \gamma_V(\mu).$$

MODELING NON-PERTURBATIVE INPUTS FOR TMD EXTRACTION

The only thing that we are left to do is modeling including the asymptotic limit

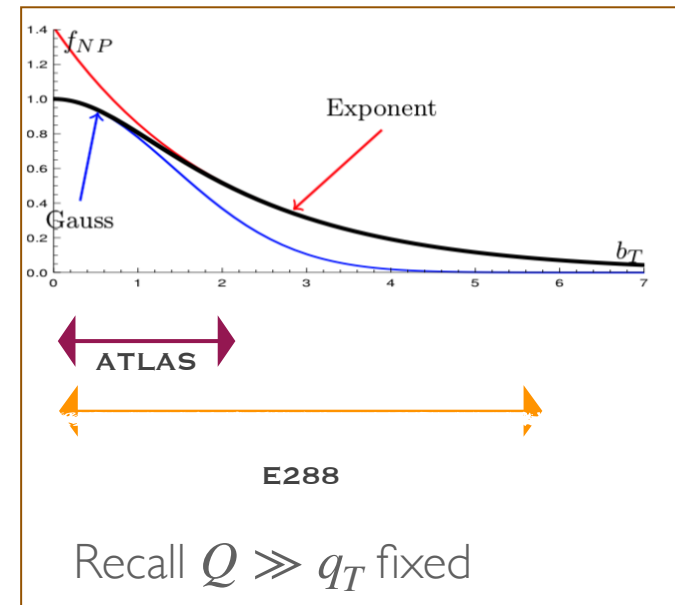
$$F_{q \leftarrow h}(x, \mathbf{b}) = \int_x^1 \frac{dz}{z} \sum_f C_{q \leftarrow f}(z, \mathbf{b}; \mu, \zeta_\mu) f_{f \leftarrow h}\left(\frac{x}{z}, \mu\right) f_{NP}(x, \mathbf{b})$$

Perturbative Wilson coefficient matching

PDF

Non-perturbative non-asymptotic part

Asymptotic (small- b) limit of TMD for large transverse momentum



Experiments at different energy allow to explore different parts of the distributions

CROSS SECTION AND TMD STRUCTURE (GENERAL SCHEME)

$$\frac{d\sigma}{dQ^2 dy d(q_T^2)} = \frac{4\pi}{3N_c} \frac{\mathcal{P}}{sQ^2} \sum_{GG'} z_{ll'}^{GG'}(q) \sum_{ff'} z_{ff'}^{GG'} |C_V(q, \mu)|^2 \int \frac{d^2 \vec{b}}{4\pi} e^{i(\vec{b} \vec{q})}$$

Lepton tensor cuts E.w. factors Hard Coefficient

$$\times F_{f \leftarrow h_1}(x_1, \vec{b}; \mu, \zeta) F_{f' \leftarrow h_2}(x_2, \vec{b}; \mu, \zeta) + \text{Power corrections at high } q_T$$

Evolution factor Low energy TMD

$$F_{f \leftarrow h}(x, \mathbf{b}; \mu_f, \zeta_f) = R^f[\mathbf{b}; (\mu_f, \zeta_f) \leftarrow (\mu_i, \zeta_i)] F_{f \leftarrow h}(x, \mathbf{b}; \mu_i, \zeta_i)$$

$$R^f[\mathbf{b}; (\mu_f, \zeta_f) \leftarrow (\mu_i, \zeta_i)] = \exp \left[\int_P \left(\gamma_F^f(\mu, \zeta) \frac{d\mu}{\mu} - \mathcal{D}^f(\mu, \mathbf{b}) \frac{d\zeta}{\zeta} \right) \right]$$

In principle $\boxed{\mu} \neq \mu$

Generic prescription

$$F_{f \leftarrow h}(x, \vec{b}; \mu, \zeta) = \sum_q \int_x^1 \frac{dz}{z} C_{f \leftarrow q}(z, \mathbf{L}_{\boxed{\mu}}, \mu, \zeta) f_{q \leftarrow h} \left(\frac{x}{z}, \boxed{\mu} \right) f_{NP}(x, \mathbf{b})$$

Matching (Wilson) coefficient

PDF

Gaussian? Exponential?

ζ -prescription

$((\mu_i, \zeta_i) = (\mu_{\text{saddle}}, \zeta_{\text{saddle}}))$

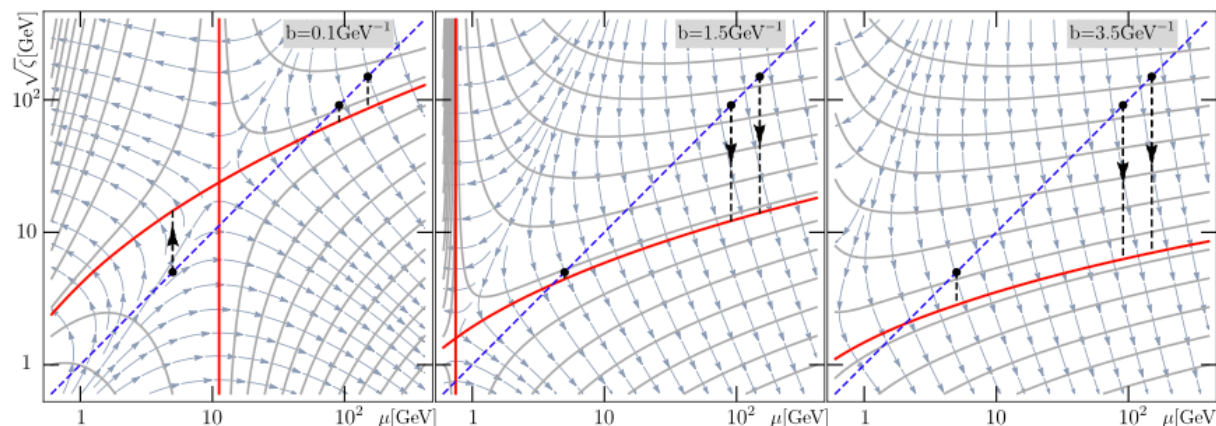
$$F_{f \leftarrow h}(x, \vec{b}) = \sum_q \int \frac{dz}{z} C_{f \leftarrow q}(z, \mathbf{L}_{\mu}) f_{q \leftarrow h} \left(\frac{x}{z}, \mu \right) f_{NP}(x, \mathbf{b})$$

CHECK THE T-SHIRT: CAN YOU RECOGNIZE EACH TERM?

$$\frac{d\sigma}{dq_T} \sim H \int d^2b \, e^{-iq_T \cdot b_T} [R \cdot F \cdot D](b)$$

SUMMARY OF RESUMMATION AND ζ -PRESCRIPTION

The potential has level lines (zero evolution curves) and a saddle point



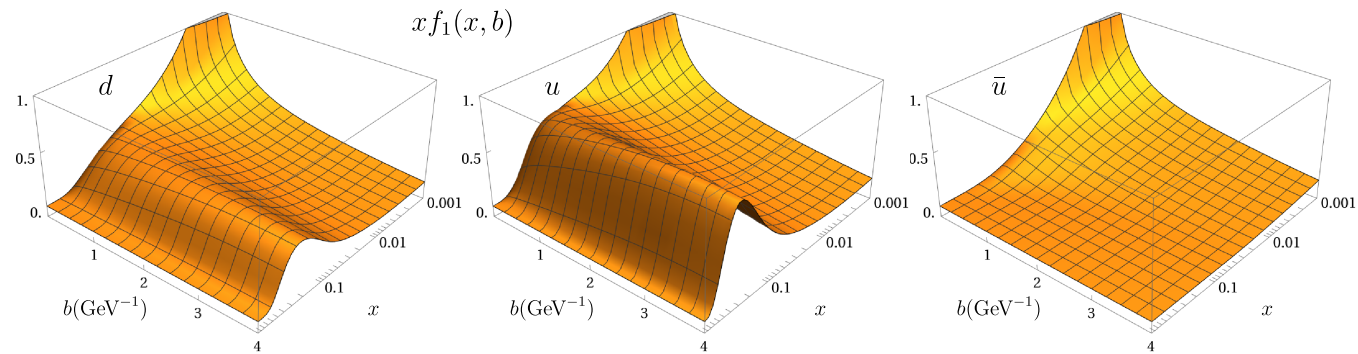
$$\vec{E}(\nu_{\text{saddle}}, b) = \vec{\nabla} U(\nu_{\text{saddle}}, b) = \frac{1}{2}(\gamma_F(\nu_{\text{saddle}}), -\mathcal{D}(\nu_{\text{saddle}}, b)) = 0$$

$$F(x, b, \mu_{\text{saddle}}, \zeta_{\text{saddle}}) = F_{\text{opt}}(x, b)$$

$$F(x, b, \mu, \zeta) = R(b, \mu, \zeta) F_{\text{opt}}(x, b) = \exp[-\mathcal{D}(b) \ln(\zeta/\zeta_{\text{saddle}})] F_{\text{opt}}(x, b)$$

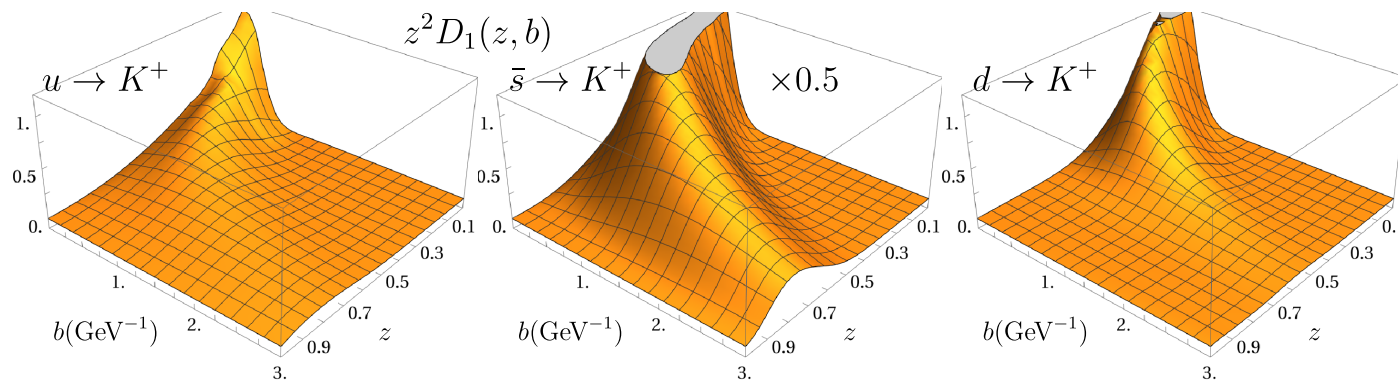
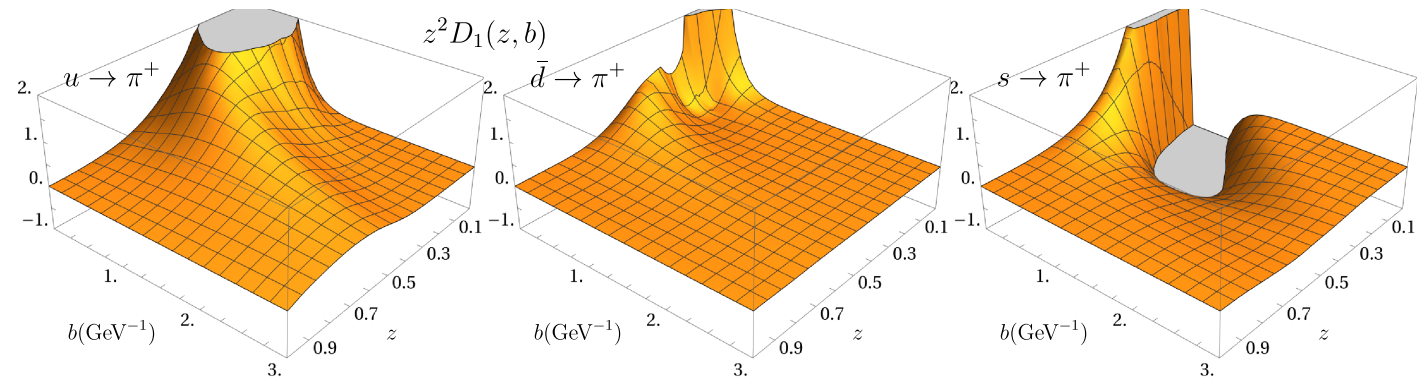
The saddle point has a non-perturbative definition calculable as a kernel function and allows defining optimal, scale-independent TMDs

TMD'S IN COORDINATE SPACE FROM ART25



TMDPDFs

π -TMDFF



K -TMDFF

.....

TMD TO COLLINEAR FUNCTION CORRESPONDENCE

Evolution of TMD and PDF. The CS kernel.

DGLAP EQUATIONS

Integro-differential equations

Non-diagonals in flavor space

$$\mu^2 \frac{d}{d\mu^2} f_q(x, \mu) = \sum_{f'} \int_x^1 \frac{dy}{y} P_{q \rightarrow q'}(y) f_{q'}\left(\frac{x}{y}, \mu\right)$$

μ = *UV renormalization scale*

ζ = *rapidity scale*

COLLINS-SOPER EQUATIONS

Double scale differential equations

Diagonals in flavor space

$$\begin{cases} \frac{d}{d \ln \mu} F(x, b; \mu, \zeta) = \gamma_F(\mu, \zeta) F(x, b; \mu, \zeta) \\ \frac{d}{d \ln \zeta} F(x, b; \mu, \zeta) = -\mathcal{D}(b, \mu) F(x, b; \mu, \zeta) \end{cases}$$

$$\frac{d}{d \ln \zeta} \mathcal{D}(b, \mu) = \Gamma_{\text{cusp}}(\mu) = -\frac{d}{d \ln \mu} \gamma_F(\mu, \zeta)$$

$$\gamma_F(\mu, \zeta) = -\gamma_V(\mu) + \Gamma_{\text{cusp}}(\mu) \ln \frac{\mu^2}{\zeta}$$

The Collins-Soper kernel $\mathcal{D}$ is a fundamental QCD quantity (as the strong coupling)

TRANSVERSE MOMENTUM MOMENTS

- TMMs are weighted integrals with an upper cut-off

$$\mathcal{M}_{\nu_1 \dots \nu_n}^{[\Gamma]}(x, \mu) \equiv \int^\mu d^2 \vec{k}_T \vec{k}_{T\nu_1} \dots \vec{k}_{T\nu_n} F^{[\Gamma]}(x, k_T)$$

for TMDs in the ζ -prescription which has no scale dependence

$$\mathcal{M}_{\nu_1 \dots \nu_n}^{*[\Gamma]}(x, \mu) \equiv \int^\mu d^2 \vec{k}_T \vec{k}_{T\nu_1} \dots \vec{k}_{T\nu_n} F^{[\Gamma]}(x, k_T; \mu, \mu^2)$$

for TMDs in the general prescription *For 0-moment: M. Ebert, J. Michel, I. Stewart, Z. Sun, JHEP 07 (2022) 129*

- 🌐 The upper cut-off becomes the scale at which the collinear functions are evaluated
- 🌐 TMMs obey DGLAP equations
- 🌐 We provide a definition for all moments

TMDS IN b -SPACE AND $\mathcal{G}$ OPERATION

Fourier transformation: angular integrations are trivial

D. Boer, L. Gamberg, B. Musch, and A. Prokudin, JHEP 10, 021 (2011)

$$\tilde{F}^{(n)}(x, b_T; \mu, \zeta) \equiv n! \left(\frac{-1}{M^2 b} \partial_b \right)^n \tilde{F}(x, b; \mu, \zeta) = \frac{2\pi n!}{(bM)^n} \int_0^\infty dk_T k_T \left(\frac{k_T}{M} \right)^n J_n(bk_T) F(x, k_T; \mu, \zeta)$$

$$\tilde{f}_1(x, b) \equiv \tilde{f}_1^{(0)}(x, b),$$

$$\tilde{g}_1(x, b) \equiv \tilde{g}_1^{\perp(0)}(x, b),$$

$$\tilde{h}_1(x, b) \equiv \tilde{h}_1^{\perp(0)}(x, b),$$

$$\tilde{h}_1^\perp(x, b) \equiv \tilde{h}_1^{\perp(1)}(x, b),$$

$$f_{1T}^\perp(x, b) \equiv \tilde{f}_{1T}^{\perp(1)}(x, b),$$

$$\tilde{g}_{1T}^\perp(x, b) \equiv \tilde{g}_{1T}^{\perp(1)}(x, b),$$

$$\tilde{h}_{1L}^\perp(x, b) \equiv \tilde{h}_{1L}^{\perp(1)}(x, b),$$

$$\tilde{h}_{1T}^\perp(x, b) \equiv \tilde{h}_{1T}^{\perp(2)}(x, b).$$

The superscript (m) determines the large k_T asymptotic

$$f(x, k_T) \propto \frac{M^{2m}}{(k_T^2)^{m+1}}$$

Formally

$$\lim_{b \rightarrow 0} \tilde{f}^{(n)}(x, b) = \int d^2 \mathbf{k}_T \left(\frac{\mathbf{k}_T^2}{2M^2} \right)^n f(x, k_T) \equiv f^{(n)}(x),$$

TMDs IN b -SPACE AND $\mathcal{G}$ OPERATION

- Operation $\mathcal{G}$ is defined as

$$\mathcal{G}_{n,m}[f](x, \mu) = \int^\mu d^2 \mathbf{k}_T \left(\frac{\mathbf{k}_T^2}{2M^2} \right)^n f(x, k_T)$$

- Without cut-off it corresponds to the conventional n^{th} moment of TMD, m is the corresponding superscript of the TMD $\tilde{f}$
- Its properties: $n = m$ logarithmic divergence, $n = m + l$ power divergence in μ

$$\begin{aligned} \mathcal{G}_{m,m}[f](x, \mu) &\propto \log(\mu) , \\ \mathcal{G}_{m+l,m}[f](x, \mu) &\propto \mu^{2l} \text{ for } m + l \geq 0 \end{aligned}$$

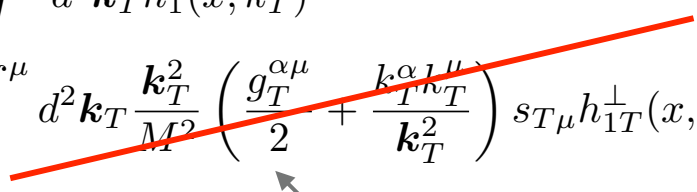
- The logarithmic divergence for $n = m$ is the UV divergence that corresponds to the divergence of the collinear functions

ZEROth TMM

► The 0th TMM is

Do we recover the PDF?

$$\begin{aligned}
 \mathcal{M}^{[\gamma^+]}(x, \mu) &= \int^\mu d^2 \mathbf{k}_T F^{[\gamma^+]}(x, k_T) = \int^\mu d^2 \mathbf{k}_T f_1(x, k_T), \\
 \mathcal{M}^{[\gamma^+ \gamma^5]}(x, \mu) &= \int^\mu d^2 \mathbf{k}_T F^{[\gamma^+ \gamma^5]}(x, k_T) = \lambda \int^\mu d^2 \mathbf{k}_T g_1(x, k_T), \\
 \mathcal{M}^{[i\sigma^{\alpha+} \gamma^5]}(x, \mu) &= \int^\mu d^2 \mathbf{k}_T F^{[i\sigma^{\alpha+} \gamma^5]}(x, k_T) = s_T^\alpha \int^\mu d^2 \mathbf{k}_T h_1(x, k_T) \\
 &\quad - \int^\mu d^2 \mathbf{k}_T \frac{\mathbf{k}_T^2}{M^2} \left(\frac{g_T^{\alpha\mu}}{2} + \frac{k_T^\alpha k_T^\mu}{\mathbf{k}_T^2} \right) s_{T\mu} h_{1T}^\perp(x, k_T),
 \end{aligned}$$


 $\propto \mu^{-2}$ so we drop it

ZEROth TMM

In practice we obtain PDF in a certain scheme (we call it TMD-scheme)

$$\begin{aligned}
 \mathcal{M}^{[\gamma^+]}(x, \mu) &= \mathcal{G}_0[f_1](x, \mu), & \mathcal{G}_0[f_1](x, \mu) &= q^{(\text{TMD})}(x, \mu) + \mathcal{O}(\mu^{-2}), \\
 \mathcal{M}^{[\gamma^+ \gamma^5]}(x, \mu) &= s_L \mathcal{G}_0[g_1](x, \mu), & \mathcal{G}_0[g_1](x, \mu) &= \Delta q^{(\text{TMD})}(x, \mu) + \mathcal{O}(\mu^{-2}), \\
 \mathcal{M}^{[i\sigma^{\alpha+} \gamma^5]}(x, \mu) &= s_T^\alpha \mathcal{G}_0[h_1](x, \mu), & \mathcal{G}_0[h_1](x, \mu) &= \delta q^{(\text{TMD})}(x, \mu) + \mathcal{O}(\mu^{-2}).
 \end{aligned}$$

Using Wilson coefficients of small- b and large- μ asymptotic expansion of Hankel transform one obtains

$$\begin{aligned}
 \mathcal{G}_0[F](x, \mu) &= \left\{ \mathbf{1} + \alpha_s C_1 + \alpha_s^2 C_2 \right. \\
 &\quad \left. + \alpha_s^3 \left[\frac{2\zeta_3}{3} (P_1 \otimes P_1 \otimes P_1 - 3\beta_0 P_1 \otimes P_1 + 2\beta_0^2 P_1) + C_3 \right] + \mathcal{O}(\alpha_s^4) \right\} \otimes f(x, \mu) + \mathcal{O}(\mu^{-2}),
 \end{aligned}$$

R. Wong, Computers & Mathematics with Applications 3, 271 (1977).
R. F. MacKinnon, Mathematics of Computation 26, 515 (1972).

ZEROth TMM

- All scales in the TMD are set to μ and we have a DGLAP equation

$$\mu^2 \frac{d}{d\mu^2} f^{(\text{TMD})}(x, \mu) = P' \otimes f^{(\text{TMD})}(x, \mu).$$

- Therefore it is the same as PDFs but computed in a different scheme.
- The difference in splitting functions is of order α_s^2 and it is calculable

$$P' - P = -\alpha_s^2 \beta_0 C_1 - \alpha_s^3 (2\beta_0 C_2 - \beta_0 C_1 \otimes C_1 + \beta_1 C_1) + \mathcal{O}(\alpha_s^4).$$

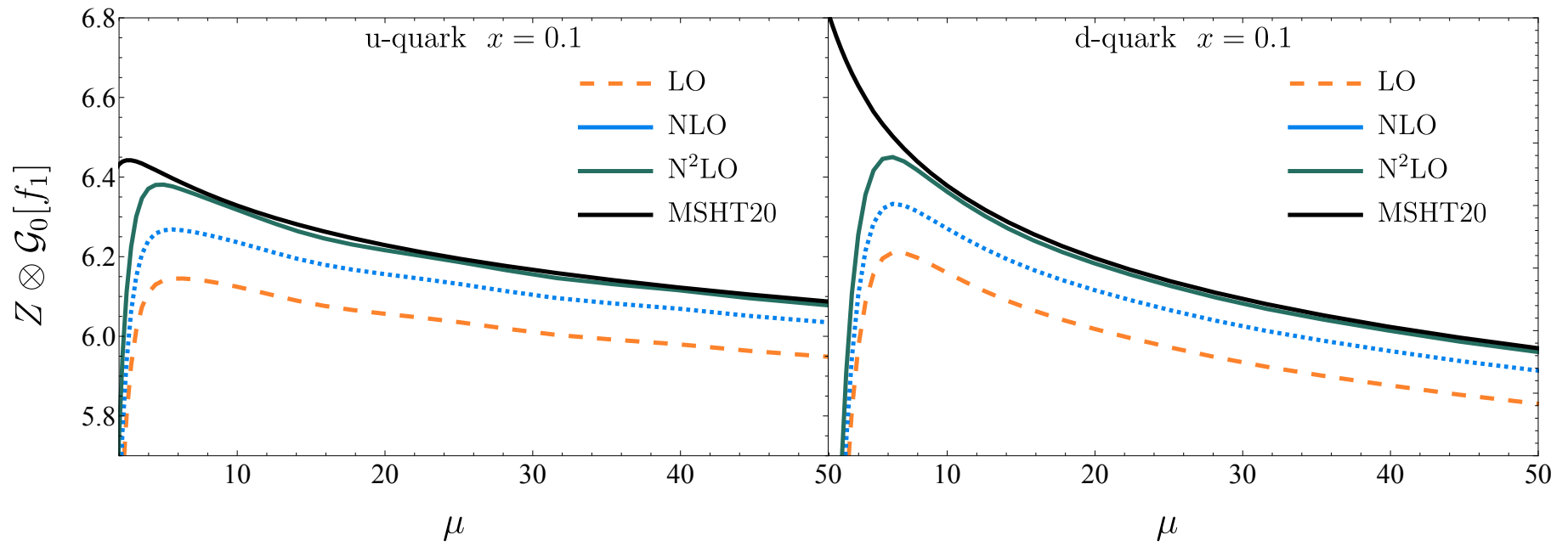
ZEROth TMM

 We call this scheme TMD-scheme and the coefficient to transform to $\overline{MS}$ scheme reads

$$f_f^{(\overline{MS})}(x, \mu) = \sum_{f'} \int_x^1 \frac{dy}{y} Z_{f \leftarrow f'}^{\overline{MS}/\text{TMD}}(y, \mu) f_{f'}^{(\text{TMD})}\left(\frac{x}{y}, \mu\right)$$

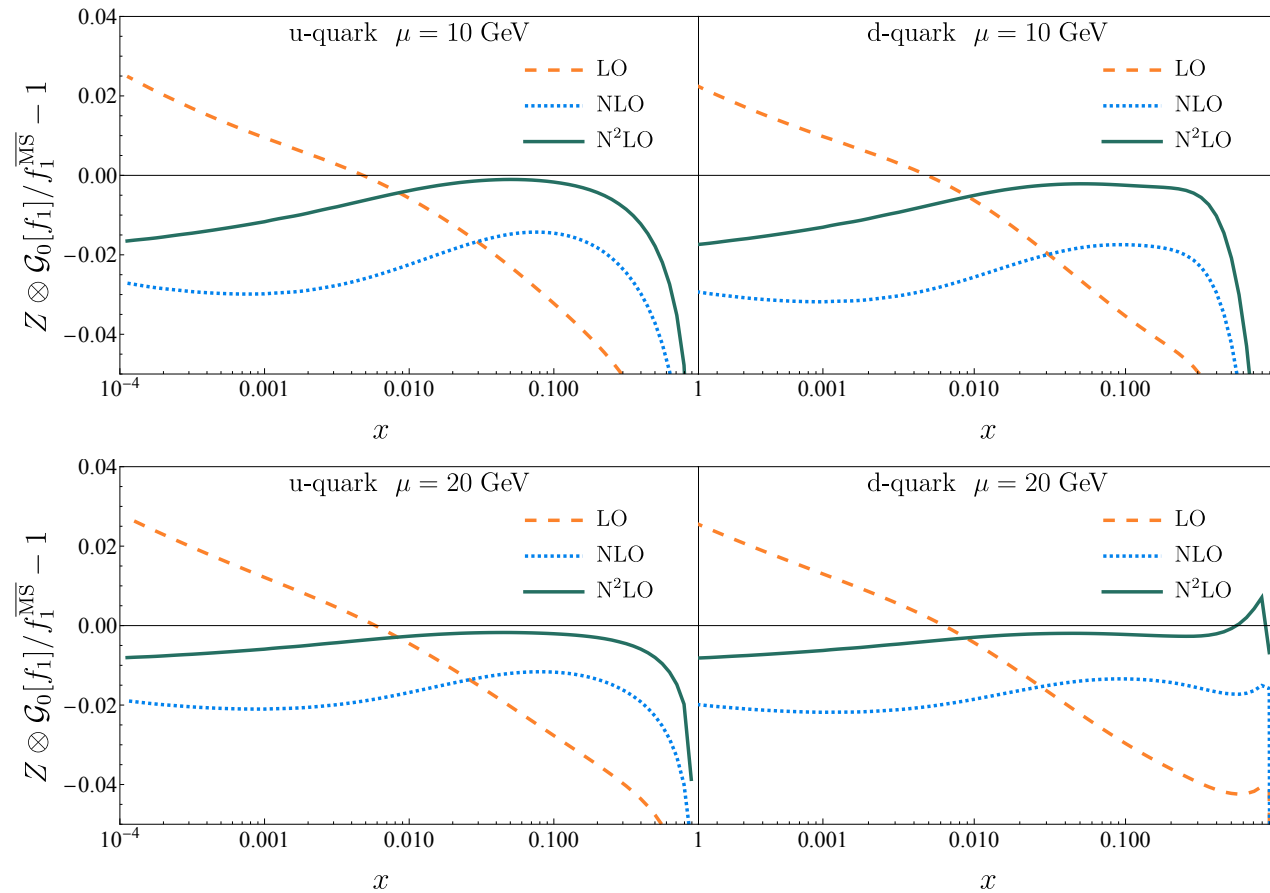
$$Z^{\overline{MS}/\text{TMD}} = \mathbf{1} - \alpha_s C_1 - \alpha_s^2 (C_2 - C_1 \otimes C_1) \\ - \alpha_s^3 \left[C_3 + C_1 \otimes C_1 \otimes C_1 - C_1 \otimes C_2 - C_2 \otimes C_1 + \frac{2\zeta_3}{3} P_1 \otimes (P_1 - \beta_0 \cdot \mathbf{1}) \otimes (P_1 - 2\beta_0 \cdot \mathbf{1}) \right] + \mathcal{O}(\alpha_s^4)$$

ZEROth TMM



🌀 Above $\mu \geq 5$ GeV the correspondence is quite precise

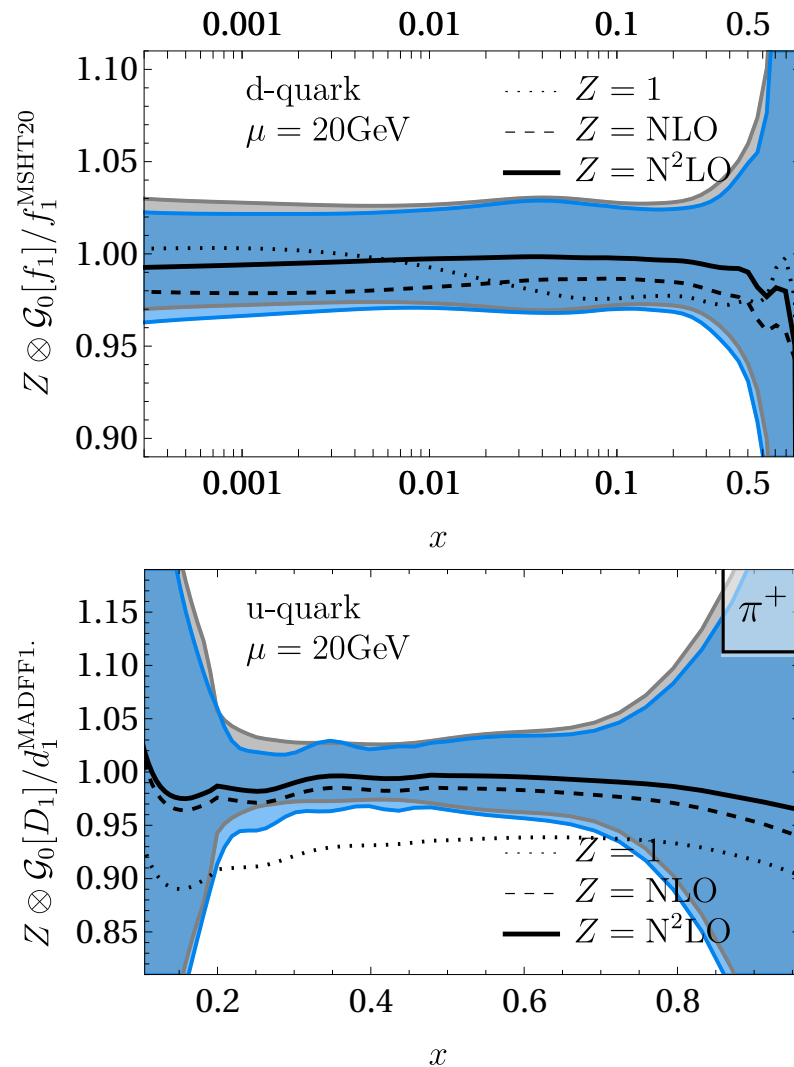
ZEROth TMM



► TMDs are from ART 23 extraction

*V. Moos, I. Scimemi, A. Vladimirov, and P. Zurita, (2023),
arXiv:2305.07473*

TMD 0-MOMENTS VS ORIGINAL PDF IN ART25



The method is robust: we recover the collinear function with consistent error.

FIRST TMM

 The 1st TMM is related to the small-b power expansion of a TMD

$$\begin{aligned}\mathcal{G}_1[f_{1T}^\perp](x, \mu) &= \pm \frac{\pi}{2} T^{(\text{TMD})}(-x, 0, x; \mu) + \mathcal{O}(\mu^{-2}), \\ \mathcal{G}_1[g_{1T}^\perp](x, \mu) &= \frac{x}{2} \int_x^1 \frac{dy}{y} \Delta q^{(\text{TMD})}(y, \mu) + x \int_{-1}^1 dy_1 dy_2 dy_3 \delta(y_1 + y_2 + y_3) \int_0^1 d\alpha \delta(x - \alpha y_3) \left[\frac{\Delta T^{(\text{TMD})}(y_{123}; \mu)}{y_2^2} + \frac{T^{(\text{TMD})}(y_{123}; \mu) - \Delta T^{(\text{TMD})}(y_{123}; \mu)}{2y_2 y_3} \right] + \mathcal{O}(\mu^{-2}), \\ \mathcal{G}_1[h_{1L}^\perp](x, \mu) &= -\frac{x^2}{2} \int_x^1 \frac{dy}{y} \delta q^{(\text{TMD})}(y, \mu) \\ &\quad - x \int_{-1}^1 dy_1 dy_2 dy_3 \delta(y_1 + y_2 + y_3) \int_0^1 d\alpha \delta(x - \alpha y_3) H^{(\text{TMD})}(y_{123}; \mu) \frac{y_3 - y_2}{y_2^2 y_3} + \mathcal{O}(\mu^{-2}), \\ \mathcal{G}_1[h_1^\perp](x, \mu) &= \mp \frac{\pi}{2} E^{(\text{TMD})}(-x, 0, x; \mu) + \mathcal{O}(\mu^{-2}),\end{aligned}$$

I. S., A. Vladimirov, Eur. Phys. J. C 78, 802 (2018), F. Rein, S. Rodini, A. Schäfer, and A. Vladimirov, JHEP 01, 116 (2023)

 The evolution is of the correct DGLAP-type...

...With a difference at NLO

$$\begin{aligned}\mu^2 \frac{d}{d\mu^2} \mathcal{G}_1[F](x, \mu) &= R_t \otimes P'_t \otimes t + \mathcal{O}(\alpha_s^2) \\ P'_t - P_t &= \mathcal{O}(\alpha_s^2)\end{aligned}$$

QIU-STERMAN FUNCTIONS

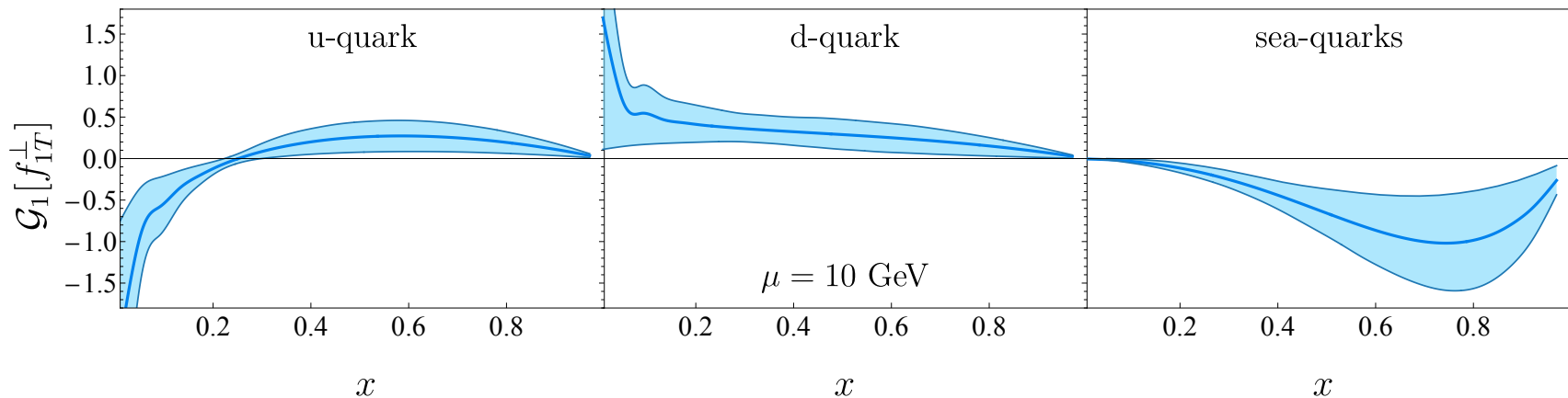
Oscar del Rio, Alexei Prokudin, Ignazio Scimemi, Alexey Vladimirov e-Print: 2402.01836 (2024)

Even though it is not possible to relate the 1st TMM of the Sivers functions to the full twist-3 functions with 3 variables $T(x_1, x_2, x_3)$, it is related to Qiu-Sterman functions $T(-x, 0, x; \mu)$

$$\langle \mathbf{k}_{T,1}^u \rangle = -0.011_{-0.023}^{+0.011} \text{ GeV}, \quad \langle \mathbf{k}_{T,1}^d \rangle = 0.17_{-0.17}^{+0.21} \text{ GeV}, \quad \langle \mathbf{k}_{T,1}^{sea} \rangle = -0.26_{-0.32}^{+0.26} \text{ GeV}$$

$$\langle \mathbf{k}_{T,1}^g \rangle \simeq 0.14_{-0.14}^{+0.31} \text{ GeV}$$

potentially sizable gluon Sivers function



Using M. Bury, A. Prokudin, A. Vladimirov, Phys.Rev.Lett. 126 (2021)

SECOND TMM

The 2nd moment is power divergent

$$\mathcal{M}_{\mu\nu,\text{div}}^{[\gamma^+]}(x, \mu) = -g_{T,\mu\nu} M^2 \mathcal{G}_{1,0}[f_1],$$

$$\mathcal{M}_{\mu\nu,\text{div}}^{[\gamma^+\gamma^5]}(x, \mu) = -\lambda g_{T,\mu\nu} M^2 \mathcal{G}_{1,0}[g_1],$$

$$\mathcal{M}_{\mu\nu,\text{div}}^{[i\sigma^{\alpha+}\gamma^5]}(x, \mu) = s_{T,\alpha} g_{T,\mu\nu} M^2 \mathcal{G}_{1,0}[h_1] + (g_{T,\mu\alpha} s_{T,\nu} + g_{T,\nu\alpha} s_{T,\mu} - g_{T,\mu\nu} s_{T,\alpha}) \frac{M^2}{2} \mathcal{G}_2[h_{1T}^\perp]$$

The asymptotic power divergence part is computed analytically ...

$$\mathcal{G}_{n+1,n}[F](x, \mu) = \frac{\mu^2}{2M^2} \text{AS}[\mathcal{G}_{n+1,n}[F]](x, \mu) + \overline{\mathcal{G}}_{n+1,n}[F](x, \mu),$$

... the width of TMDs

$$\langle \mathbf{k}_T^2 \rangle = -g_T^{\mu\nu} \mathcal{M}_{\mu\nu}^{[\gamma^+]} = 2M^2 \overline{\mathcal{G}}_{1,0}[f_1]$$

SECOND TMM

Oscar del Rio, Alexei Prokudin, Ignazio Scimemi, Alexey Vladimirov e-Print: 2402.01836 (2024)

$$\mathcal{M}_{\mu\nu}^{[\gamma^+]}(x, \mu) = \int^\mu d^2\mathbf{k}_T \mathbf{k}_{T\mu} \mathbf{k}_{T\nu} F^{[\gamma^+]}(x, k_T) = \int^\mu d^2\mathbf{k}_T \mathbf{k}_{T\mu} \mathbf{k}_{T\nu} f_1(x, k_T),$$

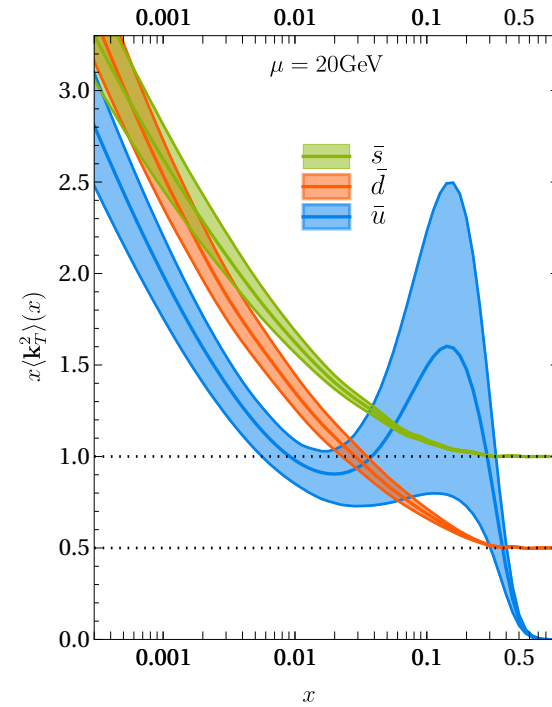
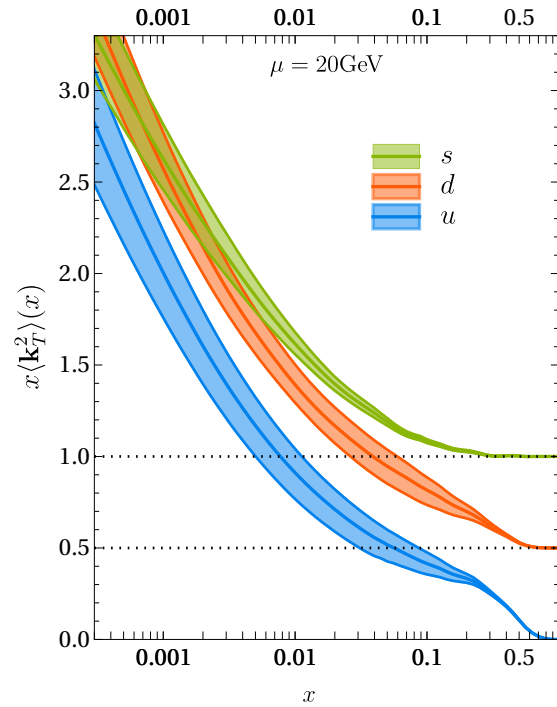
The power divergence coefficients must be subtracted up to the order N³LO

$$\langle \vec{k}_T^2 \rangle(x, \mu) = \int^\mu d^2\vec{k}_T \vec{k}_T^2 F(x, \vec{k}_T) - \mu^2 \mathbf{AS}[F](x, \mu).$$

The integration in x is finite in the combination

$$\langle \vec{k}_T^2 \rangle_{q\text{-val.}}(\mu) = \int_0^1 dx \left[\langle \vec{k}_T^2 \rangle_q(x, \mu) - \langle \vec{k}_T^2 \rangle_{\bar{q}}(x, \mu) \right]$$

AVERAGE $\vec{k}_T^2$ FROM ART25



$$\langle \vec{k}_T^2(\mu = 20 \text{ GeV}) \rangle_{d\text{-val.}} = 1.0^{+0.4}_{-0.5} \text{ GeV}^2,$$

$$\langle \vec{k}_T^2(\mu = 20 \text{ GeV}) \rangle_{u\text{-val.}} = -4.7^{+3.4}_{-3.5} \text{ GeV}^2$$

PHENOMENOLOGY

In principle we can study polarized and non-polarized processes and (many of) the angular asymmetries with their perturbative effects of higher orders, with summation and non-perturbative effects included (ask Sara...)...

$$\begin{aligned}\frac{d\sigma}{d^4q d\Omega} &= \frac{3}{16\pi} \frac{d\sigma}{d^4q} \left[(1 + \cos^2 \theta) + \frac{1 - 3 \cos^2 \theta}{2} A_0 + \sin 2\theta \cos \phi A_1 + \frac{\sin^2 \theta \cos 2\phi}{2} A_2 \right. \\ &\quad \left. + \sin \theta \cos \phi A_3 + \cos \theta A_4 + \sin^2 \theta \sin 2\phi A_5 + \sin 2\theta \sin \phi A_6 + \sin \theta \sin \phi A_7 \right] \\ &= \frac{3}{16\pi} \frac{d\sigma}{d^4q} \sum_{n=U,0,\dots,7} S_n(\theta, \phi) A_n,\end{aligned}$$

The star process is the production of vector bosons in DY and SIDIS, but there is much more...

PHENOMENOLOGY: JETS

RECALL PREVIOUS STUDIES ON TMD FACTORIZATION AND JETS ...

Based on :

D. Gutierrez-Reyes, I.S., W. Waalewijn, L. Zoppi PRL 121, 162001(2018)
 D. Gutierrez-Reyes, I.S., W. Waalewijn, L. Zoppi JHEP 1910 (2019) 031
 D. Gutierrez-Reyes, Y. Makris, I.S., V. Vaidya, L. Zoppi JHEP 1908 (2019) 161

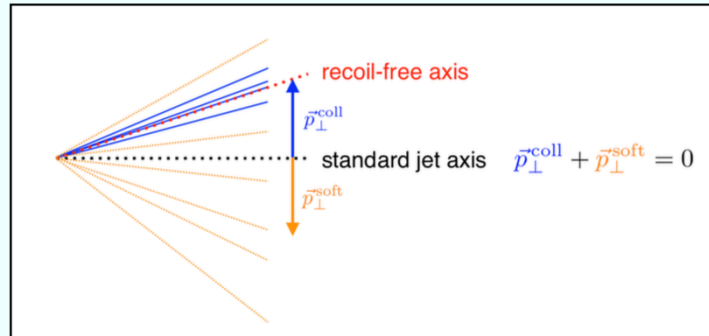
Standard jet axis (SJA)

Introduces soft-sensitivity to axis definition. In TMD factorization works for small R

Winner-take-all (WTA)

Larkoski, Neill, Thaler JHEP 04 (2014) 017

Recoil invariant. It is not sensitive to soft radiation. In TMD factorization works for all R



AND RECENT RECOIL FREE JET RECOMBINATION SCHEMES..

Y.-T. Chien, R. Rahn, D.Y. Shao, W. Waalewijn, B. Wu JHEP 02 (2023) 256

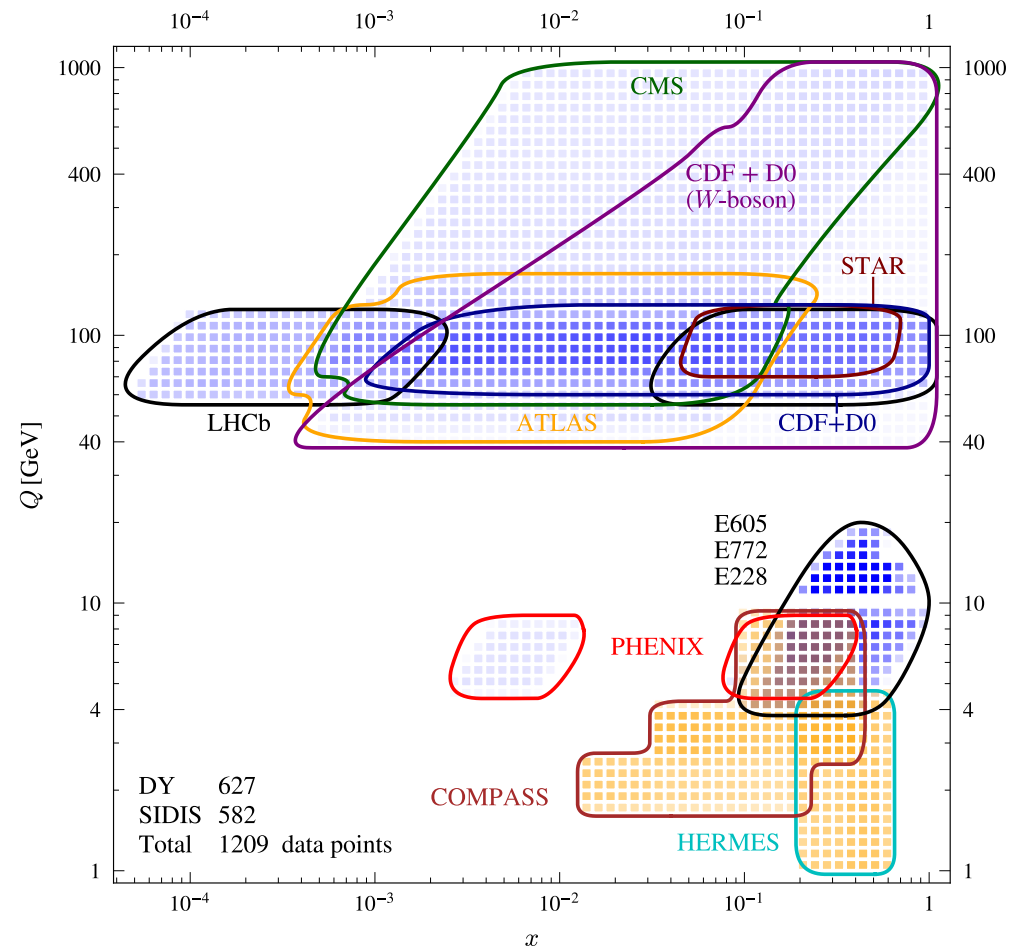
$$\hat{n} = \frac{p_{T,i}^m \hat{n}_i + p_{T,j}^m \hat{n}_j}{p_{T,i}^m + p_{T,j}^m} \quad \text{WTA for } m \rightarrow \infty$$

AND HADRON JET CORRELATORS THAT APPEAR IN EE-CORRELATIONS

I. Moutl, H.-X. Zhu JHEP 08 (2018) 160

$$J_{EEC}(b) = \sum_i \int_0^1 dz \mathbb{C}_{q \leftarrow i} \left(\frac{b}{z}, z \right)$$

Data for unpolarized TMD PDF extractions



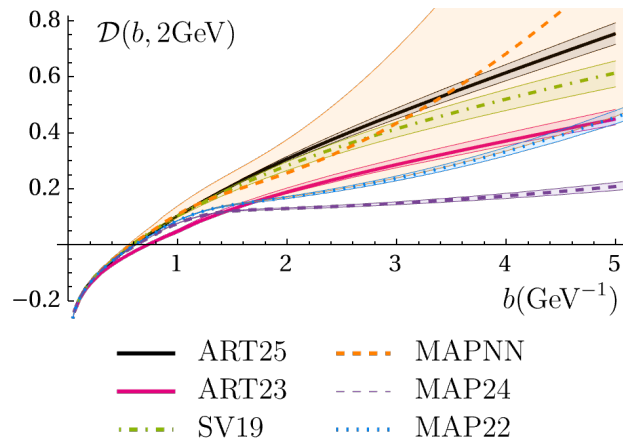
.....

ALL INPUTS: 50 YEARS OF PHYSICS HISTORY IN A FIT

.....

Element	Symbol	Comment
Common constants		
Mass and width of Z	M_Z, Γ_Z	$M_Z = 91.1876 \text{ GeV}, \Gamma_Z = 2.4942 \text{ GeV}$
Mass and width of W	M_W, Γ_W	$M_W = 80.379 \text{ GeV}, \Gamma_W = 2.089 \text{ GeV}$
Sine θ -Weinberg	s_W	$s_W^2 = 0.2312 \text{ GeV}$
Quark masses	m_q	$m_c = 1.40 \text{ GeV}, m_b = 4.75 \text{ GeV}$ same as in MSHT20
QED coupling const.	α_{EM}	$\alpha_{\text{EM}}^{-1}(M_Z) = 127.955$
QCD coupling const.	$\alpha_s \equiv 4\pi a_s$	$\alpha_s(M_Z) = 0.118$ taken from MSHT20
Hadron tensor		
Hard coef.function	$\mathcal{C}_{\text{DY}}, \mathcal{C}_{\text{SIDIS}}$	$\text{N}^4\text{LO } (a_s^4)$
Cusp AD	Γ_{cusp}	$\text{N}^4\text{LO } (a_s^5)$
TMD AD	γ_V	$\text{N}^3\text{LO } (a_s^4)$
Factoriz. scales	$\mu_H, \zeta, \bar{\zeta}$	$\mu_H^2 = \zeta = \bar{\zeta} = Q^2$
Special ζ -line	$\zeta_\mu(b)$	Exact solution at a_s^4
Collins-Soper kernel		
Pert.part	$\mathcal{D}_{\text{perp}}$	$\text{N}^3\text{LO } (a_s^4)$
NP part	$\mathcal{D}_{\text{NP}}$	2 parameters $\{c_0, c_1\}$
Defining scale	μ^*	
b -prescription	b^*	
Unpolarized TMDPDF f_1		
Matching coef.	$C_{f \leftarrow f'}$	$\text{N}^3\text{LO } (a_s^3) + \text{N}^4\text{LL}_x (a_s^4)$
Unpol. PDF	f_1	MSHT20 at $\text{N}^2\text{LO } (a_s^3)$
NP part	f_{NP}	10 parameters λ_i^f
OPE scale	μ_{OPE}	
b -prescription	b_{OPE}^*	
Unpolarized TMDFF D_1		
Matching coef.	$\mathbb{C}_{f \rightarrow f'}$	$\text{N}^3\text{LO } (a_s^3) + \text{N}^4\text{LL}_x (a_s^4)$
Unpol. FF for π^+	d_1^π	MAPFF1.0 at $\text{N}^2\text{LO } (a_s^3)$
Unpol. FF for K^+	d_1^K	MAPFF1.0 at $\text{N}^2\text{LO } (a_s^3)$
NP part for π^+	D_{NP}^π	5 parameters $\eta_{0,1}^{\pi,f}$
NP part for K^+	D_{NP}^K	5 parameters $\eta_{0,1}^{K,f}$
OPE scale	$\mu_{\text{OPE}}^{\text{FF}}$	
b -prescription	$b_{\text{OPE}}^{*(\text{FF})}$	

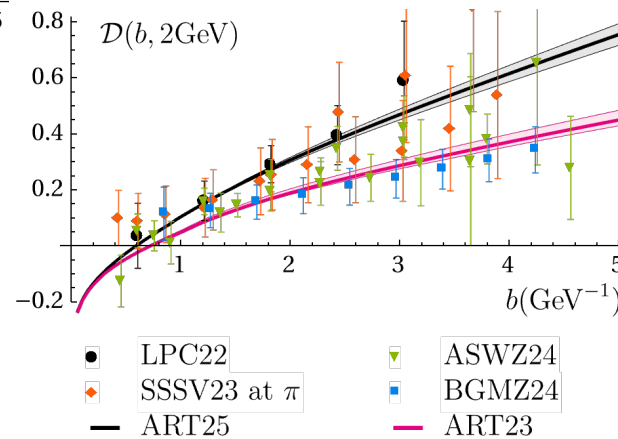
CS-KERNEL



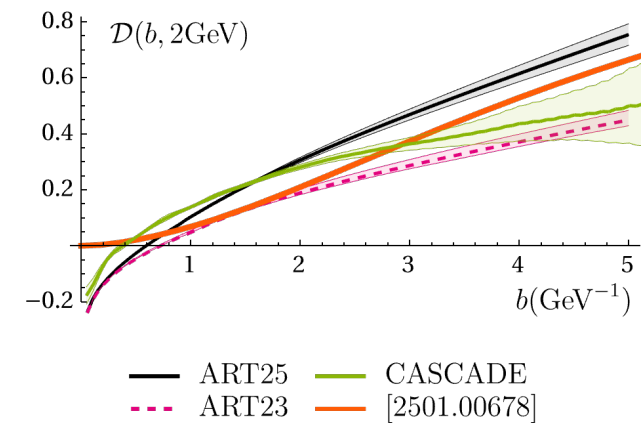
Extractions from DY and DY+SIDIS work only in the perturbative region.

MAPNN gives a reasonable error estimate

Lattice results are in between ART23 and ART25



Parton shower and instanton estimates are also in between ART23 and ART25



PERSPECTIVES

I have tried to provide you some basis to understand factorization and TMD formalism. Hands-on with Alexey.

Some parts of the formalism are still developing, mainly in higher power corrections

There is a plethora of processes that need to be studied, coded and to be discovered.

Evolution is a fundamental ingredient (it cannot be optional, unless we want a very approximate picture of Nature)

BACK UP

EVOLUTION KERNEL

En el núcleo de evolución de Collins-Soper se combinan elementos perturbativos y no perturbativos.

$$\mathcal{D}(b, \mu) = \mathcal{D}_{\text{small-b}}(b^*, \mu^*) + \int_{\mu^*}^{\mu} \frac{d\mu'}{\mu'} \Gamma_{\text{cusp}}(\mu') + \mathcal{D}_{\text{NP}}(b),$$

$$\mathcal{D}_{\text{NP}}(b) = bb^* \left[c_0 + c_1 \ln \left(\frac{b^*}{B_{\text{NP}}} \right) \right]$$

$$b^*(b) = \frac{b}{\sqrt{1 + \frac{\bar{b}^2}{B_{\text{NP}}^2}}}, \quad \mu^*(b) = \frac{2e^{-\gamma_E}}{b^*(b)}, \quad B_{\text{NP}} = 1.5/\text{GeV}$$

ART25: TMDPDF

$$f_{1,f \leftarrow h}(x, b) = \sum_{f'} C_{f \leftarrow f'}(x, b_{\text{OPE}}^*, \mu_{\text{OPE}}) \otimes f_{1,f' \leftarrow h}(x, \mu_{\text{OPE}}) F_{\text{NP}}^f(x, b)$$

📌 Coeficientes resúmdados a grande-x: *O. del Rio, A. Prokudin, I.S., A. Vladimirov e-Print: 2501.17274*

📌 Parametrización: $F_{\text{NP}}^f(x, b) = \frac{1}{\cosh \left[\left(\lambda_1^f (1-x)^{\lambda_3^f} + \lambda_2^f x \right) b \right]}, \quad f = u, \bar{u}, d, \bar{d}, \text{sea}$

$$b_{\text{OPE}}^*(b) = b e^{-0.04 b^2} + \frac{2e^{-\gamma_E}}{\mu_{\text{OPE}}(b)} (1 - e^{-0.04 b^2})$$

$$\mu_{\text{OPE}}(b) = \frac{2e^{-\gamma_E}}{b} + 5 \text{ GeV}.$$

📌 En total, 10 parámetros $\{\lambda_1^u, \lambda_2^u, \lambda_3^u, \lambda_2^{\bar{u}}, \lambda_1^d, \lambda_2^d, \lambda_3^d, \lambda_2^{\bar{d}}, \lambda_1^{\text{sea}}, \lambda_2^{\text{sea}}\}$

📌 PDFs de referencia: MSHT20

ART25: TMDFF

$$D_{1,f \rightarrow h}(z, b) = \sum_{f'} \mathbb{C}_{f \rightarrow f'}(z, b_{\text{OPE}}^{*(\text{FF})}, \mu_{\text{OPE}}^{\text{FF}}) \otimes d_{1,f' \rightarrow h}(z, \mu_{\text{OPE}}^{\text{FF}}) D_{\text{NP}}^{f/h}(z, b)$$



Parameterización for fragmentation:

$$D_{\text{NP}}^{f/h}(z, b) = \frac{1 + \eta_1^{h,f} \frac{b^2}{z^2}}{\cosh \left(\eta_0^h \frac{b}{z} \right)},$$

$$(h = \pi^+) : f = u, \bar{u}, d, \text{ rest}$$

$$(h = K^+) : f = u, \bar{u}, s, \text{ rest}$$

$$\mu_{\text{OPE}}^{\text{FF}}(b) = \mu_{\text{OPE}} \left(\frac{b}{z} \right) = \frac{2e^{-\gamma_E} z}{b} + 5 \text{ GeV},$$

$$b_{\text{OPE}}^{*(\text{FF})}(b) = b e^{-0.04b^2} + \frac{2e^{-\gamma_E}}{\mu_{\text{OPE}}^{\text{FF}}(b)} \left(1 - e^{-0.04b^2} \right)$$



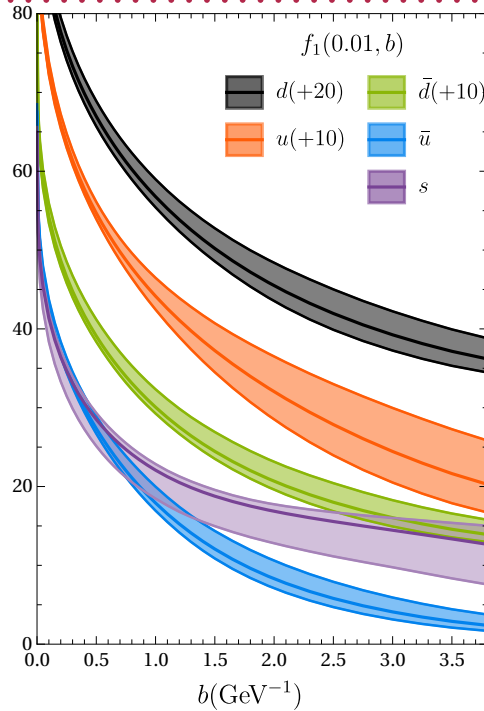
Funciones de fragmentación de referencia: MAPFF1.0



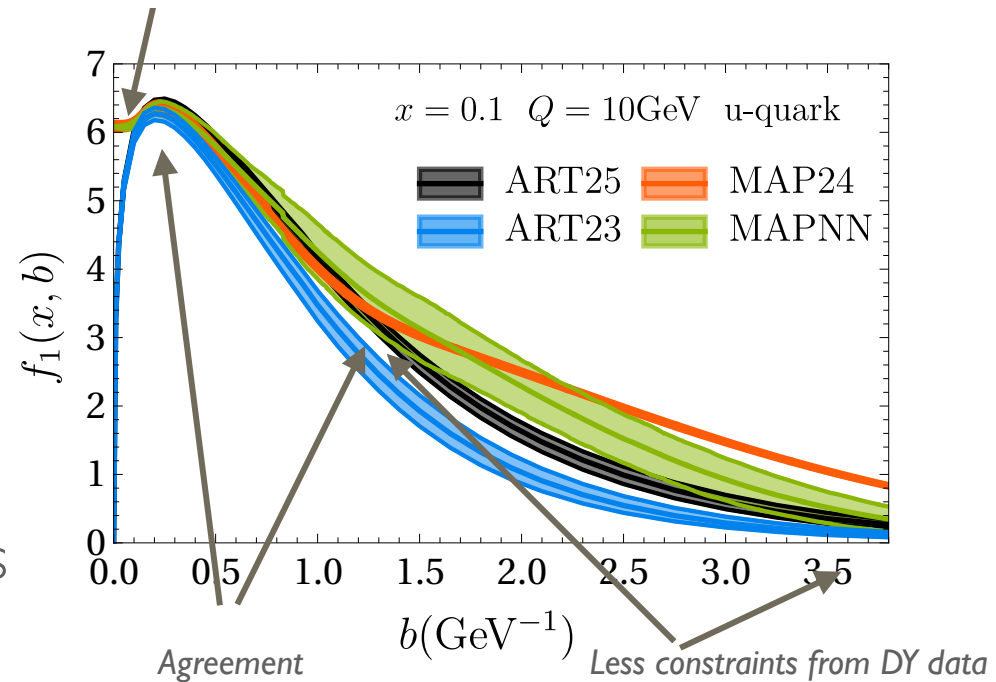
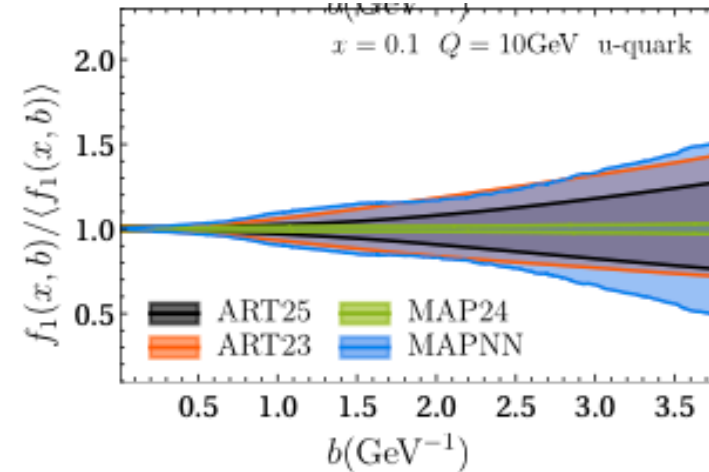
En total, 10 nuevos parámetros

$$\{\eta_0^\pi, \eta_1^{\pi,u}, \eta_1^{\pi,\bar{d}}, \eta_1^{\pi,\bar{u}}, \eta_1^{\pi,r}, \eta_0^K, \eta_1^{K,u}, \eta_1^{K,\bar{d}}, \eta_1^{K,\bar{u}}, \eta_1^{K,r}\}$$

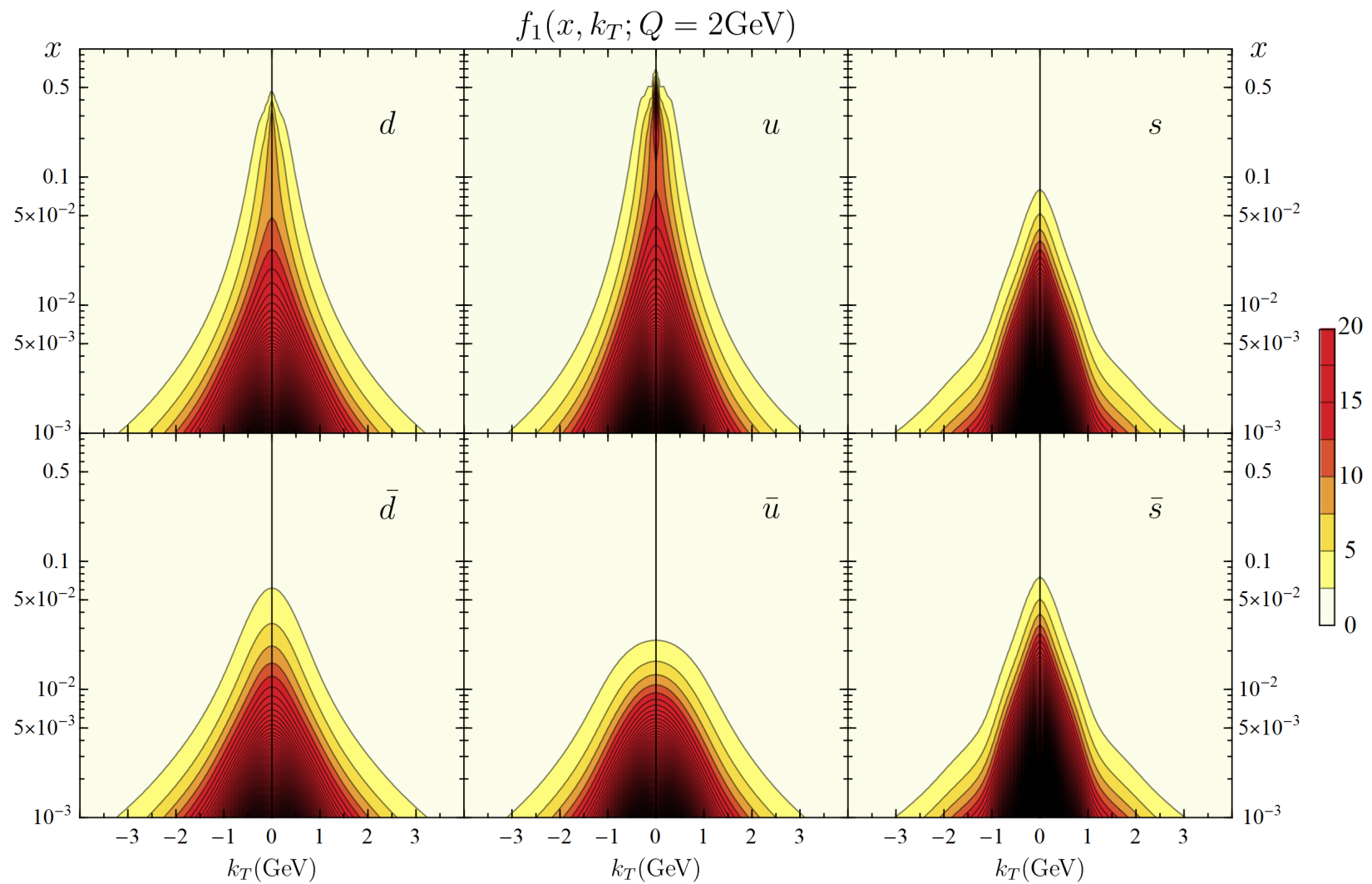
TMDPDF: RESULTS



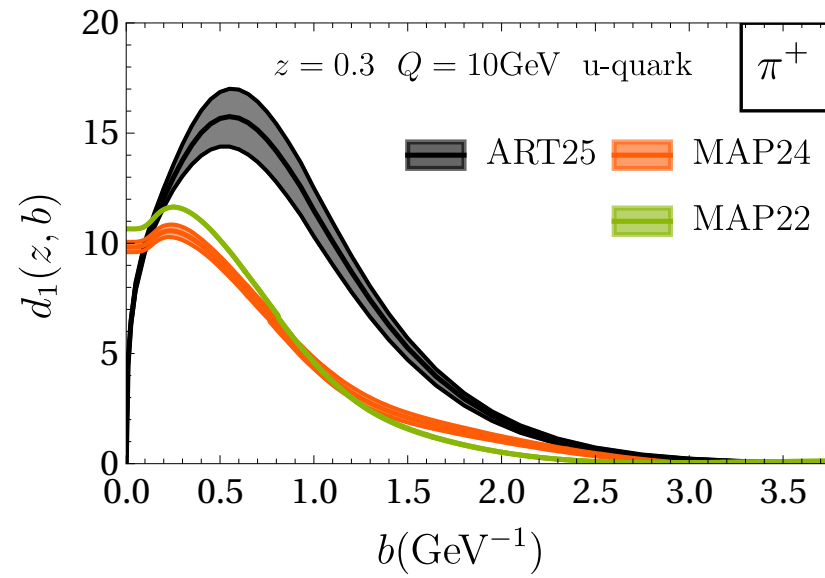
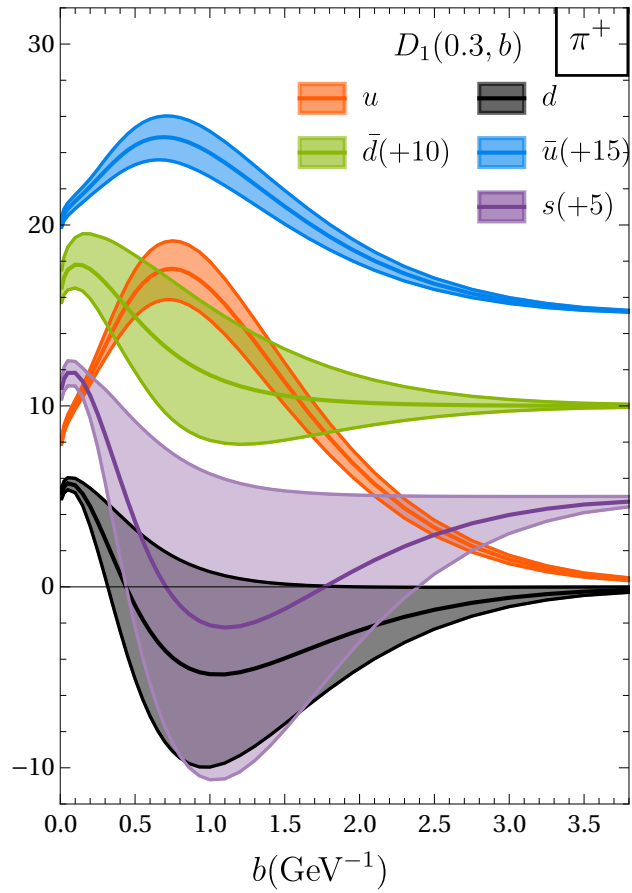
There is substantial agreement between the groups. Problems arise from the fit of the SIDIS, as it is more sensitive to high values of b ...



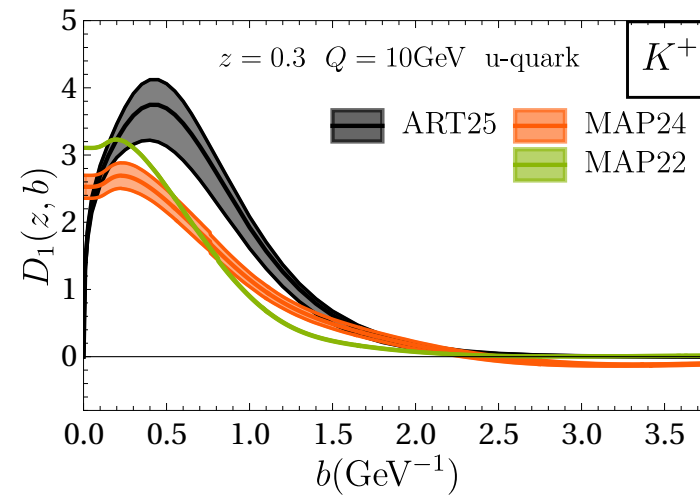
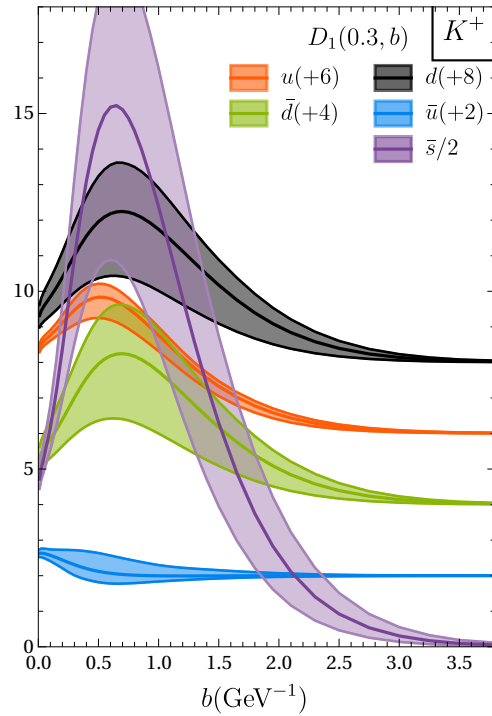
TMDPDF RESULTS: SMALL/LARGE- x VS k_T



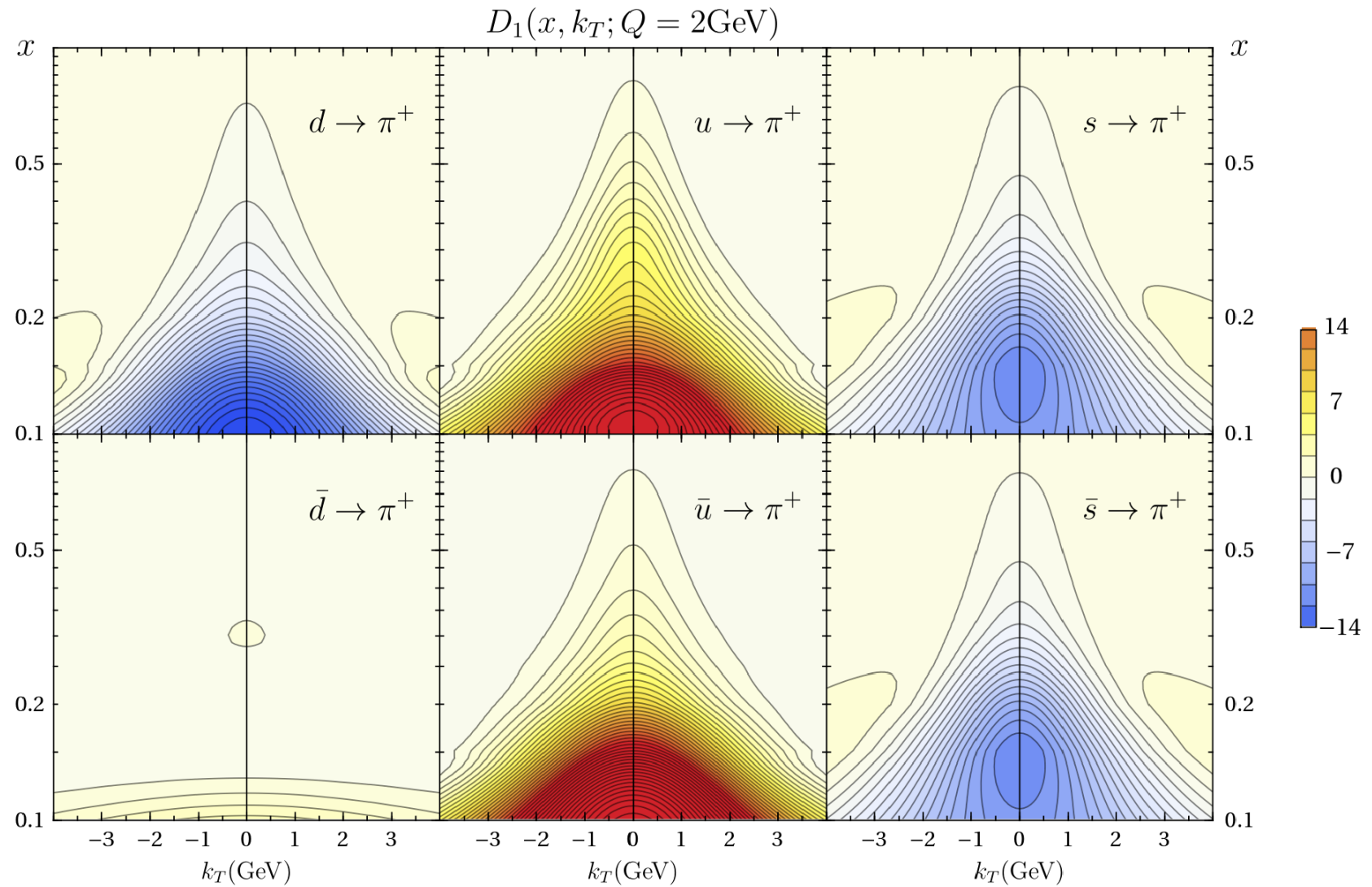
TMDFF: RESULTS (π)



TMDFF RESULTS (K)



TMDFF: RESULTADOS (π) PARA SMALL/LARGE-Z VS. k_T



DIVERGENCES AND RENORMALIZATION OF TMDS AT LP

$$O_q(x, b_T, \mu, \zeta) = Z_q(\zeta, \mu) R_q(\zeta, \mu) O_q^{bare}(x, b_T)$$

$$\mathbb{O}_q(z, b_T, \mu, \zeta) = Z_q(\zeta, \mu) R_q(\zeta, \mu) \mathbb{O}_q^{bare}(z, b_T)$$

Universal (=process independent) subtraction constants

$$R_q = \frac{\sqrt{S(b_T)}}{\text{zero} - \text{bin}}$$

Rapidity divergences subtraction

$$Z_q$$

UV divergences subtraction

Tilted WL's [Collins]

$$R_q(\zeta, \mu) = \frac{\sqrt{S(b_T; +\infty, y_s)}}{\sqrt{S(b_T; +\infty, -\infty) S(b_T; y_s, -\infty)}}$$

$$\zeta \sim m^2 e^{-2y_s}$$

δ -regularization [EIS]

zero-bin coincides with soft-factor

$$R_q(\zeta, \mu) = \frac{1}{\sqrt{S(b_T; \alpha\delta^+, \delta^+)}}$$

$$\zeta \sim \alpha Q^2$$