IPARCOS

CLASS I

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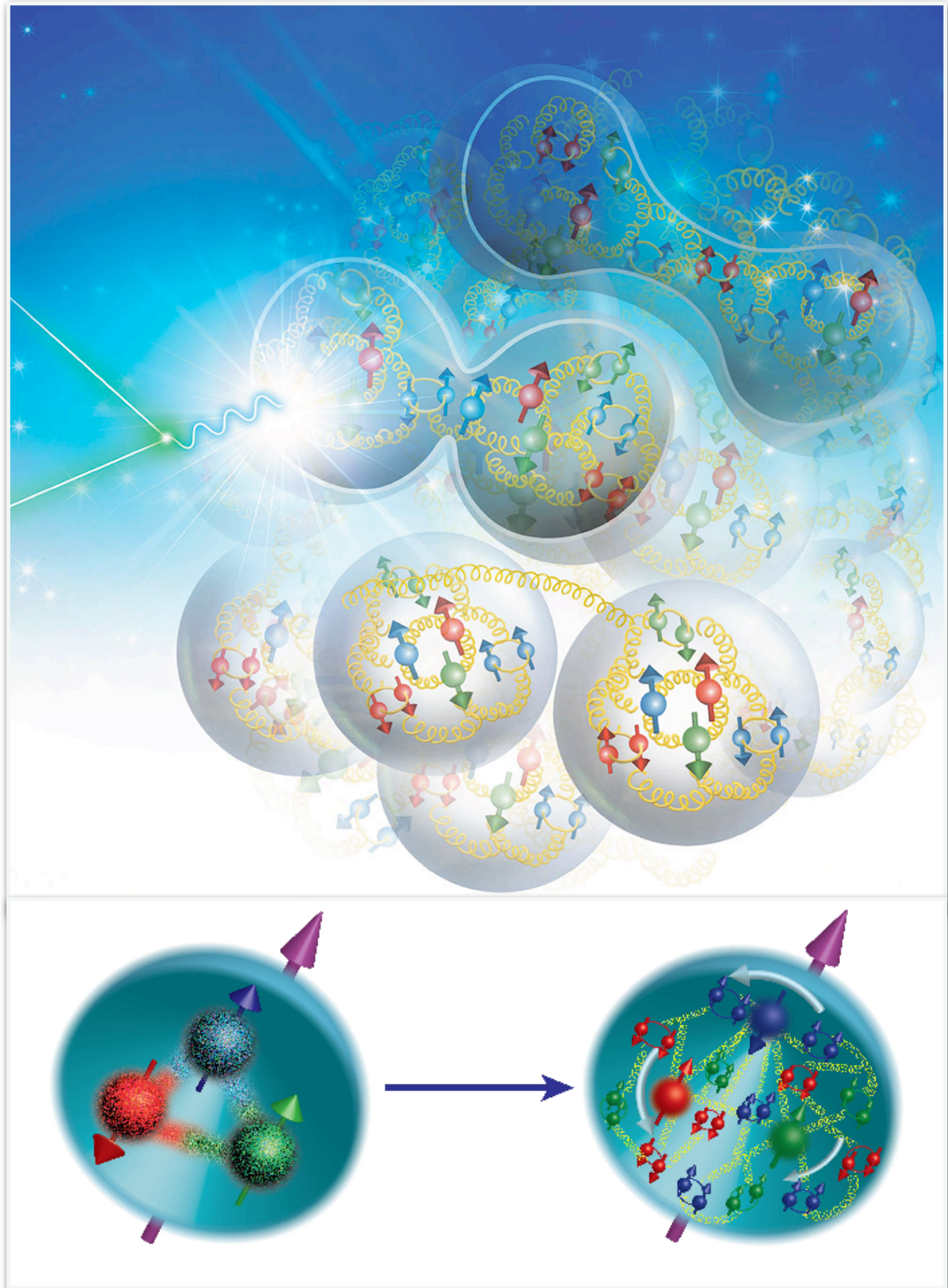
HADRONIZATION IS ONE OF NATURE'S MOST FASCINATING PROCESSES.

OUR UNDERSTANDING OF HOW HADRONS AND JETS ARE STRUCTURED AND FORMED HAS RECENTLY CHANGED THANKS TO NEW THEORETICAL FINDINGS.

NEW EXPERIMENTS ARE ON THE HORIZON: EIC, FCC-EE, AND MORE.



The question is:
HOW DO QUARKS AND GLUONS MOVE INSIDE A PROTON? (WHERE, HOW, WHY..)

??

TOPICS

- TRANSVERSE MOMENTUM DEPENDENT DISTRIBUTIONS (TMDS)
- FACTORIZATION
- EVOLUTION OF TMDS
- TMD PHENOMENOLOGY: VECTOR BOSON PRODUCTION
- TMD MOMENTS

DISCLAIMER

- THE PATH TO FINAL RESULTS IS NOT STRAIGHTFORWARD. OFTEN WE START WITH SOME DEFINITION THAT WE HAVE TO MODIFY ON A LATER STAGE...
- THIS MEANS THAT ALSO NOTATION IS IMPROVING AND CHANGING WITH TIME...

DY CROSS SECTION

The process $h_1(P_1) + h_2(P_2) \to l(l) + l'(l') + X$

There is a rich phase space too explore

Cross section $$d\sigma = \frac{2\alpha_{\rm em}^2}{s}\frac{d^3l}{2E}\frac{d^3l'}{2E'}L_{\mu\nu}W^{\mu\nu}\Delta(q)\Delta^*(q)$$

Leptonic tensor $$L_{\mu\nu} = e^{-2}\langle 0\,|\,J_\mu(0)\,|\,l,l'\rangle\langle l,l'\,|\,J_\nu^\dagger(0)\,|\,0\rangle$$

Hadronic tensor

$$\boxed{W_{\mu\nu} = e^{-2}\int\frac{d^4x}{(2\pi)^4}e^{-i(x\cdot q)}\sum_X\langle P_1,P_2\,|\,J_\mu^\dagger(x)\,|\,X\rangle\langle X\,|\,J_\nu(0)\,|\,P_1,P_2\rangle}$$

Legenda: $\Delta_G(q) = \dfrac{1}{q^2+i0}$ $\qquad J_\mu = e\bar{\psi}\gamma_\mu\psi$

SIDIS CROSS SECTION

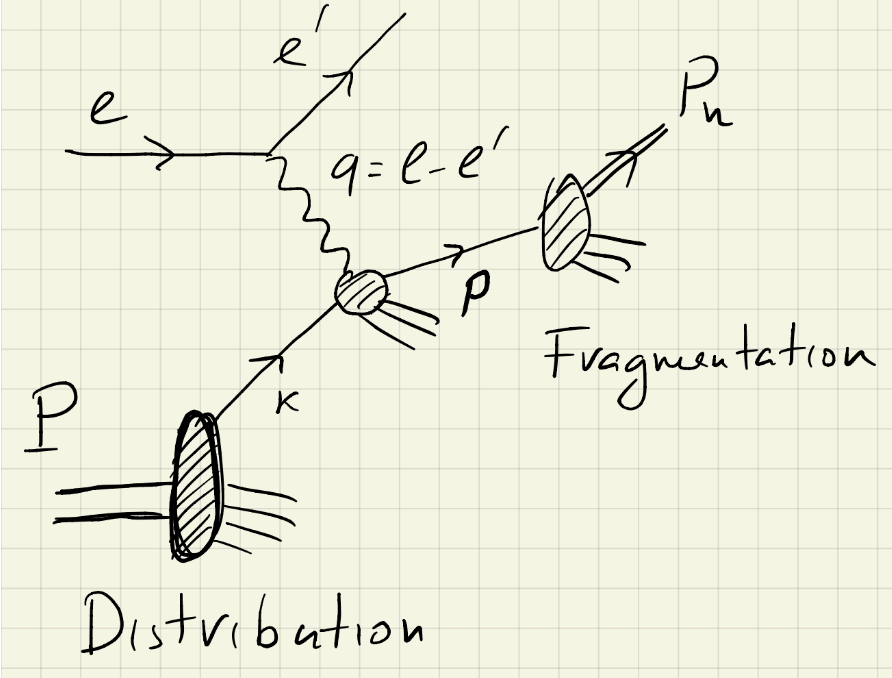
$$\ell(l) + H(P) \to \ell(l') + h(p_h) + X$$

$$d\sigma = \frac{2}{s - M^2} \frac{\alpha_{\rm em}^2}{(q^2)^2} L_{\mu\nu} W^{\mu\nu} \frac{d^3 l'}{2E'} \frac{d^3 p_h}{2E_h}$$

$$L_{\mu\nu} = e^{-2} \langle l' | J_\mu(0) | l \rangle \langle l | J_\nu^\dagger(0) | l' \rangle,$$

$$\boxed{W_{\mu\nu} = e^{-2} \int \frac{d^4 x}{(2\pi)^4} e^{-i(x\cdot q)} \sum_X \langle P | J_\mu^\dagger(x) | p_h, X \rangle \langle p_h, X | J_\nu(0) | P \rangle,}$$

SIDIS AND e^+e^- CASE: HOW TO GET THE INNER STRUCTURE OF A HADRON

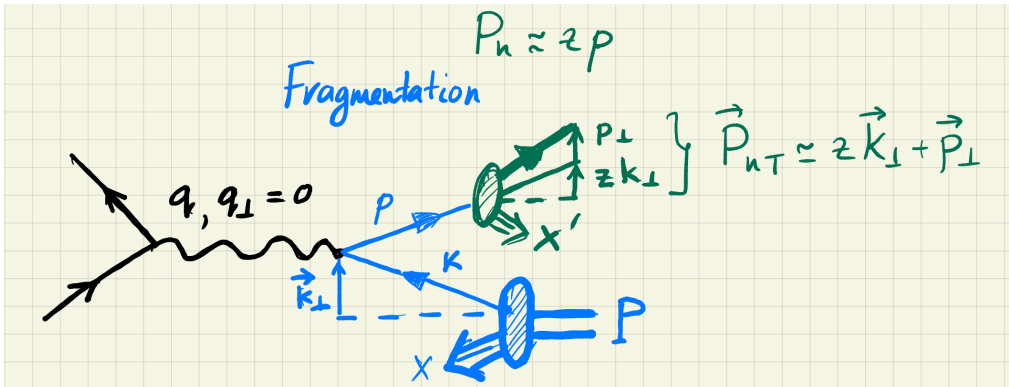


Bjorken $x = Q^2/(2P \cdot q)$

Inelasticity $y = P \cdot q/(P \cdot \ell)$

$z = P \cdot P_n/(P \cdot q)$

$s = (\ell + P)^2$



The recoil off a low energy is sensitive to the intrinsic motion $(\vec{k}_T, \vec{p}_\perp)$. We need to have QCD radiation. The distributions now depend on transverse momentum dependent distributions (3D structure), TMD.

$$f_1(z) \rightarrow f_1(z, \vec{k}_T)$$

FROM “THOMSON” PROTON TWO SOMETHING NEW

Experiments can measure also (un)polarized cross section, asymmetries in a multidimensional way

Theory can construct distribution and correlation functions to describe spin-momentum and momentum-momentum correlations

$$F_{ij} = \langle P, S \,|\, \overline{\psi}_j(x, \vec{k}_T)\psi_i(0) \,|\, P, S\rangle = \int db^- d^2 b_T \exp[-ixP^+b^- + ik_T \cdot b_T]\langle P, S \,|\, \overline{\psi}_j(b)\psi_i(0) \,|\, P, S\rangle \,|_{b^+=0}$$

(This is not the final definition yet!!)

$$F^{[\Gamma]} \equiv \frac{1}{2}\mathrm{Tr}[\Gamma F]$$

$n, \bar{n}$ (with $n^2 = \bar{n}^2 = 0,\, n \cdot \bar{n} = 1$); $\quad v^\mu = v^+\bar{n}^\mu + v^- n^\mu + v_T^\mu$

TMDS IN b-SPACE

Parity, charge conjugation, time-reversal constraint the possible observables

TMDs in b space are parametrized as ($\kappa = -1$(SIDIS),1(DY))

$$\tilde{F}^{[\gamma^+]}(x,b) = \tilde{f}_1(x,b) + i\epsilon_T^{\mu\nu} b_\mu s_{T\nu} M \tilde{f}_{1T}^{\perp}(x,b)\kappa,$$

$$\tilde{F}^{[\gamma^+\gamma^5]}(x,b) = \lambda \tilde{g}_1(x,b) + i(b\cdot s_T) M \tilde{g}_{1T}^{\perp}(x,b),$$

$$\begin{aligned}\tilde{F}^{[i\sigma^{\alpha+}\gamma^5]}(x,b) = {} & s_T^\alpha \tilde{h}_1(x,b) - i\lambda b^\alpha M \tilde{h}_{1L}^{\perp}(x,b) \\ & + i\epsilon_T^{\alpha\mu} b_\mu M \tilde{h}_1^{\perp}(x,b) + \frac{M^2}{4}(g_T^{\alpha\mu}\mathbf{b}^2 + 2b^\alpha b^\mu) s_{T\mu} \tilde{h}_{1T}^{\perp}(x,b)\kappa\end{aligned}$$

All measurements are performed in momentum space, however it is more convenient to study them in configuration space

TMDS IN k_T-SPACE

Parity, charge conjugation, time-reversal constraint the possible observables

TMDs in b space are parametrized as ($\kappa = -1$(SIDIS),1(DY))

$$F^{[\gamma^+]}(x, k_T) = f_1(x, k_T) - \epsilon_T^{\mu\nu} \frac{(k_T)_\mu s_{T\nu}}{M} f_{1T}^{\perp}(x, k_T)\kappa,$$

$$F^{[\gamma^+\gamma^5]}(x, k_T) = \lambda g_1(x, k_T) - \frac{(k_T \cdot s_T)}{M} \tilde{g}_{1T}^{\perp}(x, k_T),$$

$$F^{[i\sigma^{\alpha+}\gamma^5]}(x, k_T) = s_T^\alpha h_1(x, k_T) + \lambda \frac{k_T^\alpha}{M} h_{1L}^{\perp}(x, k_T)$$
$$-\epsilon_T^{\alpha\mu} \frac{(k_T)_\mu}{M} h_1^{\perp}(x, k_T) + \frac{k_T^2}{2M^2} \left(g_T^{\alpha\mu} - 2\frac{k_T^\alpha k_T^\mu)}{k_T^2} \right) s_{T\mu} h_{1T}^{\perp}(x, k_T)\kappa$$

FOURIER TRANSFORMS

$$\tilde{f}(x, b_T) = \int d^2k_T \exp[-i\vec{b}_T \cdot \vec{k}_T] f(x, k_T) = \int_0^\infty k_T dk_T \int_0^{2\pi} d\phi \, f(x, k_T) \exp[-i b_T k_T \cos\phi]$$

$$= 2\pi \int_0^\infty k_T dk_T J_0(|k_T||b_T|) \, f(x, k_T)$$

$$f(x, k_T) = \int_0^\infty b_T db_T \int_0^{2\pi} d\phi \;\; \exp[+i b_T \cdot k_T \cos\phi] \tilde{f}(x, b_T) = \frac{1}{2\pi} \int_0^\infty b_T db_T J_0(|k_T||b_T|) \, \tilde{f}(x, b_T)$$

TMD IN SEVERAL SPACES

$$\lim_{z\to 0} J_m(z) = \frac{1}{\Gamma[m+1]}\left(\frac{z}{2}\right)^m$$

We can take derivatives

$$\tilde{f}(x,b_T)^{(n)} \equiv n!\left(-\frac{1}{M^2 b_T}\frac{\partial}{\partial b_T}\right)^n \tilde{f}(x,b_T) = \frac{(2\pi)n!}{(M^2)^n}\int k_T dk_T \left(\frac{k_T}{b_T}\right)^n J_n(b_T k_T) f(x,k_T)$$

And evaluate moments (if integrals converge…)

$$\lim_{b_T\to 0} \tilde{f}(x,b_T)^{(n)} = = (2\pi)\int k_T dk_T \left(\frac{k_T^2}{2M^2}\right)^n f(x,k_T)$$

$$\int d^2k_T k_T^i \ldots . k_T^j f(x,k_T) e^{-i\vec{k}_T \vec{b}_T} = \left(i\frac{b_T^i}{b_T}\frac{\partial}{\partial b_T}\right)\ldots\left(i\frac{b_T^j}{b_T}\frac{\partial}{\partial b_T}\right)(2\pi)\int_0^\mu k_T dk_T\; J_0(k_T b_T) f(x,k_T)$$

AN EXAMPLE

$$\int d^2k_T\, k_T^i\, e^{-i b_T k_T} f(k_T) =$$

$$= i \frac{b_T^i}{b_T} \frac{\partial}{\partial b_T} 2\pi \int k_T\, dk_T\, J_0(k_T b_T) f(k_T) =$$

$$= i \frac{b_T^i}{b_T} 2\pi \int k_T\, dk_T \left(-J_1(k_T b_T)\right) k_T f(k_T)$$

$$= (-i)\, b_T^i\, 2\pi \int k_T\, dk_T \left(\frac{k_T}{b_T}\right) J_1(k_T b_T) f(k_T)$$

$$= (-i)\, b_T^i\, M^2\, \tilde{f}^{(1)}(b_T)$$

AN EXAMPLE

$$\Phi^{[\gamma^+]} = f_1(x,k_T) - \frac{\epsilon^{ij} k_T^i S_T^j}{M} f_{1T}^{\perp}(x,k_T)$$

$$\downarrow$$

$$\tilde{\Phi}^{[\gamma^+]} = \tilde{f}_1^{(0)}(x,b_T) + i\,\epsilon^{ij} b_T^i S_T^j M \tilde{f}_{1T}^{\perp(1)}(x,b_T)$$

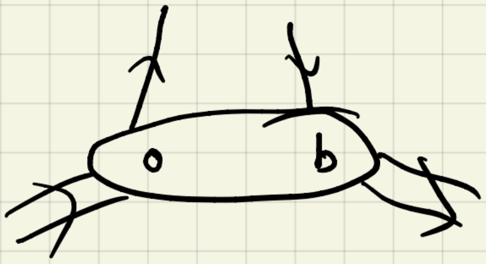
Notation: $\Phi = F$

DIFFERENT LIMITS OF TMD

$$\tilde{F}^{[\gamma^+]}(x,b) = \tilde{f}_1(x,b) + i\epsilon_T^{\mu\nu} b_\mu s_{T\nu} M \tilde{f}_{1T}^{\perp}(x,b)\kappa,$$

Sivers function

$$F^{[\gamma^+]}(x,k_T) = f_1(x,k_T) - i\epsilon_T^{\mu\nu} k_{T\mu} s_{T\nu} M f_{1T}^{\perp}(x,k_T)\kappa,$$



Assume $\tilde{\bar{\psi}}(\bar{x}_\perp, \bar{\ell}_\perp)$, $\tilde{\psi}(\bar{y}_\perp, \bar{m}_\perp)$

$$\psi(y_\perp) = \int d^2 m_\perp e^{-i y_\perp m_\perp} \tilde{\psi}(\bar{y}_\perp, \bar{m}_\perp)$$

$$\overline{\psi}(x_\perp)\psi(y_\perp) = \int d^2\ell_\perp d^2 m_\perp \exp[i\vec{x}_\perp \vec{\ell}_\perp - i\vec{y}_\perp \vec{m}_\perp]\tilde{\overline{\psi}}(x_\perp, \ell_\perp)\tilde{\psi}(y_\perp, m_\perp)$$

$$\bar{x}_\perp \bar{\ell}_\perp - \bar{y}_\perp \bar{m}_\perp = \frac{1}{2}(\bar{x}_\perp + \bar{y}_\perp)(\bar{\ell}_\perp - \bar{m}_\perp) + \frac{1}{2}(\bar{x}_\perp - \bar{y}_\perp)(\bar{\ell}_\perp + \bar{m}_\perp)$$

DIFFERENT LIMITS OF TMD

For TMDs we have a forward amplitude $\bar{e}_\perp = \bar{m}_\perp \equiv \vec{k}_T$

$$\Rightarrow \quad \frac{1}{2}(\bar{x}_\perp - \bar{y}_\perp)(\bar{e}_\perp + \bar{m}_\perp) \overset{\text{def}}{=} \bar{b}_T \cdot \bar{k}_T$$

$\bar{b}_T$ is the difference of quark field positions

$\bar{k}_T$ is the transverse momentum

TMD

b=parameter in configuration space

If the amplitude is off forward $\frac{1}{2}(x_\perp + y_\perp) \equiv b$ impact parameter w.r.t the center of mass of the hadron

$\bar{e}_\perp - \bar{m}_\perp = \bar{\Delta}$ momentum transfer $\Rightarrow$ „GPDs"

b=impact parameter

INTRODUCING WILSON LINES

Recall

$$F_{ij} = \langle P, S | \overline{\psi}_j(x, \vec{k}_T) \psi_i(0) | P, S \rangle = \int db^- d^2 b_T \exp[-ixP^+ b^- + ik_T \cdot b_T] \langle P, S | \overline{\psi}_j(b) \psi_i(0) | P, S \rangle |_{b^+=0}$$

The operator definition that we gave cannot be the final one:
Gauge transformations make it not invariant

Applying P and T to

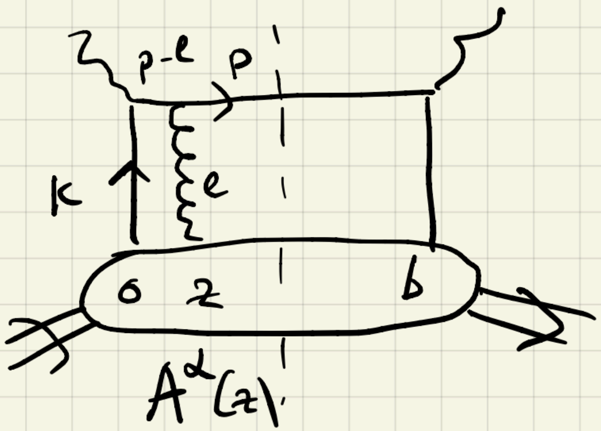
$$F^{[\gamma^+]}(x, k_T, s_T) = f_1(x, k_T) - \epsilon_T^{\mu\nu} \frac{k_{T\mu}}{M} s_{T\nu} f_{1T}^{\perp}(x, k_T) \kappa,$$

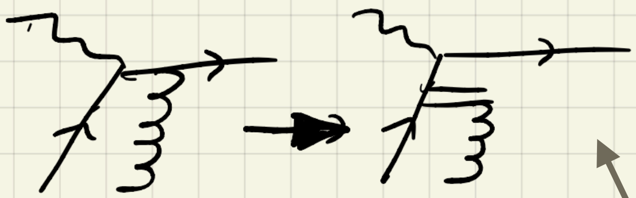
One obtains

$$F^{[\gamma^+]}(x, k_T, s_T) = F^{[\gamma^+]}(x, k_T, -s_T)$$

Which would imply that the Sivers function is vanishing

INTRODUCING WILSON LINES





$$(p-l)^2 \simeq -2p\cdot l + l^2 \simeq -2p\cdot l$$

$$\simeq -2p^- l^+$$

the eikonal approximation

$$i\,\frac{\not{p}-\not{l}}{(p-l)^2+i\epsilon} \simeq i\,\frac{p^-\gamma^+}{-2p^-l^+ + i\epsilon} = \frac{i}{2}\,\frac{\gamma^+}{-l^+ + i\epsilon}$$

The radiation emitted from final states in the eikonal approximation is collected in a Wilson line for the initial state!!

$$\tilde{W}_{\bar{n}}(0) = P\exp\left[-ig\int_0^\infty d\sigma A_-(\sigma n)\right]$$

WILSON LINES AT WORK: SMALL-B EXPANSION

Wilson line $$[a_1 n + \boldsymbol{b}, a_2 n + \boldsymbol{b}] \;=\; P\exp\left(ig\int_{a_2}^{a_1} d\sigma n^\mu A_\mu(\sigma n + \boldsymbol{b})\right).$$

DY operator $$\mathcal{U}^\Gamma_{\rm DY}(z,\boldsymbol{b}) \;=\; \bar q(zn+\boldsymbol{b})[zn+\boldsymbol{b}, -\infty n+\boldsymbol{b}]\Gamma[-\infty n-\boldsymbol{b}, -zn-\boldsymbol{b}]q(-zn-\boldsymbol{b}),$$

SIDIS operator $$\mathcal{U}^\Gamma_{\rm DIS}(x,\boldsymbol{b}) \;=\; \bar q(zn+\boldsymbol{b})[zn+\boldsymbol{b}, +\infty n+\boldsymbol{b}]\Gamma[+\infty n-\boldsymbol{b}, -zn-\boldsymbol{b}]q(-zn-\boldsymbol{b}).$$

TMD $$\Phi^{[\Gamma]}_{q\leftarrow h}(x,\boldsymbol{b}) = \int \frac{dz}{2\pi} e^{-2ixzp^+}\langle P,S|\mathcal{U}^\Gamma\left(z,\frac{\boldsymbol{b}}{2}\right)|P,S\rangle.$$

Small-b expansion $$\mathcal{U}(z,\boldsymbol{b}) = \sum_i \left[C_i * \mathcal{O}_i^{\rm tw2}\right](z) + \boldsymbol{b}^\mu \sum_i \left[\tilde C_i * \mathcal{O}^{\rm tw3}_{\mu,i}\right](z) + O(\boldsymbol{b}^2),$$

tw=Twist=Dimension -Spin; e.g. tw2=tw$(\bar q\Gamma q) = 1+1 = (1,1) = 2$

WILSON LINES AT WORK: SMALL-B EXPANSION

TMD $$\Phi^{[\Gamma]}_{q\leftarrow h}(x,\boldsymbol{b}) = \int \frac{dz}{2\pi} e^{-2ixzp^+} \langle P,S|\mathcal{U}^\Gamma\left(z,\frac{\boldsymbol{b}}{2}\right)|P,S\rangle.$$

Small-b expansion $$\mathcal{U}(z,\boldsymbol{b}) = \sum_i \left[C_i * \mathcal{O}_i^{\text{tw2}}\right](z) + \boldsymbol{b}^\mu \sum_i \left[\tilde{C}_i * \mathcal{O}_{\mu,i}^{\text{tw3}}\right](z) + O(\boldsymbol{b}^2),$$

The small-b expansion allows to “match” 3D operators to “1D” operators. This expansion should be combined with the perturbative expansion to get the Wilson coefficients (C’s). We also obtain different operators, characterized by a twist number (dimension-spin)

tw=Twist=Dimension -Spin; e.g. tw2=tw$(\bar{q}\Gamma q) = 1+1 = (1{,}1) = 2$

WILSON LINES AT WORK: SMALL-B EXPANSION

Let's see how this expansion works at tree level

$$\mathcal{U}^{\Gamma}(z,\boldsymbol{b}) = \mathcal{U}^{\Gamma}(z,\boldsymbol{0}) + b^{\mu}\frac{\partial}{\partial b^{\mu}}\mathcal{U}^{\Gamma}(z,\boldsymbol{b})\Big|_{\boldsymbol{b}=0} + O(\boldsymbol{b}^2).$$

$$\boxed{\begin{aligned}
\mathcal{U}^{\Gamma}_{\mathrm{DY}}(z,\mathbf{0}) &= \bar{q}(zn)[zn,-\infty n]\Gamma[-\infty n,-zn]q(-zn) = \bar{q}(zn)[zn,-zn]\Gamma q(-zn),\\
\mathcal{U}^{\Gamma}_{\mathrm{DIS}}(z,\mathbf{0}) &= \bar{q}(zn)[zn,+\infty n]\Gamma[+\infty n,-zn]q(-zn) = \bar{q}(zn)[zn,-zn]\Gamma q(-zn).
\end{aligned}}$$

PDF

FF

This means that the transverse momentum distribution match to PDF and fragmentation functions. The matching coefficient is basically one at tree level. This value changes with perturbative corrections

WILSON LINES AT WORK: SMALL-B EXPANSION

$$\mathcal{U}^{\Gamma}(z,\boldsymbol{b}) = \mathcal{U}^{\Gamma}(z,\boldsymbol{0}) + b^{\mu}\frac{\partial}{\partial b^{\mu}}\mathcal{U}^{\Gamma}(z,\boldsymbol{b})\Big|_{\boldsymbol{b}=0} + O(\boldsymbol{b}^2).$$

Covariant derivative and Technical tools

$$\frac{\partial}{\partial b^{\mu}}\mathcal{U}_{\Gamma}^{\mathrm{DY}}(z,\boldsymbol{b})\Big|_{\boldsymbol{b}=0} = \bar{q}(zn)[zn,-\infty n](\overleftarrow{\partial_{T\mu}} - \overrightarrow{\partial_{T\mu}})\Gamma[-\infty n,-z\lambda]q(-z),$$

We should pass to covariant derivative

$$\overrightarrow{D}_{\mu} = \overrightarrow{\partial}_{\mu} - igA_{\mu}, \qquad \overleftarrow{D}_{\mu} = \overleftarrow{\partial}_{\mu} + igA_{\mu}, \qquad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ig[A_{\mu},A_{\nu}].$$

And make the derivatives of Wilson lines

$$\begin{aligned}\partial_{\mu}\{[z_1 n, z_2 n]\} &= \frac{d}{dy^{\mu}}[z_1 n + y, z_2 n + y]\Big|_{y=0} \\ &= ig\left(A_{\mu}(z_1 n)[z_1 n, z_2 n] - [z_1 n, z_2 n]A_{\mu}(z_2 n) + \int_{z_2}^{z_1} d\tau [z_1 n, \tau n]F_{\mu+}(\tau n)[\tau n, z_2 n]\right)\end{aligned}$$

WILSON LINES AT WORK: SMALL-B EXPANSION

$$\mathcal{U}^{\Gamma}(z,\boldsymbol{b}) = \mathcal{U}^{\Gamma}(z,\boldsymbol{0}) + b^{\mu}\frac{\partial}{\partial b^{\mu}}\mathcal{U}^{\Gamma}(z,\boldsymbol{b})\Big|_{\boldsymbol{b}=0} + O(\boldsymbol{b}^2).$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial b^{\mu}}\mathcal{U}^{\Gamma}_{\text{DY}}(z,\boldsymbol{b})\Big|_{\boldsymbol{b}=0} &= \bar{q}(zn)\left(\overleftarrow{D_{\mu}}[zn,-zn] - [zn,-zn]\overrightarrow{D_{\mu}}\right)\Gamma q(-zn)\\ &+ig\left(\int_{-\infty}^{z} + \int_{-\infty}^{-z}\right)d\tau\ \bar{q}(zn)[zn,\tau n]\Gamma F_{\mu+}(\tau n)[\tau n,-zn]q(-zn).\end{aligned}}$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial b^{\mu}}\mathcal{U}^{\Gamma}_{\text{DIS}}(z,\boldsymbol{b})\Big|_{\boldsymbol{b}=0} &= \bar{q}(zn)\left(\overleftarrow{D_{\mu}}[zn,-zn] - [zn,-zn]\overrightarrow{D_{\mu}}\right)\Gamma q(-zn)\\ &-ig\left(\int_{z}^{\infty} + \int_{-z}^{\infty}\right)d\tau\ \bar{q}(zn)[zn,\tau n]\Gamma F_{\mu+}(\tau n)[\tau n,-zn]q(-zn).\end{aligned}}$$

The first term in this expansion is P-even but the second is P-odd **(sign change)**

In general we have the following general operatorial form for tw2 and tw3 operators. In the collinear limit we have

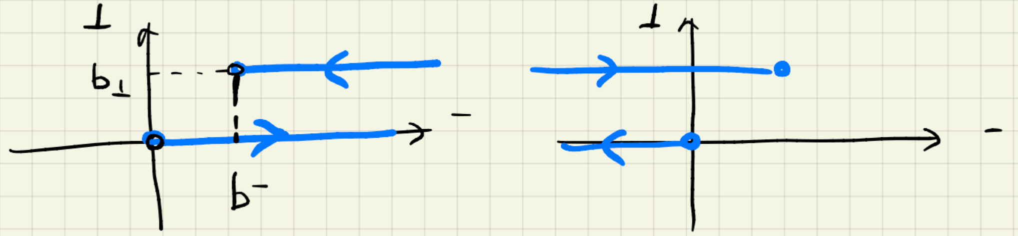
$$\begin{aligned}\mathcal{O}_{\Gamma}(z) &= \bar{q}(zn)[zn,-zn]\Gamma q(-zn),\\ \mathcal{T}^{\mu}_{\Gamma}(z_1,z_2,z_3) &= g\bar{q}(z_1 n)[z_1 n, z_2 n]\Gamma F^{\mu+}(z_2 n)[z_2 n, z_3 n]q(z_3 n).\end{aligned}$$

(Qiu-Sterman functions)

RESUME OF WILSON LINES EFFECTS IN TMD

It was found instead that

$$f^{SIDIS}_{a/p^\uparrow}(x, k_T, S_T) = f^{DY}_{a/p^\uparrow}(x, k_T, -S_T)$$



and thus

$$\underbrace{f^{SIDIS}_{q/p}(x, k_T) = f^{DY}_{q/p}(x, k_T)}_{\text{Unpolarised are the same}}$$

$$\underbrace{f^{\perp\, SIDIS}_{1T}(x, k_T) = - f^{\perp\, DY}_{1T}(x, k_T)}_{\text{Sivers function changes sign!}}$$

The path integral direction is important especially when one makes derivatives of Wilson lines.

We have to distinguish between future point and past pointing Wilson lines. Be careful.

Message: the TMD small-b limit is an asymptotic limit that can be to approximate TMD with collinear functions. More about this later.

TMD OPERATOR DEFINITIONS (INITIAL STATES, UNPOLARIZED)

$$O_q^{bare}(x,\boldsymbol{b}_T) = \frac{1}{2}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \left\{ T\left[\bar{q}_i\, \tilde{W}_n^T\right]_a \left(\frac{\xi}{2}\right) \ |X\rangle \gamma_{ij}^+ \langle X|\ \bar{T}\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(-\frac{\xi}{2}\right)\right\},$$

$$O_{\bar{q}}^{bare}(x,\boldsymbol{b}_T) = \frac{1}{2}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \left\{ T\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(\frac{\xi}{2}\right) \ |X\rangle \gamma_{ij}^+ \langle X|\ \bar{T}\left[\bar{q}_i \tilde{W}_n^T\right]_a \left(-\frac{\xi}{2}\right)\right\},$$

$$O_g^{bare}(x,\boldsymbol{b}_T) = \frac{1}{xp^+}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \left\{ T\left[F_{+\mu}\, \tilde{W}_n^T\right]_a \left(\frac{\xi}{2}\right) |X\rangle\langle X| \bar{T}\left[\tilde{W}_n^{T\dagger} F_{+\mu}\right]_a \left(-\frac{\xi}{2}\right)\right\},$$

Matrix elements

$$\Phi_{q\leftarrow N}(x,\boldsymbol{b}_T) = \frac{1}{2}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \times \langle N| \left\{ T\left[\bar{q}_i\, \tilde{W}_n^T\right]_a \left(\frac{\xi}{2}\right) |X\rangle \gamma_{ij}^+ \langle X| \bar{T}\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(-\frac{\xi}{2}\right)\right\} |N\rangle,$$

$$\Phi_{\bar{q}\leftarrow N}(x,\boldsymbol{b}_T) = \frac{1}{2}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \times \langle N| \left\{ T\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(\frac{\xi}{2}\right) |X\rangle \gamma_{ij}^+ \langle X| \bar{T}\left[\bar{q}_i \tilde{W}_n^T\right]_a \left(-\frac{\xi}{2}\right)\right\} |N\rangle,$$

$$\Phi_{g\leftarrow N}(x,\boldsymbol{b}_T) = \frac{1}{xp^+}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \times \langle N| \left\{ T\left[F_{+\mu}\, \tilde{W}_n^T\right]_a \left(\frac{\xi}{2}\right) |X\rangle\langle X| \bar{T}\left[\tilde{W}_n^{T\dagger} F_{+\mu}\right]_a \left(-\frac{\xi}{2}\right)\right\} |N\rangle,$$

TMD OPERATOR DEFINITIONS (FINAL STATES, UNPOLARIZED)

$$\mathbb{O}_q^{bare}(z,\boldsymbol{b}_T) = \frac{1}{4zN_c}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z}$$
$$\times \langle 0|T\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(\frac{\xi}{2}\right)|X,\frac{\delta}{\delta J}\rangle \gamma_{ij}^+ \langle X,\frac{\delta}{\delta J}|\bar{T}\left[\bar{q}_i\,\tilde{W}_n^T\right]_a\left(-\frac{\xi}{2}\right)|0\rangle,$$

$$\mathbb{O}_{\bar{q}}^{bare}(z,\boldsymbol{b}_T) = \frac{1}{4zN_c}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z}$$
$$\times \langle 0|T\left[\bar{q}_i\,\tilde{W}_n^T\right]_a \left(\frac{\xi}{2}\right)|X,\frac{\delta}{\delta J}\rangle \gamma_{ij}^+ \langle X,\frac{\delta}{\delta J}|\bar{T}\left[\tilde{W}_n^{T\dagger} q_j\right]_a\left(-\frac{\xi}{2}\right)|0\rangle,$$

$$\mathbb{O}_g^{bare}(z,\boldsymbol{b}_T) = \frac{-1}{2(1-\epsilon)p^+(N_c^2-1)}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z}$$
$$\times \langle 0|T\left[\tilde{W}_n^{T\dagger} F_{+\mu}\right]_a \left(\frac{\xi}{2}\right)|X,\frac{\delta}{\delta J}\rangle \langle X,\frac{\delta}{\delta J}|\bar{T}\left[F_{+\mu}\,\tilde{W}_n^T\right]_a\left(-\frac{\xi}{2}\right)|0\rangle,$$

Matrix elements

$$\Delta_{q\to N}(z,\boldsymbol{b}_T) = \frac{1}{4zN_c}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z}$$
$$\times \langle 0|T\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(\frac{\xi}{2}\right)|X,N\rangle \gamma_{ij}^+ \langle X,N|\bar{T}\left[\bar{q}_i\,\tilde{W}_n^T\right]_a\left(-\frac{\xi}{2}\right)|0\rangle,$$

$$\Delta_{\bar{q}\to N}(z,\boldsymbol{b}_T) = \frac{1}{4zN_c}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z}$$
$$\times \langle 0|T\left[\bar{q}_i\,\tilde{W}_n^T\right]_a \left(\frac{\xi}{2}\right)|X,N\rangle \gamma_{ij}^+ \langle X,N|\bar{T}\left[\tilde{W}_n^{T\dagger} q_j\right]_a\left(-\frac{\xi}{2}\right)|0\rangle,$$

$$\Delta_{g\to N}(z,\boldsymbol{b}_T) = \frac{-1}{2(1-\epsilon)p^+(N_c^2-1)}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z} \tag{2.4}$$
$$\times \langle 0|T\left[\tilde{W}_n^{T\dagger} F_{+\mu}\right]_a \left(\frac{\xi}{2}\right)\sum_X |X,N\rangle \langle X,N|\bar{T}\left[F_{+\mu}\,\tilde{W}_n^T\right]_a\left(-\frac{\xi}{2}\right)|0\rangle,$$

INITIAL STATE DISTRIBUTIONS

Quark

$$\Phi^{[\Gamma]}(x,b) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-ix\lambda P^+} \langle P,s\,|\,\bar{q}W^\dagger(\lambda n + b)\frac{\Gamma}{2}Wq(0)\,|\,P,s\rangle$$

$$W(x) = [-\infty n + x, x] = P\exp\left(-ig\int_{-\infty}^{0} d\sigma A_+(x+\sigma n)\right)$$

Anti-Quark

$$\overline{\Phi}^{[\Gamma]}(x,b) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-ix\lambda P^+} \mathrm{Tr}\langle P,s\,|\,\frac{\Gamma}{2}Wq(\lambda n + b)\bar{q}W^\dagger(0)\,|\,P,s\rangle\,.$$

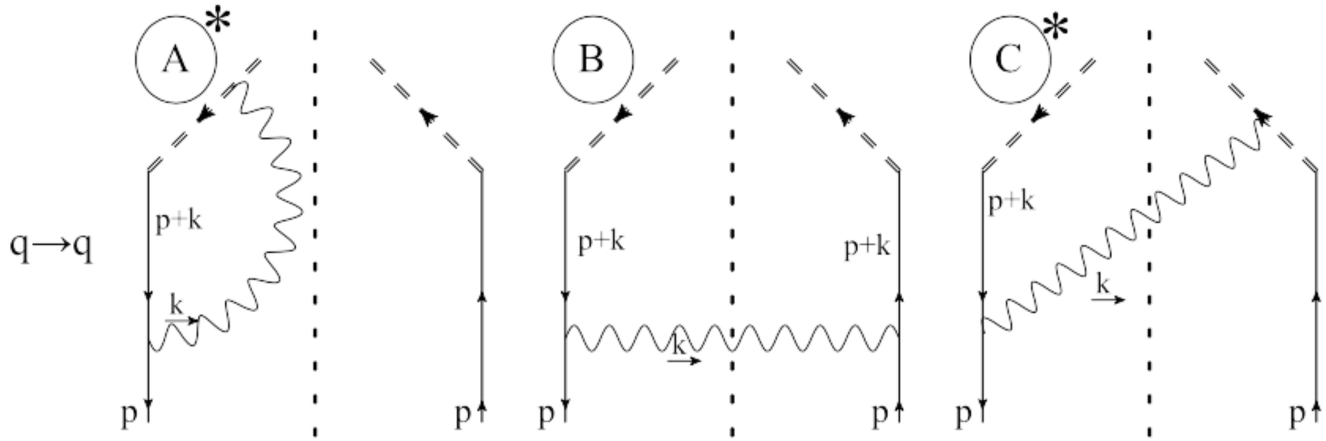
Gluon

$$\Phi_{\mu\nu}(x,b) = \frac{1}{xp_+}\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-ix\lambda P^+} \langle P,s\,|\,F_{\mu+}W^\dagger(\lambda n + b)\frac{\Gamma}{2}WF_{\nu+}(0)\,|\,P,s\rangle,$$

DOES ALL THIS WORK AT ONE LOOP?

We can check what happens assuming the external state is just a quark

$$\Phi^{[\Gamma]}(x,b) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-ix\lambda P^+} \langle q \,|\, \bar{q} W^\dagger(\lambda n + b) \frac{\Gamma}{2} W q(0) \,|\, q \rangle$$



+ mirror diagr.

Diagram A gives the same result for PDF.

For PDF we have A+B+C+mirror diagr.=0, so we can avoid too many calculations.

RAPIDITY REGULATOR (HERE DELTA-REGULATOR)

We need to regularize integrals in the collinear direction.

We also want to remember the future/past pointing directions, e.g.

$$P\exp\left[-ig\int_0^\infty d\sigma A_+(\sigma\bar{n})\right] \to P\exp\left[-ig\int_0^\infty d\sigma A_+(\sigma\bar{n})e^{-\delta^+\sigma}\right],$$
$$P\exp\left[ig\int_{-\infty}^0 d\sigma A_-(\sigma n)\right] \to P\exp\left[ig\int_{-\infty}^0 d\sigma A_-(\sigma n)e^{+\delta^-\sigma}\right],$$

e.g. for TMDPDF

$$W_n(0) = P\exp\left[ig\int_{-\infty}^0 d\sigma A_-(\sigma n)\right] \to P\exp\left[ig\int_{-\infty}^0 d\sigma A_-(\sigma n)e^{+\delta^- x\sigma}\right]$$

Feynman rules changes $\frac{1}{k^+ \pm i0} \to \frac{1}{k^+ \pm i\delta^+}, \qquad \frac{1}{k^- \pm i0} \to \frac{1}{k^- \pm i\delta^-},$

$$\frac{1}{(k_1^+ - i0)(k_2^+ - i0)\ldots(k_n^+ - i0)} \to \frac{1}{(k_1^+ - i\delta^+)(k_2^+ - 2i\delta^+)\ldots(k_n^+ - ni\delta^+)},$$

ELEMENTARY INTEGRALS

$$\int \frac{d^{d-2}k_T}{(2\pi)^{d-2}} \frac{1}{(k_T^2+\Delta_1)^{1+\delta}} = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \frac{\Gamma(\epsilon+\delta)}{\Gamma(1+\delta)} \Delta_1^{-\epsilon-\delta}, \qquad \mathrm{Re}[\delta+\epsilon] > 0.$$

$$\int \frac{d^{d-2}k_T}{(2\pi)^{d-2}} \frac{e^{i(kb)_T}}{(k_T^2)^{1+\delta}} = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \left(\frac{b_T^2}{4}\right)^{\epsilon+\delta} \frac{\Gamma(-\epsilon-\delta)}{\Gamma(1+\delta)} \qquad \mathrm{Re}[\delta+\epsilon] < 0,$$

$$\int \frac{d^{d-2}k_T}{(2\pi)^{d-2}} \frac{e^{i(kb)_T}}{(k_T^2+i\Delta)^{1+\delta}} = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \left[\left(\frac{b_T^2}{4}\right)^{\epsilon+\delta} \frac{\Gamma(-\epsilon-\delta)}{\Gamma(1+\delta)} + (i\Delta)^{-\epsilon-\delta} \frac{\Gamma(\epsilon+\delta)}{\Gamma(1+\delta)}\right] + \mathcal{O}(\Delta)$$

$$\int \frac{d^{d-2}k_T}{(2\pi)^{d-2}} \frac{e^{i(kb)_T}}{(k_T^2+i\Delta_1)^{\delta}(k_T^2+i\Delta_2)} = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \Bigg[\left(\frac{b_T^2}{4}\right)^{\epsilon+\delta} \frac{\Gamma(-\epsilon-\delta)}{\Gamma(1+\delta)}$$
$$+ (i\Delta_2)^{-\epsilon-\delta} \frac{\Gamma(\epsilon+\delta)}{\Gamma(1+\delta)} \, {}_2F_1\left(\delta, \delta+\epsilon, 1+\delta; 1-\frac{\Delta_1}{\Delta_2}\right)\Bigg] + \mathcal{O}(\Delta)$$

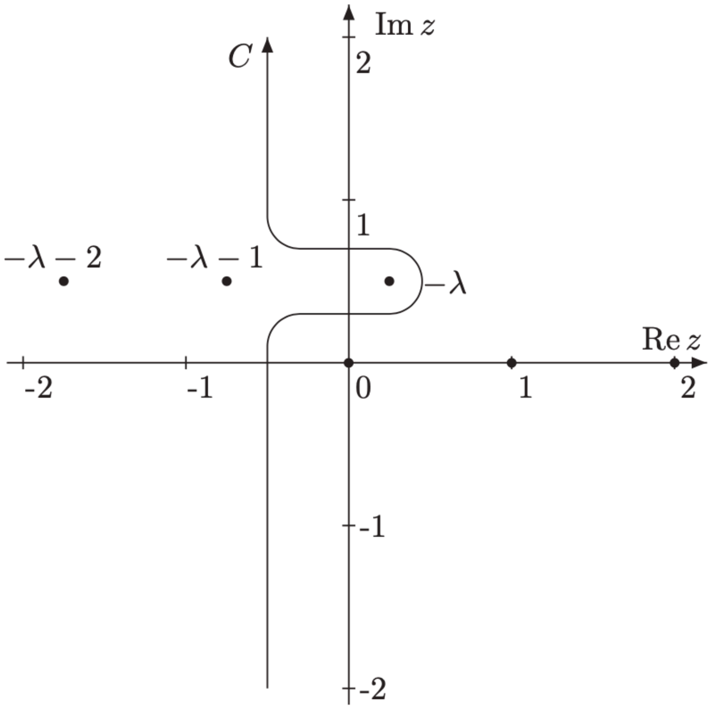
Cut gluon and fermion propagators (imaginary parts or discontinuity)

$$\mathrm{Im} D^{\mu\nu}(q) = -2\pi\theta(q^+)\delta(q^2)g^{\mu\nu}$$

$$\mathrm{Disc}\hat{D}(k) = 2\pi \not{k}\,\delta(k^2)\theta(k^+) = 2\pi \not{k}\,\delta(k^2)\theta(k^-).$$

ELEMENTARY INTEGRALS TECHNIQUE: E.G. MELLIN-BARNES

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda+z)\Gamma(-z) \frac{Y^z}{X^{\lambda+z}} \,.$$



Vladimir A. Smirnov

Evaluating
Feynman Integrals

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Many books in the market

ELEMENTARY 1-LOOP RESULT FOR TMD

$$\Phi_{q\leftarrow q}(x,\mathbf{b},\mu)=\delta(1-x)+a_s 2C_F\mathbf{B}^{\varepsilon}\Gamma(-\varepsilon)\left(p_{qq}(x)-\varepsilon(1-x)-2\delta(1-x)\lambda_\delta\right)+\ldots$$

Complete separation of UV and IR singularities is not achieved!

We have δ^+ instead of δ^-

$$a_s=\frac{\alpha_s}{4\pi}$$

$$p_{qq}=\text{splitting function}$$

$$\mathbf{B}=\mathbf{b}^2/4$$

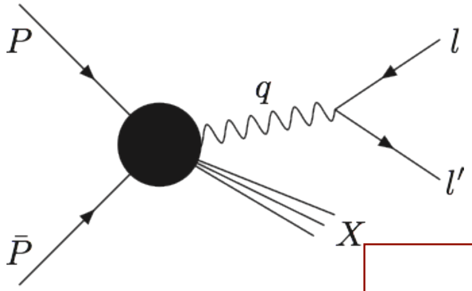
$$\lambda_\delta=\ln\delta^+/p^+$$

In standard QCD calculations, this new divergent part cancels out only in the cross-section calculation: it appears that, with a small transverse momentum, the initial states are entangled.

What is missing? The proper TMD definition needs to be understood starting from a cross section factorization

CLASSICAL KNOWLEDGE ON DRELL-YAN AND DEEP INELASTIC SCATTERING (DIS)

Factorization!



$$\frac{d\sigma}{dQ^2} = \sum_{i,j=q,\bar{q},g} \int_0^1 dx_1\, dx_2\, \mathcal{H}_{ij}(x_1, x_2, Q^2, \mu^2)\, f_{i/P}(x_1, \mu^2)\, f_{j/\bar{P}}(x_2, \mu^2)$$

PDF: Parton Distribution Functions

PDFs describe the probability of finding a parton "i" in a hadron "P":

We can define the probability of a parton in a hadron independently of the probability of the other parton.

The quark states are completely disentangled!

Unpolarized distributions only couple to each other.

PDFs are the only nonperturbative inputs.

TRYING TO EXPAND OUR KNOWLEDGE NAIVELY

Suppose we measure differential cross sections, can we write (Collins, Soper, Sterman '80)?

$$\frac{d\sigma}{dQ^2 dq_T dy} = \sum_q \sigma_q^\gamma H(Q^2,\mu^2) \int \frac{d^2\mathbf{b}}{4\pi} e^{-i\mathbf{q_T}\cdot\mathbf{b}} \Phi_{q/A}(x_A,\mathbf{b},\mu)\Phi_{q/B}(x_B,\mathbf{b},\mu)?$$

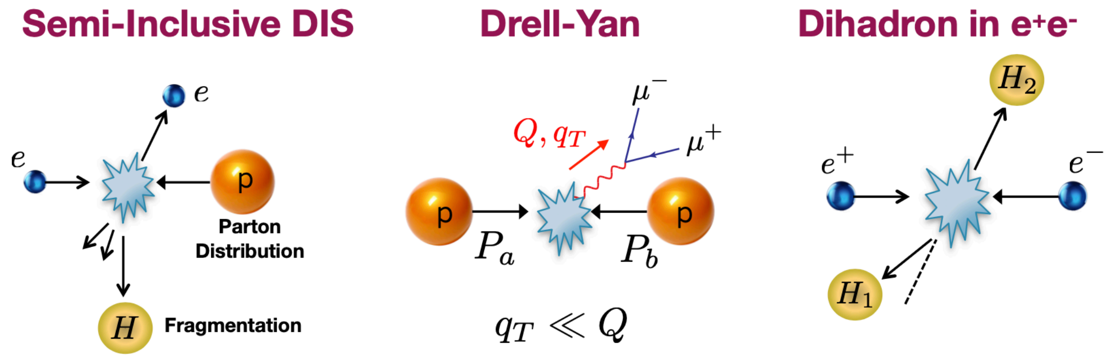
Unfortunately we have new divergences that cannot be regularized with dimensional regularization, recall

$$\Phi_{q\leftarrow q}(x,\mathbf{b},\mu) = \delta(1-x) + a_s 2C_F \mathbf{B}^\varepsilon \Gamma(-\varepsilon)\left(p_{qq}(x) - \varepsilon(1-x) - 2\delta(1-x)\lambda_\delta\right) + \ldots$$

Dead end: collinear factorization cannot work as usual

FACTORIZATION

A physical process is typically characterized by several energy scales. By expanding the cross sections into powers of ratios of small scales to large ones, fundamental QCD distributions/correlations are discovered.



Leading Quark TMDPDFs

Nucleon Spin — Quark Spin

Nucleon Polarization	Quark Polarization: Un-Polarized (U)	Quark Polarization: Longitudinally Polarized (L)	Quark Polarization: Transversely Polarized (T)
U	f_1 = Unpolarized		$h_1^\perp$ = Boer-Mulders
L		g_1 = Helicity	$h_{1L}^\perp$ = Worm-gear
T	$f_{1T}^\perp$ = Sivers	$g_{1T}^\perp$ = Worm-gear	h_1 = Transversity $h_{1T}^\perp$ = Pretzelosity

FACTORIZATION OF THE CROSS SECTION

$$W^{\mu\nu}_{\mathrm{DY}} = \int \frac{d^4y}{(2\pi)^4} e^{-i(yq)} \sum_X \langle p_1, p_2 \,|\, J^{\mu\dagger}(y) \,|\, X \rangle \langle X \,|\, J^{\nu}(0) \,|\, p_1, p_2 \rangle$$

In order to arrive to some factorization we need some kinematical limits

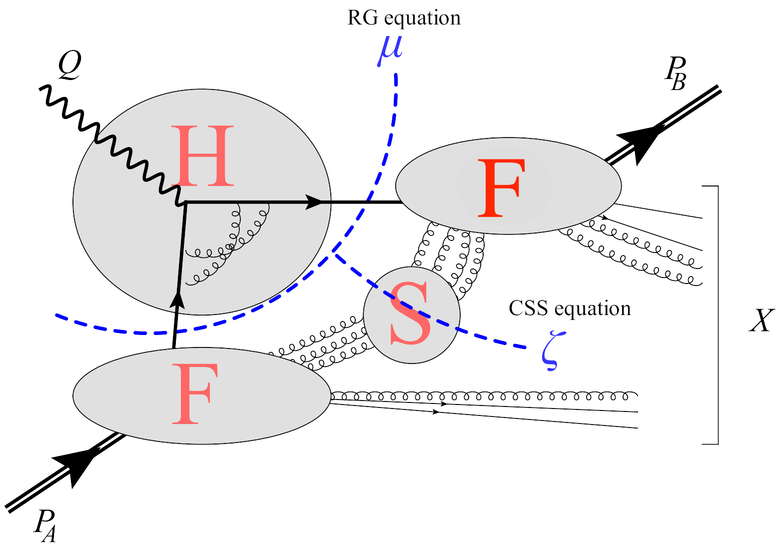
$$Q^2 \gg \Lambda^2, \qquad Q^2 \gg \mathbf{q_T}^2 = \text{fixed}$$

$$Q^2 = q^2 = 2q^+q^- - \mathbf{q_T}^2 \quad q_\mu W^{\mu\nu} = 0$$

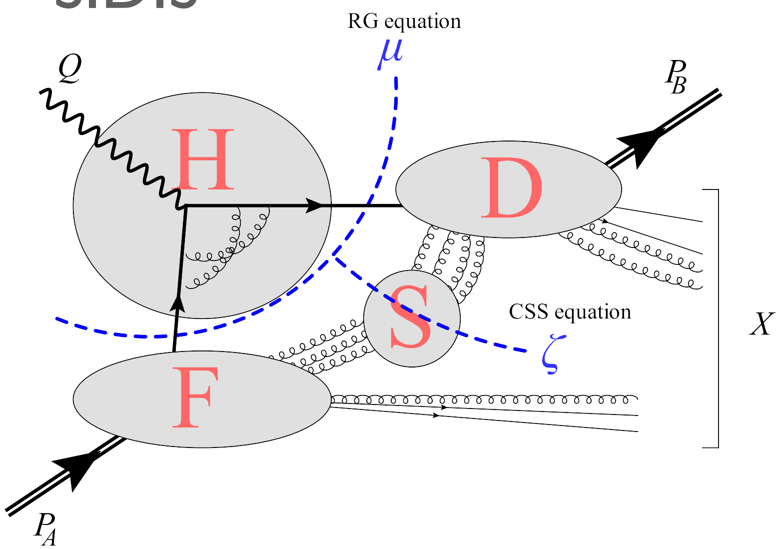
We would also like to be consistent with transversality of the hadronic tensor

SOFT RADIATION

DY



SIDIS

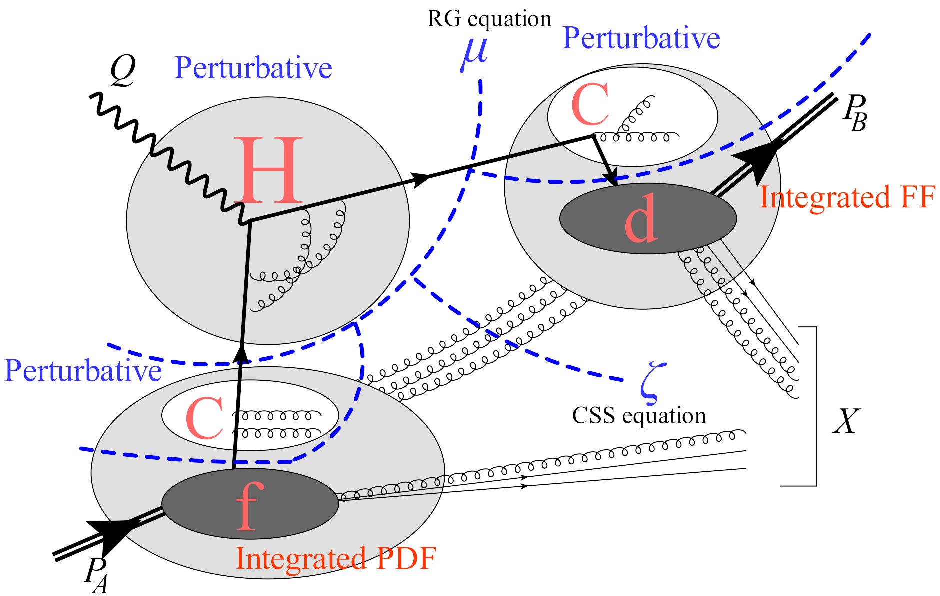


The μ-scale allows to separate hard interactions from the rest.
We need another scale to divide soft interactions

There are several ways to implement this separation:

Diagrammatic (Collins), SCET, Background field method

RECOVERING COLLINEAR FACTORIZATION PICTURES



We can also do a small-b asymptotic limit and recover collinear distributions. This can be useful for phenomenology in some cases.

BASICS: MODES AND EXPANSIONS

$\lambda \sim q_T/Q$ Expansion parameter

We have several modes, fixed by momenta

n-collinear $\sim (\lambda^2, 1, \lambda)$, $\bar{n}$-collinear $\sim (1, \lambda^2, \lambda)$, soft $\sim (\lambda, \lambda, \lambda)$

$$\partial^\mu \phi \sim (\lambda^2, 1, \lambda)\phi\,, \qquad \partial^\mu \phi \sim (1, \lambda^2, \lambda)\phi\,, \qquad \partial^\mu \phi \sim (\lambda, \lambda, \lambda)\phi$$

$$A^\nu \sim (\lambda^2, 1, \lambda)\,, \qquad A^\nu \sim (1, \lambda^2, \lambda)\,, \qquad A^\nu \sim (\lambda, \lambda, \lambda)$$

The explicit Lagrangian formulation differs slightly among authors in the literature: SCET (Becher, Neubert; Echevarria, I.S., Idilbi; Chiu et al…, Ebert et al., 2011-2022), Background method without soft modes (Vladimirov, Moos, I.S., 2022), Background method with soft modes (Del Rio, Jaarsma, I.S. ,Waalewijn, next week, 2025). There is a substantial agreement but details differ (it may cause confusion).

SOFT MODES IN SCET

SCET proves that different modes do not interact at the Lagrangian level, based on the "method of regions" (Beneke, Smirnov). The basic concept is that every perturbative integral can be reconstructed as a sum of integrals that contain only fields of a single mode.

$$|\Psi\rangle = \Psi[\phi;\phi;\phi]\,|0\rangle = |\Psi_c\rangle\,|\Psi_s\rangle\,|\Psi_{\bar{c}}\rangle$$

This statement however must be refined:

- ***Naive calculations of collinear modes actually overlap with soft modes. We need to understand zero-bin subtraction to define "pure collinear modes"***
- ***The zero-bin subtraction is rapidity-regulator dependent***
- ***After all this we proceed to separate partially include soft radiation into TMD definitions to be construct functions rapidity regulator independent***

ZERO-BIN SUBTRACTION

It is necessary to give an explicit definition of collinear fields, where soft interactions are explicitly subtracted.

$$\chi \to S_{\bar{n}}^{\dagger} S_n \chi + \Psi_{\bar{n}}$$

$$\mathscr{A}^{\mu} \to S_{\bar{n}}^{\dagger} S_n \mathscr{A}^{\mu} S_n^{\dagger} S_{\bar{n}} + \mathscr{A}_{\bar{n}}^{\mu}$$

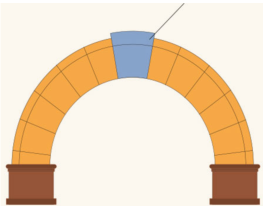
$$\langle P | O_{\alpha\beta}^{\text{sub, bare}}(xp^+, b_\perp) | P \rangle = \langle P | \bar{\chi}_\alpha(b_\perp + xp^+ \bar{n}) | X \rangle \langle X | \chi_\beta(0) | P \rangle^{\text{sub, bare}}$$

$$= \frac{\langle P | \bar{\chi}_\alpha(b_\perp + xp^+ \bar{n}) | X \rangle \langle X | \chi_\beta(0) | P \rangle^{\text{uns, bare}}}{\langle 0 | S_n^{\dagger} S_{\bar{n}}(b_\perp) S_{\bar{n}}^{\dagger} S_n(0) | 0 \rangle} = \frac{O_{\alpha\beta}^{\text{uns, bare}}(xp^+, b_\perp)}{S(b_\perp)}$$

With ***soft factor***

$$\boxed{S(b_\perp) = \frac{\mathrm{Tr}_c}{N_c} \langle 0 | T \left[S_n^{\dagger} S_{\bar{n}} \right](b_\perp) \bar{T}[S_{\bar{n}}^{\dagger} S_n](0) | 0 \rangle}$$

RAPIDITY DIVERGENCES AND THE SOFT FUNCTION

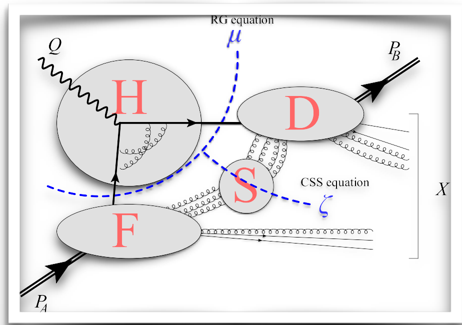


$$S(b_T) = \frac{1}{N_c}\langle 0|[-\infty n, b_T, \infty\bar{n}][\infty\bar{n}, 0, -\infty n]|0\rangle, \quad [\gamma] \sim P\exp\left(-ig\int_\gamma A_\gamma\right)$$

- The rapidity divergences arise in the limit $k^+ \to \infty,\ k^- \to 0,\ k^+k^-$ fixed
- The Soft Function is undefined without a regulator for rapidity divergences
- The log of the Soft Function is linear in the rapidity regulator at all orders

$$\tilde{S}(\mathbf{L}_\mu, \mathbf{L}_{\sqrt{\delta^+\delta^-}}) = \tilde{S}^{\frac{1}{2}}(\mathbf{L}_\mu, \mathbf{L}_{\delta^+})\, \tilde{S}^{\frac{1}{2}}(\mathbf{L}_\mu, \mathbf{L}_{\delta^-})$$

- By definition the Soft Function is just 1 in the limit $b_T \to 0$

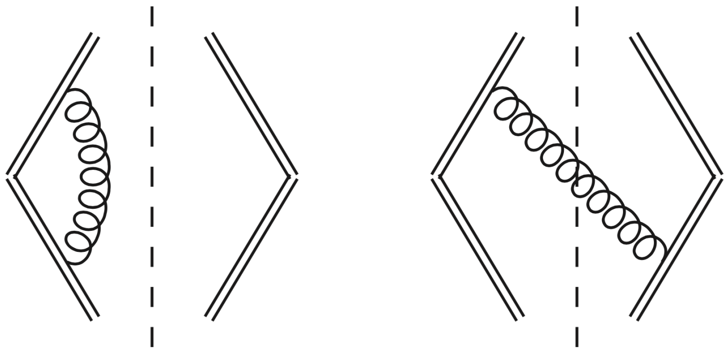


$$\mathbf{L}_X \equiv \ln(X^2 b_T^2 e^{2\gamma_E}/4). \qquad \mathbf{l}_\zeta = \ln(\mu^2/\zeta)$$

ONE LOOP SOFT FUNCTION

$$S_A = C_F g^2 (-i)(\mu^2)^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{(n\bar{n})}{(k^+ + i\delta^+)(k^- + i\delta^-)(k^2 + i0)} =$$

$$S_B = C_F g^2 (-2\pi)(\mu^2)^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{(n\bar{n})e^{i(kb)_T}}{(k^+ + i\delta^+)(-k^- - i\delta^-)} \theta(k^+)\delta(k^2)$$



A B

d: UV-Dim.Reg.

$\delta^\pm$: rapidity div.

ONE LOOP SOFT FUNCTION

$$S_A = -\frac{2g^2}{(4\pi)^{\frac{d}{2}}}\boldsymbol{\delta}^{-\epsilon}\Gamma^2(\epsilon)\Gamma(1-\epsilon) + h.c.$$

Mixed divergences cancel at all orders

$$S_B = \frac{2g^2}{(4\pi)^{\frac{d}{2}}}\left[\boldsymbol{\delta}^{-\epsilon}\Gamma^2(\epsilon)\Gamma(1-\epsilon) - \boldsymbol{B}^{\epsilon}\Gamma(-\epsilon)\left(L_+ - \psi(-\epsilon) - \gamma_E\right)\right] + h.c.$$

Summing up only UV and rapidity divergences survive:

$$S^{[1]} = -4\boldsymbol{B}^{\epsilon}\Gamma(-\epsilon)\left(L_0 - \psi(-\epsilon) - \gamma_E\right)$$

The trivial limit (b_T =0) easily recovered

$$\boldsymbol{\delta} = \pm\delta^+\delta^-, \quad \boldsymbol{B} = \frac{b_T^2}{4}$$

$$L_0 = \ln\left(\frac{\boldsymbol{B}|\boldsymbol{\delta}|}{e^{-2\gamma_E}}\right)$$

ONE LOOP SOFT FUNCTION

The divergences appearing in the calculations of the soft factor are not infrared in the final result. But there are infrared pieces in intermediate steps (check with gluon mass)

Origin of divergences: A.Idilbi, M.G. Echevarria, I.S. arXiv:1310.8541, Int.J.Mod.Phys.Conf.Ser. 25 (2014) 1460005

$$\boxed{S_A = -2ig_s^2 C_F \delta^{(2)}(\vec{k}_{s\perp})\mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^+ - i\delta^+][k^- + i\delta^-][k^2 - \lambda^2 + i0]} + h.c.}$$

$$\boxed{S_B = 4\pi g_s^2 C_F \mu^{2\varepsilon} \int d^d\boldsymbol{k}_s\, e^{i\boldsymbol{k}_s \boldsymbol{b}} \int \frac{d^d k}{(2\pi)^d} \frac{\delta^{(2)}(\boldsymbol{k} + \boldsymbol{k}_s)\delta(k^2 - \lambda^2)\theta(k^+)}{(k^+ + i\delta^+)(k^- - i\delta^-)} + h.c.}$$

d: UV-Dim.Reg.

λ : IR div.

$\delta^{\pm}$: rapidity div.

ONE LOOP SOFT FUNCTION

$$S_A = \frac{\alpha_s C_F}{2\pi}\left[\frac{-2}{\varepsilon_{\rm UV}^2} + \frac{2}{\varepsilon_{\rm UV}}\ln\frac{\delta^+\delta^-}{\mu^2} + \ln^2\frac{\lambda^2}{\mu^2} - 2\ln\frac{\lambda^2}{\mu^2}\ln\frac{\delta^+\delta^-}{\mu^2} + \frac{\pi^2}{6}\right]$$

$$S_B = \frac{\alpha_s C_F}{2\pi}\left[L_\perp^2 + 2L_\perp\ln\frac{\delta^+\delta^-}{\mu^2} + 2\ln\frac{\lambda^2}{\mu^2}\ln\frac{\delta^+\delta^-}{\mu^2} - \ln^2\frac{\lambda^2}{\mu^2}\right]$$

ε_{UV} : Dim. Reg.

λ : IR div.

$\delta^\pm$: rapidity div.

$$L_\perp = \ln\frac{\mu^2 b_T^2 e^{2\gamma_E}}{4}$$

Summing up only UV and rapidity divergences survive:

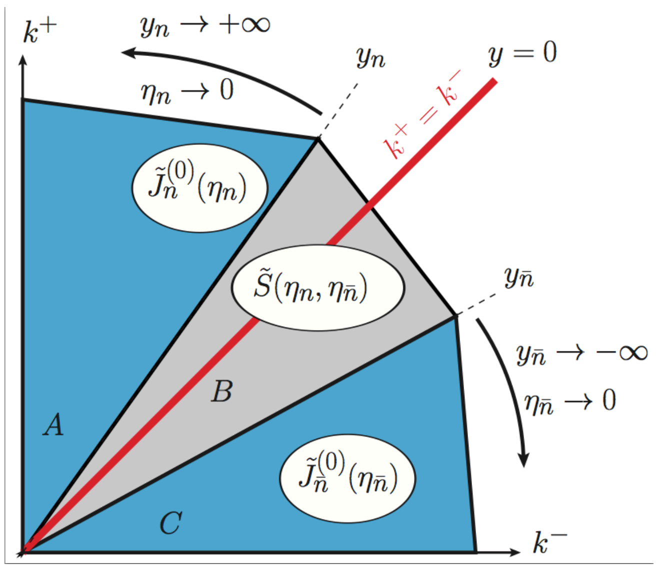
$$\tilde{S}_1 = \frac{\alpha_s C_F}{2\pi}\left[\frac{-2}{\varepsilon_{\rm UV}^2} + \frac{2}{\varepsilon_{\rm UV}}\ln\frac{\delta^+\delta^-}{\mu^2} + L_\perp^2 + 2L_\perp\ln\frac{\delta^+\delta^-}{\mu^2} + \frac{\pi^2}{6}\right]$$

To compute a NNLO calculation we need this result at all orders in ε

PURE COLLINEAR OPERATORS

With delta-regulator the zero-bin subtraction of collinear functions corresponds to a re-writing $\delta^+ \leftrightarrow \delta^-$. In this way each collinear functions depends on the correct regulator for its sector.

$$\text{Tr}(\overline{\Gamma}\overline{\mathcal{O}}^{\text{sub, bare}}(Q^2,\mu^2,\delta_+)) = \frac{\text{Tr}(\overline{\Gamma}\overline{\mathcal{O}}^{\text{uns, bare}}(Q^2,\mu^2,\delta_-))}{S(b,\mu^2,\delta^+\delta^-)}$$



A=pure-anticollinear

C=pure- collinear

B=Soft

A+B=naive anticollinear

C+B=naive collinear

FACTORIZED HADRONIC TENSOR

FINALLY THE HARD FACTOR IS JUST THE SQUARE OF THE CURRENT CALCULATED PERTURBATEVELY $H(Q^2, \mu^2) = |C_V(Q^2, \mu^2)|^2$

$$W_{LP} \sim |C_V(Q^2, \mu^2)|^2 \mathrm{Tr}(\Gamma \mathcal{O}^{\text{uns, bare}}(Q^2, \mu^2, \delta_+)) \mathrm{Tr}(\overline{\Gamma \mathcal{O}}^{\text{uns, bare}}(Q^2, \mu^2, \delta_-)) S(\mu^2, \delta_+ \delta_-)$$

$$\to |C_V(Q^2, \mu^2)|^2 \mathrm{Tr}(\Gamma \mathcal{O}^{\text{sub, bare}}(Q^2, \mu^2, \delta_-)) \mathrm{Tr}(\overline{\Gamma \mathcal{O}}^{\text{sub, bare}}(Q^2, \mu^2, \delta_+)) S(\mu^2, \delta_+ \delta_-)$$

$$= |C_V(Q^2, \mu^2)|^2 \mathrm{Tr}(\Gamma \mathcal{O}^{\text{bare}}(Q^2, \mu^2, \zeta)) \mathrm{Tr}(\overline{\Gamma \mathcal{O}}^{\text{bare}}(Q^2, \mu^2, \bar{\zeta}))$$

And TMD are defined as

$$\mathrm{Tr}(\Gamma \mathcal{O}^{\text{bare}}(Q^2, \mu^2, \zeta)) = \mathrm{Tr}(\Gamma \mathcal{O}^{\text{sub, bare}}(Q^2, \mu^2, \delta_-)) \sqrt{S(\mu^2, \alpha \delta_- \delta_-)}$$

$$\mathrm{Tr}(\Gamma \mathcal{O}^{\text{bare}}(Q^2, \mu^2, \bar{\zeta})) = \mathrm{Tr}(\Gamma \mathcal{O}^{\text{sub, bare}}(Q^2, \mu^2, \delta_+)) \sqrt{S(\mu^2, \bar{\alpha} \delta_+ \delta_+)}$$

With

$$\alpha = \frac{\zeta}{(p^-)^2}; \bar{\alpha} = \frac{\bar{\zeta}}{(p^+)^2}; \qquad \zeta \bar{\zeta} = Q^4$$

SOME REFERENCES TO WORKS THAT INSPIRED THESE CLASSES

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