RG EQUATIONS AND HIGH ENERGY BEHAVIOUR IN NON-RENORMALIZABLE THEORIES

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Motivation:

• The Standard Model is renormalizable
• Gravity is not renormalizable

Non-renormalizable theories are not accepted due to:

• UV divergences are not under control - infinite number of new types of divergences
• The amplitudes increase with energy (in PT) and violate unitarity

Suggestion (novel approach to NR interactions):

• To replace the multiplicative renormalization procedure by a new one, where the renormalization constant $Z$ is replaced by an operator $\hat{Z}$, which depends on kinematics
• To sum up the leading asymptotics in all orders of PT (generalized RG) and to study the high-energy behavior
Renormalization

Bogolyubov-Parasiuk Theorem: In any local quantum field theory to get the UV finite S-matrix one has to introduce local counter terms to the Lagrangian in each order of perturbation theory - R-operation

$$\mathcal{L} \Rightarrow \mathcal{L} + \Delta \mathcal{L}$$

In renormalizable case this is equivalent to the operation of multiplication by a renormalization constant $Z$

Consider 2->2 scattering amplitude on shell

$$\Gamma_4(s, t, u) = \Gamma_4^{tree}\bar{\Gamma}_4(s, t, u) \quad \bar{\Gamma}_4 = 1 + \lambda... + \lambda^2...$$

Renormalization (dimensional regularization)

$$\bar{\Gamma}_4 = Z_4(\lambda)\bar{\Gamma}_4^{bare}|_{\lambda_{bare} \rightarrow \lambda Z_4},$$

$$\lambda_{bare} = \mu^e Z_4(\lambda)\lambda$$

BPHZ R-operation

$$RG = (1 - K)R'G$$

$$Z = 1 - \sum_i KR'_i G_i$$
Renormalization

In non-renormalizable case the BP theorem is still valid and the counter terms are also local (at maximum are polynomial over momenta)

- **Multiplication** operation is replaced by acting of an operator $Z \rightarrow \hat{Z}$

$\hat{Z}$ is a function (polynomial) of momenta ($s,t,u$ for the 4-point case)

- When acting on the diagram the $\hat{Z}$ factor has to inserted inside the diagram and integrated over the internal loop

Example (taken from D=8 YM theory)

Exactly follows the BPHZ $\mathcal{R}$-operation

$\hat{Z} = 1 + g^2 \frac{st}{\epsilon} \quad \Rightarrow \quad g^2 s t \quad \Rightarrow \quad g^2 \left( s \quad + \quad t \right)$

Either $s$ or $t$ are to be inserted into the loop and integrated
\[ \mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^n\epsilon}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)}\epsilon}{\epsilon^n} + \ldots + \frac{A_1^{(n)}(\mu^2)\epsilon}{\epsilon^n} \]

lower pole terms

\[ A_k^{(n)}(\mu^2)^{k\epsilon} \] terms appear after subtraction of (n-k) loop counter terms

Statement: \( R'G_n \) is local, i.e. terms like \( \log^k \mu^2/\epsilon^m \) should cancel for any \( k \) and \( m \)

Consequence: \( A_n^{(n)} = (-1)^{n+1} \frac{A_1^{(n)}}{n} \)

\[ KR'G_n = \sum_{k=1}^{n} \left( \frac{A_k^{(n)}}{\epsilon^n} \right) \equiv \frac{A_n^{(n)'}\epsilon^n}{\epsilon^n} \]

\[ A_n^{(n)'} = (-1)^{n+1} A_n^{(n)} = \frac{A_1^{(n)}}{n}. \]

\( A_1^{(n)} \) is the contribution to the leading pole in n-loops from the diagrams appearing in due course of \( \mathcal{R} \)-operation after subtraction of (n-1) loop counter terms

The leading divergences are governed by 1 loop diagrams!
Two loop example

$$\phi_4^4 = \left( \frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon} \right) \left( \frac{\mu^2}{s} \right)^{2\epsilon}$$

$$\mathcal{R}' = \phi_4^4 - \text{non-local terms to be cancelled}$$

$$= \frac{A_2^{(2)}}{\epsilon^2} - \left( \frac{A_1^{(1)}}{\epsilon^2} \right)^2 + 2 \frac{A_2^{(2)}}{\epsilon} \log(\mu^2/s) - \frac{A_1^{(1)}}{\epsilon} \log(\mu^2/s) = -\frac{1}{2} \frac{(A_1^{(1)})^2}{\epsilon^2} + ...$$

Leading divergence is given by the one-loop term

$$A_2^{(2)} = \frac{1}{2} (A_1^{(1)})^2$$

• These statements are universal and are valid in non-renormalizable theories as well.
• The only difference is that the counter term \( A_1^{(1)} \) depends on kinematics and has to be integrated through the remaining one-loop graph.
• As a result \( A_2^{(2)} \) is not the square of \( A_1^{(1)} \) anymore but is the integrated square (see below).
• This last statement is the general feature of any QFT irrespective of renormalizability
The Recurrence Relation

\[ n \quad A_n = -2 \quad A_{n-1} - \sum_{k=1}^{n-2} A_k \quad A_{n-1-k} \]

• This is the general recurrence relation that reflects the locality of the counter terms in any theory
• In renormalizable theories \( A_n \) is a constant and this relation is reduced to the algebraic one
• In non-renormalizable theories \( A_n \) depends on kinematics and one has to integrate through the one loop diagrams

Taking the sum \( \sum_n A_n(-z)^n = A(z) \) one can transform the recurrence relation into integro-diff equation

\[ \frac{d}{dz} A(z) = -1 - 2 \int_{\Delta} A(z) - \int_{\bigcirc} A^2(z) \]

This is the generalized RG equation valid in any (even non-renormalizable) theory!
The Scalar theory example

\( \phi^4_D \)

\[ D = 4, 6, 8, 10 \quad [\lambda] = 2 - D/2 \]

Kazakov, 19

2->2 scattering amplitude on shell

\[ m = 0 \quad s + t + u = 0 \]

\[ \Gamma_4(s, t, u) = \lambda (1 + \Gamma_s(s, t, u) + \Gamma_t(s, t, u) + \Gamma_u(s, t, u)) \]

PT:

\[ \Gamma_s = \sum_{n=1}^{\infty} (-z)^n S_n, \quad \Gamma_t = \sum_{n=1}^{\infty} (-z)^n T_n, \quad \Gamma_u = \sum_{n=1}^{\infty} (-z)^n U_n, \quad z \equiv \frac{\lambda}{\epsilon} \]

PT expansion (only s-channel is shown)
Recurrence Relations for the Leading Poles

\[
R' = \sum_{k=1}^{n-2} (n-k-1)\text{-loop} - \cdots \text{terms with higher loop remaining diagrams}
\]

\[
KR' = \frac{A^{(k)}_k}{\epsilon^k} = (-1)^{k+1} \frac{A^{(k)}_k}{\epsilon^k}
\]

\[
nS_n(s, t, u) = \frac{s^{D/2-2}}{\Gamma(D/2-1)} \int_0^1 dx [x(1-x)]^{D/2-2} (S_{n-1}(s, t', u') + T_{n-1}(s, t', u') + U_{n-1}(s, t', u'))
\]

\[+ \frac{1}{2 \Gamma(D/2-1)} \int_0^1 dx [x(1-x)]^{D/2-2} \sum_{k=1}^{n-2} (D/2-2)^k \sum_{p=0}^k \sum_{l=0}^p \frac{1}{p!(p + D/2 - 2)!} \times \]

\[\times \frac{d^p}{dt'^l du'^p-l} (S_k + T_k + U_k) \frac{d^p}{dt'^l du'^p-l} (S_{n-k-1} + T_{n-k-1} + U_{n-k-1}) s^p [x(1-x)]^p t'^l u'^{p-l}
\]

\[t' = -xs, u' = -(1-x)s\]
Differential Equation

Summing up the recurrence relation \( \sum_{n=2}^{\infty} (-z)^n \) one gets the diff equation

\[
\begin{align*}
- \frac{d\Gamma_s(s, t, u)}{dz} &= \frac{1}{2} \frac{\Gamma(D/2 - 1)}{\Gamma(D - 2)} s^{D/2 - 2} \\
+ \frac{1}{\Gamma(D/2 - 1)} \int_0^1 dx [x(1-x)]^{D/2 - 2} \left[ \Gamma_s(s, t', u') + \Gamma_t(s, t', u') + \Gamma_u(s, t', u') \right] \\
& \quad \left| t' = -xs, \quad u' = -(1-x)s \right. \\
+ \frac{1}{2} \frac{s^{D/2 - 2}}{\Gamma(D/2 - 1)} \int_0^1 dx [x(1-x)]^{D/2 - 2} \sum_{p=0}^{\infty} \sum_{l=0}^{p} \frac{1}{p!(p + D/2 - 2)!} \\
\times \left( \frac{d^p}{dt^n du'^p-l} \left( \Gamma_s + \Gamma_t + \Gamma_u \right) \bigg| t' = -xs, \quad u' = -(1-x)s \right)^2 s^p [x(1-x)]^p t^l u^{p-l}
\end{align*}
\]

\[
\begin{align*}
\frac{d\Gamma_s(s, t, u)}{d \log \mu^2} &= - \frac{\lambda}{2} \frac{s^{D/2 - 2}}{\Gamma(D/2 - 1)} \int_0^1 dx [x(1-x)]^{D/2 - 2} \sum_{p=0}^{\infty} \sum_{l=0}^{p} \frac{1}{p!(p + D/2 - 2)!} \\
& \quad \times \left( \frac{d^p \Gamma_4(s, t', u')}{dt^n du'^p-l} \bigg| t' = -xs, \quad u' = -(1-x)s \right)^2 s^p [x(1-x)]^p t^l u^{p-l}
\end{align*}
\]

\( \Gamma_s (\log \mu^2 = 0) = 0 \)
Solution of RG Equation

\[ D = 4 \]

\[
\frac{d \bar{\Gamma}_4}{d \log \mu^2} = -\lambda \frac{3}{2} \bar{\Gamma}_4, \quad \bar{\Gamma}_4(\log \mu^2 = 0) = 1
\]

\[
\bar{\Gamma}_4 = \frac{1}{1 + \frac{3}{2} \lambda \log(\mu^2/E^2)}
\]

General Solution for any D

\[
\bar{\Gamma}_4(s, t, u) = \mathcal{P} \frac{1}{1 + \frac{1}{2} \frac{\Gamma(D/2-1)}{\Gamma(D-2)} \lambda (s^{D/2-2} + t^{D/2-2} + u^{D/2-2}) \log(\mu^2/E^2)}
\]

\( \mathcal{P} \) is the symbol of ordering in a sense of recurrence relation

\[
\Gamma_4(s, t, u) = \mathcal{P} \frac{\lambda}{1 + \lambda A_1^{(1)} \log(\mu^2/E^2)} = \mathcal{P} \sum_{n=0}^{\infty} (-\lambda)^n \log^n(\mu^2/E^2) (A_1^{(1)})^n
\]

\[
\mathcal{P}(A_1^{(1)})^n = \int_0^1 dx \sum_{k=0}^{n-1} \mathcal{P}(A_1^{(1)})^k A_1^{(1)} \mathcal{P}(A_1^{(1)})^{n-1-k},
\]
High Energy Behaviour of the scattering amplitude in $\phi_4^D$ theory

$$\Gamma_4(s,t,u) = \mathcal{P} \frac{\lambda}{1 + \frac{1}{2} \frac{\Gamma(D/2-1)}{\Gamma(D-2)} \lambda (s^{D/2-2} + t^{D/2-2} + u^{D/2-2}) \log(\mu^2/E^2)}$$

$$s \sim t \sim u \sim E^2$$

$D = 4 \quad 3/2 > 0 \quad$ As a result one has a Landau pole as $E \to \infty$

$D = 6 \quad s + t + u = 0 \quad$ All the leading divergences (logs) cancel in all loops

One can explicitly check that $S_2$ given above vanishes

$D = 8 \quad s^2 + t^2 + u^2 > 0 \quad$ has a Landau pole as $E \to \infty$

$D = 10 \quad s^3 + t^3 + u^3 = 3stu > 0 \quad s > 0, t, u < 0 \quad$ has a Landau pole as $E \to \infty$

Conclusion: $\phi_4^D$ has a Landau pole as $E \to \infty$
Maximal SUSY theories in various dimensions

- **D=4** N=4
  - Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)

- **D=6** N=2
  - First UV divergent diagrams at D=4+6/L

- **D=8** N=1
  - Conformal or dual conformal symmetry

- **D=10** N=1
  - Common structure of the integrands

**D=4 N=8** Supergravity

- On-shell finite up to 7 loops
- Similar to higher dim SYM

**Object**: Helicity Amplitudes on mass shell with arbitrary number of legs and loops

**The case**: Planar limit

\[ N_c \to \infty, \quad g_{YM}^2 \to 0 \quad \text{and} \quad g_{YM}^2 N_c \quad \text{- fixed} \]

**The aim**: to get all loop (exact) result

Study of higher dim SYM gives insight into quantum gravity
Perturbation Expansion for the 4-point Amplitudes for any D

\[ \frac{A_4}{A_4^{\text{tree}}} \]

\(-g^2\)  
\(\text{st} \)

\(g^4\)  
\(s^2t \)  +  \(st^2 \)

No bubbles
No Triangles

\(-g^6\)  
\(s^3t \)  +  \(s^2t \)  +  \(st^2 \)  +  \(st^3 \)

First UV div at \(L = [6/(D-4)] \) loops

\(g^8\)  
\(s^4t \)  +  \(s^3t \)  +  \(s^2t \)  +  \(s^1t^4 \)  +  \(st^4 \)

IR finite

\(-g^{10}\)  
\(s^5t \)  +  \(s^4t \)  +  \(s^3t \)  +  \(s^2t \)  +  \(st^5 \)

Universal expansion for any D in maximal SYM due to Dual conformal invariance

T. Dennen Yu-yin Huang, S. Caron-Huot, D. O'Connell
Recursion relations and RG equations

Leading logs \( s \to \infty, t \to \infty \)

UV divergences \( \mathbb{D}=6 \ \mathbb{N}=2 \)

\[
M_4(s, t) = 1 + \Sigma_s(s, t) + \Sigma_t(s, t)
\]

\[
\Sigma_s = \sum_{n=3}^{\infty} (-z)^n S_n, \quad \Sigma_t = \sum_{n=3}^{\infty} (-z)^n T_n
\]

s-channel term \( S_n(s, t) \) \quad t-channel term \( T_n(s, t) \)

\[
T_n(s, t) = S_n(t, s)
\]

Exact relation for all diagrams

\[
nS_n(s, t) = -2s \int_0^1 dx \int_0^x dy \ (S_{n-1}(s, t') + T_{n-1}(s, t'))
\]

\[
t' = t(x - y) - sy
\]

\[
S_3 = -s/3, \quad T_3 = -t/3
\]

Diff equation \quad Generlized RG equation

\[
z \frac{d}{dz} \Sigma_s(s, t, z) = sz - 2 \Sigma_s(s, t, z) + 2sz \int_0^1 dx \int_0^x dy (\Sigma_s(s, t', z) + \Sigma_t(s, t', z)) |_{t' = xt + yu}
\]

\[
z \equiv \frac{g^2}{\epsilon} \leftrightarrow g^2 \log(\mu^2)
\]

Bork, Kazakov, Kompaneets, Vlasenko, 13
Solution of RG equation \[ D=6 \ N=2 \]

Horizontal ladder + tennis court

\[
\Sigma_L(s, z) = \frac{2}{s^2 z^2} (e^{sz} - 1 - sz - \frac{s^2 z^2}{2})
\]

\[
\Sigma_{L2} = \frac{1}{2s^2 z^2} \left[ 27(e^{z/3} - 1 - \frac{z}{3} - \frac{1}{2} \frac{z^2}{9} - \frac{1}{6} \frac{z^3}{27}) (1 + 2\frac{t}{s}) - (e^z - 1 - sz - \frac{1}{2} z^2 - \frac{1}{6} z^3) \right]
\]

In general case - numerical solution similar to the ladder approximation

\[
\Sigma_s + \Sigma_t \sim e^{(s+t)z}
\]

- \[ s + t = -u > 0, \ \Sigma \to \infty \]
- \[ z \to \infty \]
- \[ s + u = -t > 0, \ \Sigma \to \infty \]
- \[ t + u = -s < 0, \ \Sigma \to \text{const} \]
**D=8 N=1**

**Recursion relations and RG equations**

Bork, Kazakov, Tolkachev, Vlasenko, 14

**Leading logs** \( s \to \infty, \ t \to \infty \)  

**UV divergences**

\[
\begin{align*}
nS_n(s, t) &= -2s^2 \int_0^1 dx \int_0^x dy \ y(1-x) \ (S_{n-1}(s, t') + T_{n-1}(s, t')) |_{t'=tx+yu} \\
+ s^4 \int_0^1 dx \ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \\
&\times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t')) |_{t'=-sx} (tsx(1-x))^p
\end{align*}
\]

**Diff equation**

\[
S_1 = \frac{1}{12}, \quad T_1 = \frac{1}{12}
\]

\[
\begin{align*}
\frac{d}{dz} \Sigma(s, t, z) &= -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy \ y(1-x) \ (\Sigma(s, t', z) + \Sigma(t', s, z)) |_{t'=tx+yu} \\
- s^4 \int_0^1 dx \ x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!} \left( \frac{d^p}{dt'^p} (\Sigma(s, t', z) + \Sigma(t', s, z)) |_{t'=-sx} \right)^2 (tsx(1-x))^p
\end{align*}
\]
Solution of RG equation

Horizontal ladder

Diff equation
\[ \frac{d}{dz} \Sigma_A = -\frac{1}{3!} + \frac{2}{4!} \Sigma_A - \frac{2}{5!} \Sigma_A^2 \]
\[ z = g^2 s^2 / \epsilon \]

\[ \Sigma_A(z) = -\sqrt{\frac{5}{3}} \frac{4 \tan(z/(8\sqrt{15}))}{1 - \tan(z/(8\sqrt{15})) \sqrt{5/3}} = \sqrt{10} \frac{\sin(z/(8\sqrt{15}))}{\sin(z/(8\sqrt{15}) - z_0)} \]

\[ \Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \ldots) \]
\[ z_0 = \arcsin(\sqrt{3/8}) \]

In general case - numerical solution similar to the ladder approximation possessing infinite number of poles in both directions
Resume
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Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations that define all loop divergences starting from one loop.
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The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories.
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The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories.

The RG equations allow one to sum up the leading (subleading, etc) divergences in all loops and define the high-energy behaviour.