Hyperasymptotic approximation to the Operator Product Expansion (OPE)

Based on

Ayala, Lobregat, Pineda: 1902.07736; 1909.01370; Montpellier proceedings

Antonio Pineda

Universitat Autònoma de Barcelona & IFAE

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OPE

$$\text{Observable}(\frac{Q}{\Lambda_{\rm QCD}}) = S_{\rm pert}(\alpha_X(Q)) + \sum_d C_{O,d}(\alpha_X(Q))\frac{\langle O_d \rangle}{Q^d}\,.$$

What is $S_{\rm pert}(\alpha_X(Q))$?
What is $C_{O,d}(\alpha_X(Q))\frac{\langle O_d\rangle}{Q^d}$?

$$\text{Observable}(\frac{Q}{\Lambda_{\rm QCD}}) =$$

$$= \sum_{n=0}^{\infty} p_n^{(X)}\alpha_X^{n+1}(Q) + \left(K + \sum_{n=0}^{\infty} p_n^{(X,d)}\alpha_X^{n+1}(Q)\right)\alpha_X^{\gamma}(Q)\frac{\Lambda_X^d}{Q^d} + \cdots$$

$$= \sum_{n=0}^{\infty} p_n^{(X)}\alpha_X^{n+1}(Q) + \left(K' + \sum_{n=0}^{\infty} p_n^{'(X,d)}\alpha_X^{n+1}(Q)\right)\alpha_X^{\gamma-db}(Q)e^{-d\frac{2\pi}{\beta_0\alpha_X(Q)}} + \cdots$$

$$= \sum_{n=0}^{\infty} p_n^{(X)}(\frac{\mu}{Q})\alpha_X^{n+1}(\mu) + \left(K' + \sum_{n=0}^{\infty} p_n^{'(X,d)}(\frac{\mu}{Q})\alpha_X^{n+1}(\mu)\right)\alpha_X^{\gamma-db}(\mu)\frac{\mu^d}{Q^d}e^{-d\frac{2\pi}{\beta_0\alpha_X(\mu)}}$$

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$\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q)$ is divergent: $p_n \sim n!$

One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations.
It allows for a parametric control of the error.

$$\mathrm{Observable}\left(\frac{Q}{\Lambda_{\mathrm{QCD}}}\right) - \sum_{n=0}^{N} p_n^{(X)}\left(\frac{\mu}{Q}\right)\alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!
Truncate the sum at the minimal term $\rightarrow$ superasymptotic approximation:

1) N and $\mu \sim Q$ large but finite:

$$N = N_P \equiv |d|\frac{2\pi}{\beta_0 \alpha_X(\mu)}\left(1 - c\,\alpha_X(\mu)\right),$$

2) $N \rightarrow \infty$ and $\mu \rightarrow \infty$ in a correlated way.

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How to go beyond that?

1. Predict observables with $e^{-A\frac{2\pi}{\beta_0\alpha(Q)}}$ precision ($A > |d|$).
2. Avoid spurious renormalon problems.

Borel transform

$$\sum_{n=0}^{\infty} p_n^{(X)}(\frac{\mu}{Q})\alpha_X^{n+1}(\mu) \to B[O](t) = \sum_{n=0}^{\infty} \frac{p_n^{(X)}(\frac{\mu}{Q})}{n!} t^n$$

Inverse Borel transform

$$\int_0^\infty dt e^{-t/\alpha_X(Q)} B[O](t)$$

PV regularization

$$S_{\rm PV}(\alpha(Q)) \equiv \int_{0,\rm PV}^\infty dt e^{-t/\alpha_X(Q)} B[O](t)$$

Scale and Scheme independent

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Assumption

$$\text{Observable}(\frac{Q}{\Lambda_{\rm QCD}}) = S_{\rm PV}(\alpha(Q))$$
$$+K_X^{(\rm PV)}\alpha_X^{\gamma}(Q)\frac{\Lambda_X^d}{Q^d}\left(1+\mathcal{O}(\alpha_X(Q))\right)+\mathcal{O}(\frac{\Lambda_X^{d'}}{Q^{d'}}),$$

$S_{\rm PV}$ can only be computed in an approximated way
$B[O](t) \rightarrow$ analytic function in the complex plane plus cuts.

$$\delta B[O](t) = Z_{O_d}^X \frac{\mu^d}{Q^d}\frac{1}{(1-2u/d)^{1+db-\gamma}}\left(1+b_1(1-2u/d)+\cdots\right)$$

where $u \equiv \beta_0 t/(4\pi)$.
These singularities are linked to the asymptotic behavior of perturbation theory

$$I(db) \equiv \int_{0,\rm PV}^{\infty} dt e^{-t/\alpha}\frac{1}{(1-2u/d)^{1+db-\gamma}} = \alpha D_{db-\gamma}(-(2\pi d)/(\beta_0\alpha))$$
$$\sim \sum_{n=0}^{\infty}\frac{\Gamma(1+db-\gamma+n)}{\Gamma(1+db-\gamma)}\left(\frac{\beta_0}{2\pi d}\right)^n \alpha^{n+1}(\mu),$$

$$
\begin{aligned}
\delta S_{\rm PV} &= Z^X_{O_d}\frac{\mu^d}{Q^d}\left[I(db)+b_1 I(db-1)+\cdots\right] \\
&= Z^X_{O_d}\frac{\mu^d}{Q^d}\sum_{n=0}^{N}\frac{\Gamma(1+db-\gamma+n)}{\Gamma(1+db-\gamma)}\left[1+b_1\frac{db-\gamma}{db-\gamma+n}\right. \\
&\quad \left.+b_2\frac{(db-\gamma)^2}{(n+db-\gamma)(n+db-\gamma-1)}+\cdots\right]\left(\frac{\beta_0}{2\pi d}\right)^n\alpha_X^{n+1}(\mu)+\Omega\,,
\end{aligned}
$$

The finite sum stands for the contribution to

$$
S_P \equiv \sum_{n=0}^{N_P(|d_{min}|)} p_n\alpha^{n+1}(\mu)
$$

associated with the leading renormalon.

Ω_d is the terminant (Dingle) of the asymptotic series when we truncate at α^{N+1}:

$$\Omega_d = \Delta\Omega(db) + b_1 \Delta\Omega(db-1) + w_2 \Delta\Omega(db-2) + \cdots$$

where

$$w_2 = \frac{b_2(db-\gamma)}{db-\gamma-1}$$

and $\Delta\Omega$ admits the following integral (but not a Borel integral) representation

$$\Delta\Omega(db) \equiv Z^X_{O_d} \frac{\mu^d}{Q^d} \frac{1}{\Gamma(1+db-\gamma)} \left(\frac{\beta_0}{2\pi d}\right)^{N+1} \alpha_X^{N+2}(\mu) \int_{0,\mathrm{PV}}^{\infty} dx \frac{x^{db-\gamma+N+1} e^{-x}}{1-x\frac{\beta_0 \alpha_X(\mu)}{2\pi d}} .$$

$$\Omega_d = \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\mu^d}{Q^d} e^{-\frac{d2\pi}{\beta_0 \alpha_X(\mu)}} \left(\frac{\beta_0 \alpha_X(\mu)}{4\pi}\right)^{-db} \alpha_X^{\gamma}(\mu)$$

$$\times \left(1 + \bar{K}_{X,1}^{(P)} \alpha_X(\mu) + \bar{K}_{X,2}^{(P)} \alpha_X^2(\mu) + O(\alpha_X^3(\mu))\right),$$

$$\Omega_d = \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\Lambda_X^d}{Q^d} \alpha_X^{\gamma}(\mu) \left(1 + K_{X,1}^{(P)} \alpha_X(\mu) + K_{X,2}^{(P)} \alpha_X^2(\mu) + O(\alpha_X^3(\mu))\right),$$

$$K_X^{(P)} = \frac{-Z_{O_d}^X}{\Gamma(1 + bd - \gamma)} \left(\frac{2\pi d}{\beta_0}\right)^{bd-\gamma+1} \left(\frac{\beta_0}{4\pi}\right)^{bd} \left(\frac{\beta_0}{d}\right)^{1/2} \left[-\eta_c + \frac{1}{3}\right]$$

$$\bar{K}_{X,1}^{(P)} = \frac{\beta_0/(\pi d)}{-\eta_c + \frac{1}{3}} \left[-b_1 (bd - \gamma) \left(\frac{1}{2}\eta_c + \frac{1}{3}\right) - \frac{1}{12}\eta_c^3 + \frac{1}{24}\eta_c - \frac{1}{1080}\right]$$

$$K_{X,1}^{(P)} = \bar{K}_{X,1}^{(P)} - \frac{b\beta_0 d s_1}{2\pi}$$

$$\ldots$$

where $\eta_c \equiv -db + \gamma + \frac{2\pi|d|c}{\beta_0} - 1$

Ultraviolet renormalon $d < 0$

$$p_n^{(as)} = Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{\Gamma(n+b'+1)}{\Gamma(b'+1)} \left(\frac{\beta_0}{2\pi d}\right)^n \left\{1 + c_1 \frac{b'}{n+b'} + c_2 \frac{b'^2}{(n+b')(n+b'-1)} + \dots\right\},$$

$\Omega_{d<0}$ reads

$$\Omega_{d<0} = \Delta\Omega_{UV}(db) + c_1 \Delta\Omega_{UV}(db-1) + \cdots,$$

where $b' = db - \gamma$

$$\Delta\Omega_{UV}(db) = Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{(-1)^{N_P+1}}{\Gamma(b'+1)} \left(\frac{\beta_0}{2\pi d}\right)^{N_P+1} \alpha^{N_P+2} \int_0^\infty dx \frac{e^{-x} x^{N_P+1+b'}}{1 + \frac{x\beta_0\alpha}{2\pi|d|}}$$

$$= Z_{O_d}^X \frac{\mu^d}{Q^d} (-1)^{N_P+1} \frac{\pi}{\Gamma(b'+1)} \left(\frac{\beta_0}{|d|}\right)^{-b'-1/2} \alpha(\mu)^{1/2-b'} e^{\frac{-2\pi|d|}{\beta_0\alpha(\mu)}} \left\{1 + \frac{\alpha(\mu)}{\pi} \frac{\beta_0}{12|d|} \left[-1 + 3\eta_c^2\right] + \frac{\alpha^2(\mu)}{\pi^2} \frac{\beta_0^2}{1152|d|^2} \left[13 - 48\eta_c - 60\eta_c^2 + 48\eta_c^3 + 36\eta_c^4\right] + \mathcal{O}(\alpha^3)\right\}.$$

Joining all terms together we have

$$\Omega_{d<0} = \sqrt{\alpha(\mu)} K_X^{(P)} \frac{Q^{|d|}}{\mu^{|d|}} e^{\frac{-2\pi|d|}{\beta_0\alpha(\mu)}} \left(\frac{\beta_0\alpha(\mu)}{4\pi}\right)^{-b'} \times \left\{1 + \bar{K}_{X,1}^{(P)}\alpha(\mu) + \bar{K}_{X,2}^{(P)}\alpha^2(\mu) + \mathcal{O}\left(\alpha^3(\mu)\right)\right\},$$

where

$$K_X^{(P)} \equiv Z_{\mathcal{O}_d}^X (-1)^{N_p+1} \left(\frac{\beta_0}{\pi^2|d|}\right)^{-1/2} \frac{1}{\Gamma(b'+1)} \left(\frac{2}{|d|}\right)^{-b'},$$

$$\bar{K}_{X,1}^{(P)} \equiv \left(\frac{2}{\pi}\right)^{1/2} \left(c_1 \frac{\beta_0 b'}{2\sqrt{2\pi}|d|} + \frac{\beta_0}{12|d|\sqrt{2\pi}}(-1+3\eta_c^2)\right),$$

$$\bar{K}_{X,2}^{(P)} \equiv \left(\frac{2}{\pi}\right)^{1/2} \left(c_2 \frac{b'^2\beta_0^2}{4\sqrt{2}|d|^2\pi^{3/2}} + c_1 \frac{b'\beta_0^2(-1+3(\eta_c+1)^2)}{24\sqrt{2}|d|^2\pi^{3/2}} + \frac{\beta_0^2}{1152|d|^2 2^{1/2}\pi^{3/2}}\left[13 - 48\eta_c - 60\eta_c^2 + 48\eta_c^3 + 36\eta_c^4\right]\right).$$

$$\Omega_{d<0} \sim \sqrt{\alpha(\mu)} \frac{\Lambda_{\rm QCD}^{|d|} Q^{|d|}}{\mu^{2|d|}}$$

$S_{\rm PV}$ will be computed truncating the hyperasymptotic expansion in a systematic way. This means truncating as follows (note that we always define D to be positive):

$$S_{\rm PV}(Q) = S_P(Q;\mu) + \Omega(\mu) + \sum_{n=N_P+1}^{N'_P} (p_n - p_n^{(as)})\alpha_X^{n+1}(\mu) + \Omega'(\mu) + \cdots ,$$

$$S_{\rm PV}^{(D,N)}(Q) = \sum_{\{|d|\}} S_{|d|<D} + \sum_{\{|d|\leq D\}} \Omega_d + \sum_{n=N_P(D)+1}^{N_P(D)+N} (p_n - p_n^{(as)})\alpha^{n+1}(\mu) .$$

where

$$S_P \equiv \sum_{n=0}^{N_P(|d_{min}|)} p_n \alpha^{n+1}(\mu) \equiv S_{|d|=0} ,$$

and ($|d| > 0$)

$$S_{|d|} = \sum_{n=N_P(|d|)+1}^{N_P(|d'|)} (p_n - p_n^{(as)})\alpha^{n+1}(\mu) ,$$

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For each value of the couple (D, N) we can state the parametric accuracy of $S_{\rm PV}^{(D,N)}(Q)$. For $S^{(0,N_P)}$ the error would be (up to a numerical and a $\sqrt{\alpha_X}$ factor)

$$\delta S^{(0,N_P)} \sim \mathcal{O}\left(e^{-|d_{min}|\frac{2\pi}{\beta_0\alpha_X(Q)}}\right) ,$$

and for $S^{(|d_{min}|,0)}$ (up to a numerical and a possible $\alpha_X^{3/2}$ factor):

$$\delta S^{(|d_{min}|,0)} \sim \mathcal{O}\left(\left(|\frac{d}{d_{min}}|e\right)^{-|d_{min}|\frac{2\pi}{\beta_0\alpha_X(Q)}} = e^{-|d_{min}|\frac{2\pi}{\beta_0\alpha_X(Q)}(1+\ln(|d/d_{min}|)}\right) ,$$

where d is the location of the next renormalon closest to the origin. This corresponds to the first term in the hyperasymptotic approximation. The expression for the error in the general case $S_{\rm PV}^{(D,N)}(Q)$ reads ($N \neq N_P$ but large)

$$\delta S^{(D,N)} \sim \mathcal{O}\left(e^{-D\frac{2\pi}{\beta_0\alpha_X(Q)}(1+\ln(|d/D|)}\alpha_X^N\right) ,$$

where d is the location of the next renormalon closest to the origin after D.

Static potential in the large β_0 approximation

$$V(r) = -\frac{2C_F}{\pi}\int_0^\infty dq \frac{\sin qr}{qr}\alpha_v(q)\,.$$

This defines $\alpha_v(q)$ in the V-scheme.
In the large-β_0 approximation

$$\alpha_v(q) = \alpha_X \sum_{n=0}^{\infty} L^n = \alpha_X \frac{1}{1-L}\,,$$

where $L = \frac{\beta_0 \alpha_X}{2\pi} \ln(\frac{\mu e^{-c_X/2}}{q})$. $c_{\overline{\rm MS}} = -5/3$; $c_V = 0$. If $X = \mathrm{latt}$, we take the $n_f = 0$ number for a Wilson action of $d_1 = 5.88359$ Hasenfratz:1980 and use $c_{\rm latt} = -2(\frac{5}{6} + \frac{2\pi d_1}{\beta_0})$: $c_{\rm latt}(n_f = 0) = -8.38807$ and $c_{\rm latt}(n_f = 0) = -9.88171$.
$\tilde{\Lambda} = \Lambda_X e^{-c_X/2}$ and $\rho = \tilde{\Lambda} r$. Note that $\tilde{\Lambda}$ is scheme independent.

$V(r)$ is ill defined but not its Borel transform. It reads Aglietti:1995

$$B[V](t(u)) = B(t(u)) = \frac{-C_F}{\pi^{1/2}}\frac{1}{r}e^{-c_X u}\left(\frac{\mu^2 r^2}{4}\right)^u \frac{\Gamma(1/2-u)}{\Gamma(1+u)},$$

We then define (where the single poles of the Borel transform are regulated using the PV prescription)

$$V_{\rm PV}(r) = \int_{0,\rm PV}^{\infty} dt e^{-t/\alpha(\mu)} B[V](t(u)) .$$

We define

$$V_P \equiv \sum_{n=0}^{N_P} V_n \alpha^{n+1} .$$

Hyperasymptotic yields

$$V_{\rm PV} = V_P + \frac{1}{r}\Omega_V + \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(\rm as)})\alpha^{n+1} + \frac{1}{r}\Omega'_V + o(\Lambda_{\rm QCD}^3 r^2) ,$$

where Ω_V reads for this case

$$\Omega_V = \sqrt{\alpha_X(\mu)} K_X^{(P)} r\,\mu e^{-\frac{2\pi}{\beta_0 \alpha_X(\mu)}} \times \left(1 + \bar{K}_{X,1}^{(P)} \alpha_X(\mu) + \bar{K}_{X,2}^{(P)} \alpha_X^2(\mu) + \mathcal{O}\left(\alpha_X^3(\mu)\right)\right),$$

and $K_X^{(P)}$ and $\bar{K}_{X,i}^{(P)}$ read

$$K_X^{(P)} = -2\pi Z_V^X \beta_0^{-1/2}\left[-\eta_c + \frac{1}{3}\right],$$

$$\bar{K}_{X,1}^{(P)} = \frac{\beta_0/(\pi)}{-\eta_c + \frac{1}{3}}\left[-\frac{1}{12}\eta_c^3 + \frac{1}{24}\eta_c - \frac{1}{1080}\right],$$

$$\bar{K}_{X,2}^{(P)} = \frac{\beta_0^2/\pi^2}{-\eta_c + \frac{1}{3}}\left[-\frac{1}{160}\eta_c^5 - \frac{1}{96}\eta_c^4 + \frac{1}{144}\eta_c^3 + \frac{1}{96}\eta_c^2 - \frac{1}{640}\eta_c - \frac{25}{24192}\right],$$

where

$$Z_V^X = -2\frac{C_F}{\pi} e^{-\frac{c_X}{2}} \quad \text{and} \quad \eta_c = \frac{2\pi c}{\beta_0} - 1\,.$$

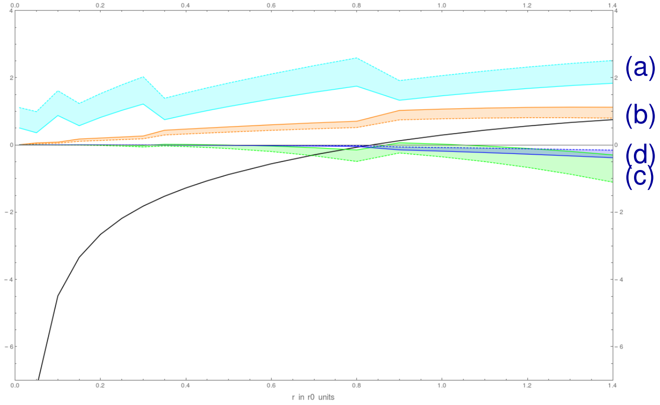


Figure: We plot V_{PV} (black line) and the differences: (a) $V_{PV} - V_P$ (cyan), (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue) in the lattice scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of c that yields integer values for N_P.

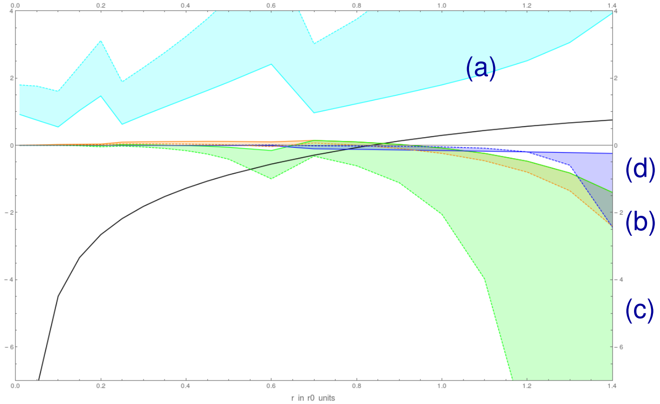


Figure: We plot V_{PV} (black line) and the differences: (a) $V_{PV} - V_P$ (cyan), (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue) in the $\overline{\mathrm{MS}}$ scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of c that yields integer values for N_P.

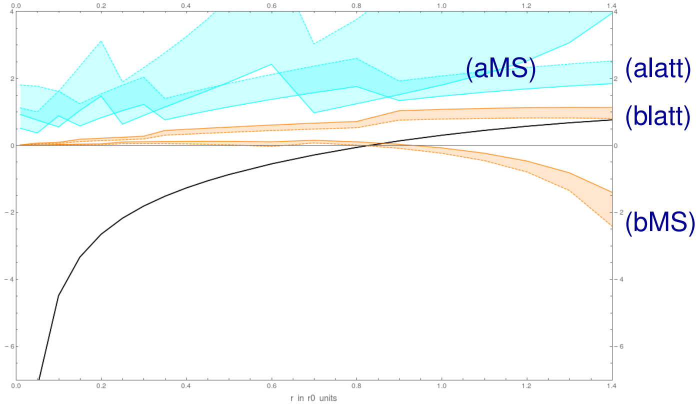


Figure: Comparison of lattice and $\overline{\mathrm{MS}}$ scheme results for $n_f = 0$. We plot V_{PV} and the differences: (a) $V_{PV} - V_P$, and (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ in the lattice and $\overline{\mathrm{MS}}$ scheme with $n_f = 0$ light flavours.

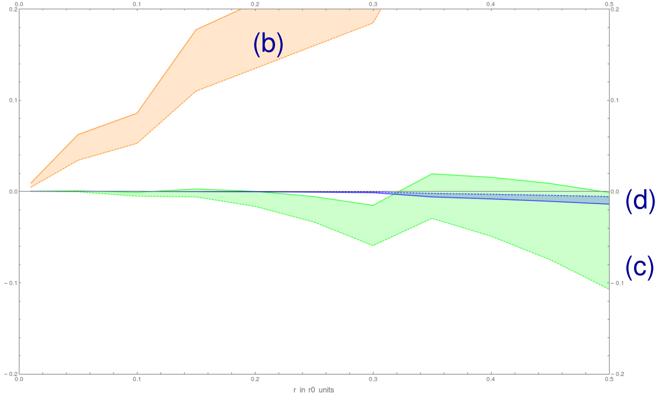


Figure: (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue) in the lattice scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of c that yields integer values for N_P.

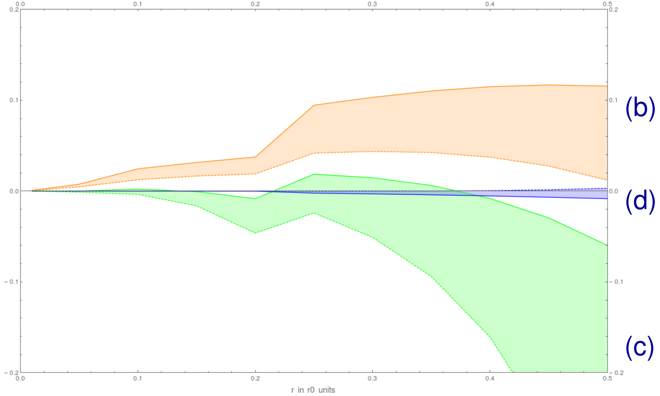


Figure: (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue) in the $\overline{\mathrm{MS}}$ scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of c that yields integer values for N_P.

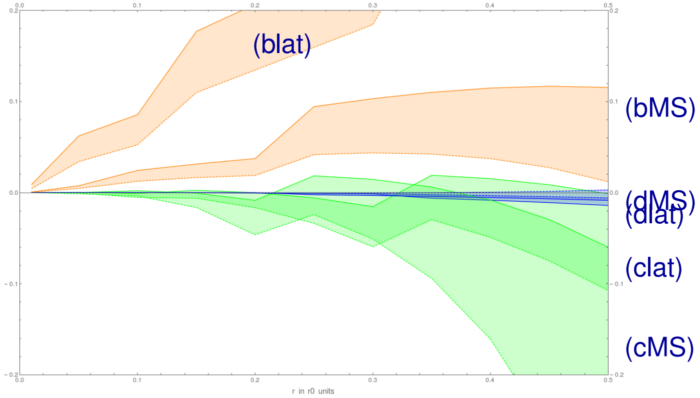


Figure: Comparison of lattice and $\overline{\mathrm{MS}}$ scheme results for $n_f = 0$.

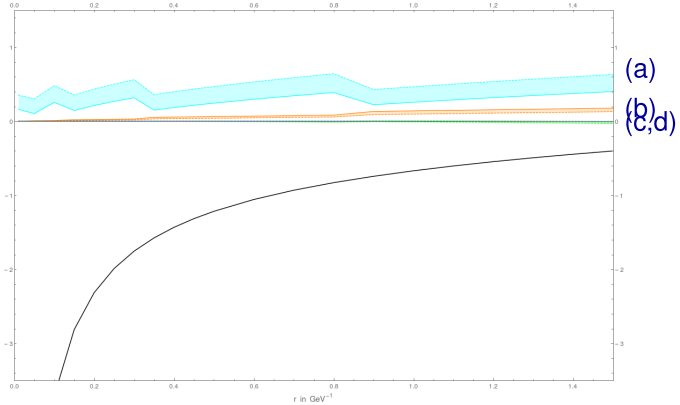
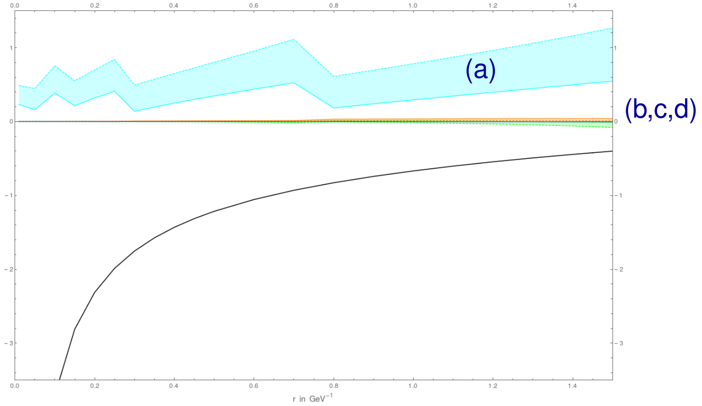


Figure: We plot V_{PV} (black line) and the differences: (a) $V_{PV} - V_P$ (cyan), (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\rm as)})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\rm as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue) in the lattice scheme with $n_f = 3$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of c that yields integer values for N_P.



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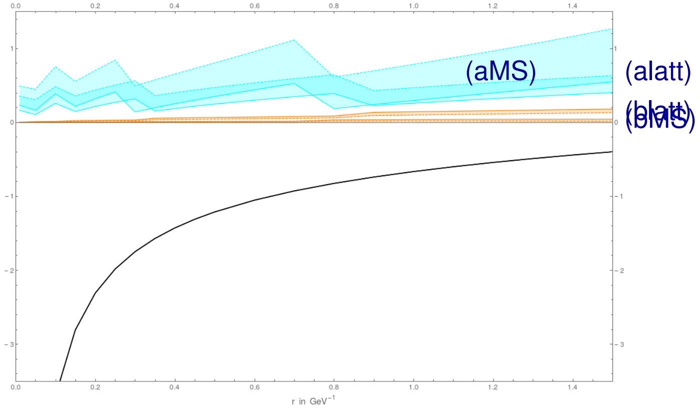

Figure: Comparison of lattice and $\overline{\text{MS}}$ scheme results for $n_f = 3$. We plot V_{PV} and the differences: (a) $V_{PV} - V_P$, and (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ in the lattice and $\overline{\text{MS}}$ scheme with $n_f = 3$ light flavours.

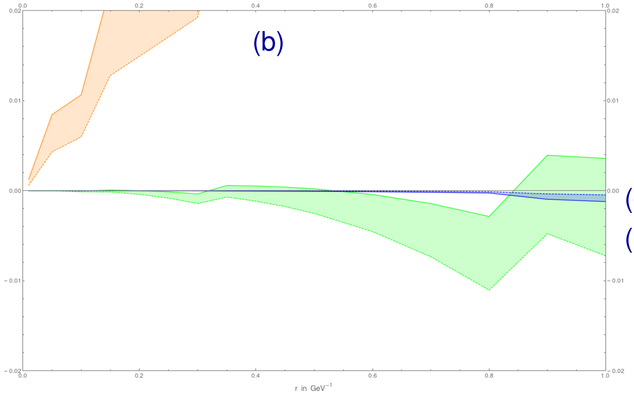

Figure: (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue) in the lattice scheme with $n_f = 3$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of c that yields integer values for N_P.

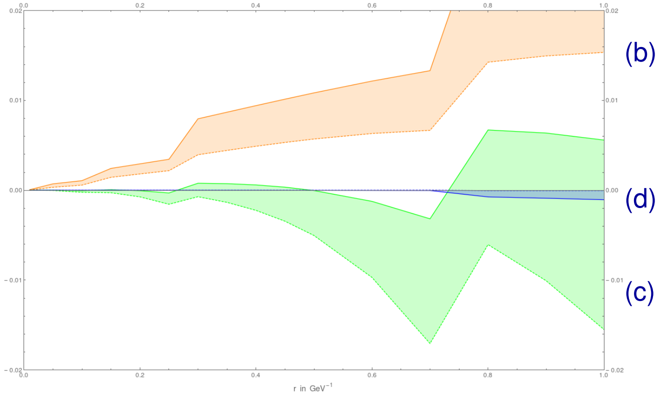

Figure: (b) $V_{PV} - V_P - \frac{1}{r}\Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{(\mathrm{as})})\alpha^{n+1} - \frac{1}{r}\Omega'_V$ (blue) in the $\overline{\mathrm{MS}}$ scheme with $n_f = 3$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of c that yields integer values for N_P.

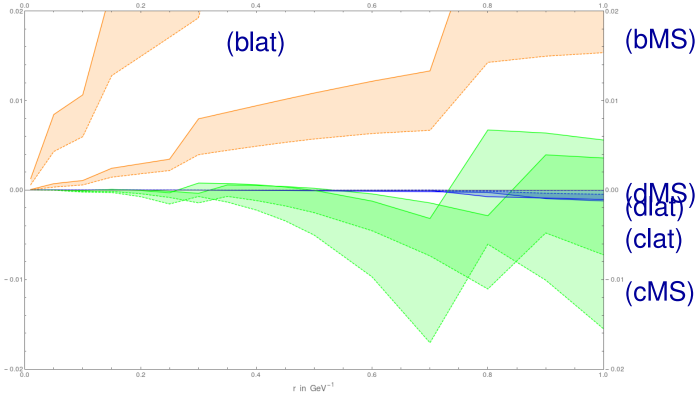

Figure: Comparison of lattice and $\overline{\rm MS}$ scheme results for $n_f = 3$.

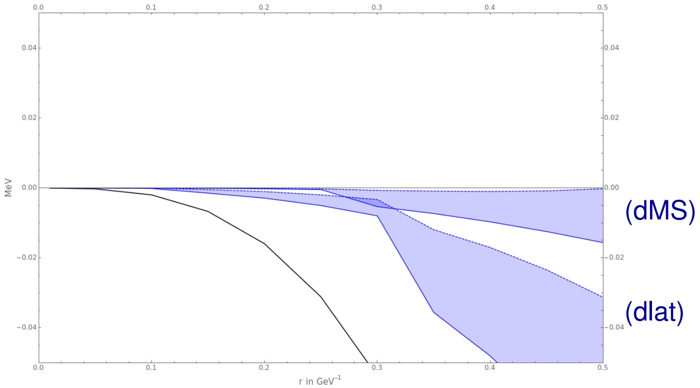

Figure: $\mathcal{O}(\Lambda_{\rm QCD}^3)$ accuracy. Comparison of lattice and $\overline{\rm MS}$ scheme results for $n_f = 3$. Vertical axis in MeV.

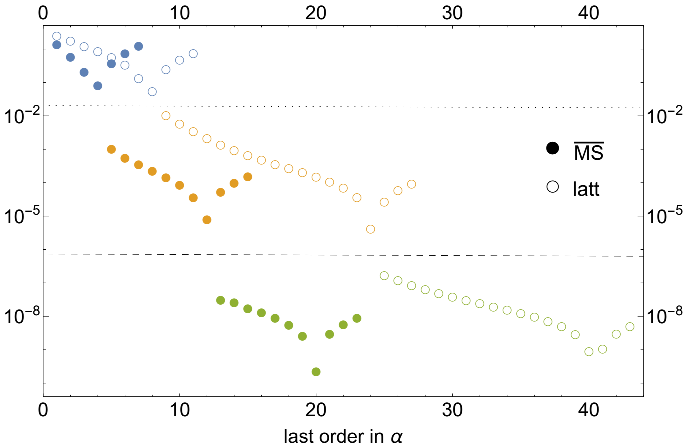

Figure: $|V_{\rm PV} - V_{\rm PV}^{\rm Hyperasymptotic}|$ for $r = 0.1$ GeV^{-1}. Points above the horizontal dotted line are $|V_{\rm PV} - V_N|$. Points between the horizontal dotted and horizontal dashed lines are $|V_{\rm PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{N}(V_n - V_n^{\rm (as)})\alpha^{n+1}|$. Points below the horizontal dashed lines are
$|V_{\rm PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P}(V_n - V_n^{\rm (as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V - \sum_{n=3N_P+1}^{N}(V_n - V_n^{\rm (as)})\alpha^{n+1}|$.

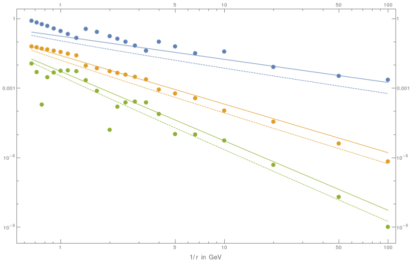

Figure: Blue points are $|r(V_{\rm PV}-V_P)|$. Orange points are $|r(V_{\rm PV}-V_P)-\Omega_V|$, Green $|r(V_{\rm PV}-V_P)-\Omega_V-r\sum_{n=N_P+1}^{3N_P}(V_n-V_n^{\rm (as)})\alpha^{n+1}|$. They are plotted as functions of $1/r$ in logarithmic scale (which is equivalent to plotting them in terms of $1/\alpha$). The continuous lines are $e^{-\frac{2\pi}{\beta_0\alpha}}$ (blue), $e^{-(1+\ln 3)\frac{2\pi}{\beta_0\alpha}}$ (orange), $e^{-3\frac{2\pi}{\beta_0\alpha}}$ (green). The dashed lines are the same functions multiplied by $\sqrt{\alpha}$. In the notation (D,N) of they correspond to $(0,N_P)$, $(1,0)$ and $(1,2N_P)$ respectively. The computation has been done with $n_f=3$, in the $\overline{\rm MS}$ scheme, and taking the smallest positive value possible of c that yields integer values for N_P.

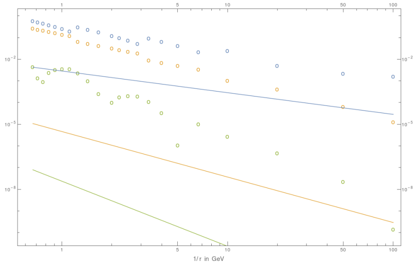

Figure: Blue points are $|r(V_{\rm PV}-V_P)|$. Orange points are $|r(V_{\rm PV}-V_P)-\Omega_V|$, Green $|r(V_{\rm PV}-V_P)-\Omega_V-r\sum_{n=N_P+1}^{3N_P}(V_n-V_n^{\rm (as)})\alpha^{n+1}|$. They are plotted as functions of $1/r$. The continuous lines are $e^{-\frac{2\pi}{\beta_0\alpha}}$ (blue), $e^{-(1+\ln 3)\frac{2\pi}{\beta_0\alpha}}$ (orange), $e^{-3\frac{2\pi}{\beta_0\alpha}}$ (green). In the notation (D,N) of they correspond to $(0,N_P)$, $(1,0)$ and $(1,2N_P)$ respectively. The computation has been done with $n_f=3$, in the lattice scheme, and taking the smallest positive value possible of c that yields integer values for N_P.

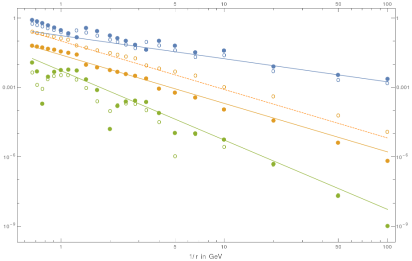

Figure: Blue points are $|r(V_{\rm PV}-V_P)|$. Orange points are $|r(V_{\rm PV}-V_P)-\Omega_V|$, Green $|r(V_{\rm PV}-V_P)-\Omega_V - r\sum_{n=N_P+1}^{3N_P}(V_n - V_n^{\rm (as)})\alpha^{n+1}|$. They are plotted as functions of $1/r$. Full points are computed in the $\overline{\rm MS}$ scheme and empty points in the lattice scheme. The continuous lines are $e^{-\frac{2\pi}{\beta_0\alpha_{\overline{\rm MS}}}}=e^{-5/6}e^{-c_{\rm latt}/2}e^{-\frac{2\pi}{\beta_0\alpha_{\rm latt}}}$ (blue), $e^{-(1+\ln 3)\frac{2\pi}{\beta_0\alpha_{\overline{\rm MS}}}}=e^{-(1+\ln 3)(5/6+c_{\rm latt}/2)}e^{-(1+\ln 3)\frac{2\pi}{\beta_0\alpha_{\rm latt}}}$ (orange), $e^{-3\frac{2\pi}{\beta_0\alpha_{\overline{\rm MS}}}}=e^{-5/2}e^{-3c_{\rm latt}/2}e^{-3\frac{2\pi}{\beta_0\alpha_{\rm latt}}}$ (green). In the notation (D,N) of they correspond to $(0,N_P)$, $(1,0)$ and $(1,2N_P)$ respectively. The dashed orange line corresponds to $4\times e^{-(1+\ln 3)\frac{2\pi}{\beta_0\alpha_{\overline{\rm MS}}}}$. The computation has been done with $n_f=3$ and taking the smallest positive value possible of c that yields integer values for N_P.

Bottom pole mass

$$m_{B(D)} = m_{\rm PV} + \bar{\Lambda}_{\rm PV} + O\left(\frac{1}{m_{\rm PV}}\right).$$

$$\bar{\Lambda}_{\rm PV} = M_{B/D} - m_{\rm P}(\overline{m}_{b/c}) - \overline{m}_{b/c}\Omega_m - \sum_{N_P+1}^{N'=2N_P} [r_n - r_n^{\rm (as)}]\alpha^{n+1} + \cdots .$$

$$m_{b,\rm PV} = 4836(\mu)^{+8}_{-17}(Z_m)^{-11}_{+12}(\alpha)^{+8}_{-9}\ {\rm MeV}.$$

$$\bar{\Lambda}_{\rm PV} = 477(\mu)^{-8}_{+17}(Z_m)^{+11}_{-12}(\alpha)^{-8}_{+9}\ {\rm MeV}.$$

$\mu \in (\overline{m}_b/2, 2\overline{m}_b)$. For Z_m we take $Z_m^{\overline{\rm MS}}(n_f = 3) = 0.5626(260)$.

Top pole mass

$$m_{\rm PV}(\overline{m}) = \overline{m} + \sum_{n=0}^{N_{max}} r_n^{(n_f)}(\overline{m};\nu=\overline{m})\alpha_{(n_f)}^{n+1}(\overline{m}) + \delta m_b^{(n_f)}(\overline{m}) + \delta m_c^{(n_f)}(\overline{m}) + \delta m_{bc}^{(n_f)}(\overline{m})\,,$$

$$\begin{aligned} m_{\rm PV}(\overline{m}_t) = \overline{m}_t \quad &+ \int_{\mu_b}^{\overline{m}_t} d\overline{m}\left(\mathcal{F}(\overline{m},5) + \frac{d}{d\overline{m}}(\delta m_b^{(5)}(\overline{m}) + \delta m_c^{(5)}(\overline{m}) + \delta m_{(bc)}^{(5)}(\overline{m}))\right) \\ &+ \int_{\mu_c}^{\mu_b} d\overline{m}\left(\mathcal{F}(\overline{m},4) + \frac{d}{d\overline{m}}(\delta m_b^{(4)}(\overline{m}) + \delta m_c^{(4)}(\overline{m}) + \delta m_{(bc)}^{(4)}(\overline{m}))\right) \\ &+ m_{\rm PV}(\mu_c) - \mu_c\,. \end{aligned}$$

$$\begin{aligned} \mathcal{F}(\overline{m},n_f) &\equiv \frac{d}{d\overline{m}}(m_{\rm PV}(\overline{m}) - \overline{m}) \simeq \frac{d}{d\overline{m}}\sum_{n=0} r_n^{(n_f)}(\overline{m};\nu=\overline{m})\alpha_{(n_f)}^{n+1}(\overline{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\overline{m})\left(\frac{\alpha_{(n_f)}(\overline{m})}{\pi}\right)^n. \end{aligned}$$

$$m_{\rm PV}(\mu_c) = m_P(\mu_c) + \mu_c\Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}\left(\mu_c e^{-\frac{2\pi}{\beta_0\alpha(\mu_c)}(1+\ln(2))}\right)$$

Top pole mass

$$m_{\rm PV}(\overline{m}) = \overline{m} + \sum_{n=0}^{N_{max}} r_n^{(n_f)}(\overline{m}; \nu = \overline{m}) \alpha_{(n_f)}^{n+1}(\overline{m}) + \delta m_b^{(n_f)}(\overline{m}) + \delta m_c^{(n_f)}(\overline{m}) + \delta m_{bc}^{(n_f)}(\overline{m}),$$

$$\begin{aligned} m_{\rm PV}(\overline{m}_t) = \overline{m}_t &+ \int_{\mu_b}^{\overline{m}_t} d\overline{m} \left(\mathcal{F}(\overline{m}, 5) + \frac{d}{d\overline{m}} (\delta m_b^{(5)}(\overline{m}) + \delta m_c^{(5)}(\overline{m}) + \delta m_{(bc)}^{(5)}(\overline{m})) \right) \\ &+ \int_{\mu_c}^{\mu_b} d\overline{m} \left(\mathcal{F}(\overline{m}, 4) + \frac{d}{d\overline{m}} (\delta m_b^{(4)}(\overline{m}) + \delta m_c^{(4)}(\overline{m}) + \delta m_{(bc)}^{(4)}(\overline{m})) \right) \\ &+ m_{\rm PV}(\mu_c) - \mu_c \,. \end{aligned}$$

$$\begin{aligned} \mathcal{F}(\overline{m}, n_f) &\equiv \frac{d}{d\overline{m}} (m_{\rm PV}(\overline{m}) - \overline{m}) \simeq \frac{d}{d\overline{m}} \sum_{n=0} r_n^{(n_f)}(\overline{m}; \nu = \overline{m}) \alpha_{(n_f)}^{n+1}(\overline{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\overline{m}) \left(\frac{\alpha_{(n_f)}(\overline{m})}{\pi} \right)^n . \end{aligned}$$

$$m_{\rm PV}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0 \alpha(\mu_c)}(1+\ln(2))})$$

Top pole mass

$$m_{\rm PV}(\overline{m}) = \overline{m}+\sum_{n=0}^{N_{max}} r_n^{(n_f)}(\overline{m};\nu=\overline{m})\alpha_{(n_f)}^{n+1}(\overline{m})+\delta m_b^{(n_f)}(\overline{m})+\delta m_c^{(n_f)}(\overline{m})+\delta m_{bc}^{(n_f)}(\overline{m})\,,$$

$$\begin{aligned} m_{\rm PV}(\overline{m}_t) = \overline{m}_t \quad &+ \int_{\mu_b}^{\overline{m}_t} d\overline{m}\left(\mathcal{F}(\overline{m},5)+\frac{d}{d\overline{m}}(\delta m_b^{(5)}(\overline{m})+\delta m_c^{(5)}(\overline{m})+\delta m_{(bc)}^{(5)}(\overline{m}))\right) \\ &+ \int_{\mu_c}^{\mu_b} d\overline{m}\left(\mathcal{F}(\overline{m},4)+\frac{d}{d\overline{m}}(\delta m_b^{(4)}(\overline{m})+\delta m_c^{(4)}(\overline{m})+\delta m_{(bc)}^{(4)}(\overline{m}))\right) \\ &+ m_{\rm PV}(\mu_c)-\mu_c\,. \end{aligned}$$

$$\begin{aligned} \mathcal{F}(\overline{m},n_f) &\equiv \frac{d}{d\overline{m}}(m_{\rm PV}(\overline{m})-\overline{m}) \simeq \frac{d}{d\overline{m}}\sum_{n=0} r_n^{(n_f)}(\overline{m};\nu=\overline{m})\alpha_{(n_f)}^{n+1}(\overline{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\overline{m})\left(\frac{\alpha_{(n_f)}(\overline{m})}{\pi}\right)^n . \end{aligned}$$

$$m_{\rm PV}(\mu_c) = m_P(\mu_c)+\mu_c\Omega_m+\delta m_b^{(3)}(\mu_c)+\delta m_c^{(3)}(\mu_c)+\delta m_{(bc)}^{(3)}(\mu_c)+\mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0\alpha(\mu_c)}(1+\ln(2))})$$

Top pole mass

$$m_{\rm PV}(\overline{m}) = \overline{m} + \sum_{n=0}^{N_{max}} r_n^{(n_f)}(\overline{m}; \nu = \overline{m})\alpha_{(n_f)}^{n+1}(\overline{m}) + \delta m_b^{(n_f)}(\overline{m}) + \delta m_c^{(n_f)}(\overline{m}) + \delta m_{bc}^{(n_f)}(\overline{m}) \,,$$

$$\begin{aligned} m_{\rm PV}(\overline{m}_t) = \overline{m}_t \quad &+ \int_{\mu_b}^{\overline{m}_t} d\overline{m} \left(\mathcal{F}(\overline{m}, 5) + \frac{d}{d\overline{m}}(\delta m_b^{(5)}(\overline{m}) + \delta m_c^{(5)}(\overline{m}) + \delta m_{(bc)}^{(5)}(\overline{m})) \right) \\ &+ \int_{\mu_c}^{\mu_b} d\overline{m} \left(\mathcal{F}(\overline{m}, 4) + \frac{d}{d\overline{m}}(\delta m_b^{(4)}(\overline{m}) + \delta m_c^{(4)}(\overline{m}) + \delta m_{(bc)}^{(4)}(\overline{m})) \right) \\ &+ m_{\rm PV}(\mu_c) - \mu_c \,. \end{aligned}$$

$$\begin{aligned} \mathcal{F}(\overline{m}, n_f) &\equiv \frac{d}{d\overline{m}}(m_{\rm PV}(\overline{m}) - \overline{m}) \simeq \frac{d}{d\overline{m}} \sum_{n=0} r_n^{(n_f)}(\overline{m}; \nu = \overline{m})\alpha_{(n_f)}^{n+1}(\overline{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\overline{m}) \left(\frac{\alpha_{(n_f)}(\overline{m})}{\pi} \right)^n . \end{aligned}$$

$$m_{\rm PV}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0 \alpha(\mu_c)}(1+\ln(2))})$$

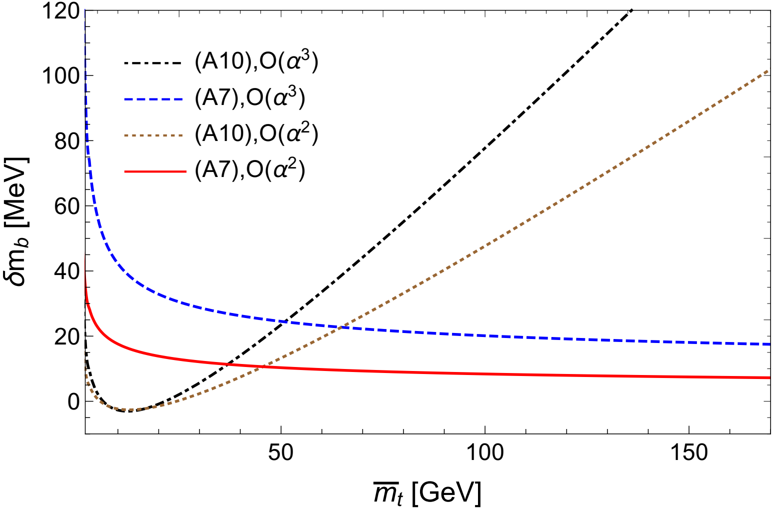

Figure: **Upper panel**: Plot of the correction to the PV mass of a top mass with varying $\overline{m}_t$ mass due to a heavy quark with $\overline{\rm MS}$ mass equal to 4.185 GeV (bottom) with and without decoupling (assuming a single heavy quark).

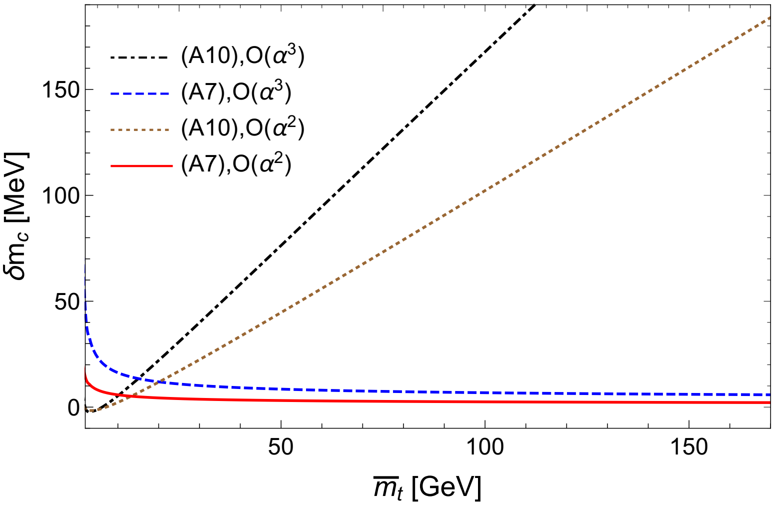

Figure: Plot of the correction to the PV mass of a top mass with varying $\overline{m}_t$ mass due to a heavy quark with $\overline{\rm MS}$ mass equal to 1.223 GeV (charm).

$$\int_{\mu_b}^{\overline{m}_t} d\overline{m}\frac{d}{d\overline{m}}(\delta m_b^{(5)}(\overline{m})+\delta m_c^{(5)}(\overline{m})+\delta m_{(bc)}^{(5)}(\overline{m}))$$

$$+\quad \int_{\mu_c}^{\mu_b} d\overline{m}\frac{d}{d\overline{m}}(\delta m_b^{(4)}(\overline{m})+\delta m_c^{(4)}(\overline{m})+\delta m_{(bc)}^{(4)}(\overline{m}))$$

$$+\quad \delta m_b^{(3)}(\mu_c)+\delta m_c^{(3)}(\mu_c)+\delta m_{(bc)}^{(3)}(\mu_c) = -2.5\Big|_{\mathcal{O}(\alpha^2)} + 0.8\Big|_{\mathcal{O}(\alpha^3)} = -1.7 \text{ MeV}\,.$$

$$\int_{\mu_b}^{\overline{m}_t} d\overline{m}\mathcal{F}(\overline{m},5)+\int_{\mu_c}^{\mu_b} d\overline{m}\mathcal{F}(\overline{m},4) = 8445+837+53-43 = 9291(22) \text{ MeV}\,.$$

$$(m_P(\mu_c)+\mu_c\Omega_m)\Big|_{\mu_c=5 \text{ GeV}} = 5744(\mu)^{+7}_{-15}(Z_m)^{+9}_{-9} \text{ MeV}\,.$$

Finally, we also include the error associated to α. Combining all errors we obtain

$$m_{t,\mathrm{PV}}(163\mathrm{MeV}) = 173033(\mathrm{h.o.})^{+22}_{-22}(\mu)^{+7}_{-15}(Z_m)^{+9}_{-9}(\alpha)^{+119}_{-123}\ \mathrm{MeV}\,.$$

$$[\frac{m_{t,\mathrm{PV}}}{\overline{m}_t} - 1] \times 10^5 = 6155(\mathrm{h.o.})^{+13}_{-13}(\mu)^{+4}_{-9}(Z_m)^{+6}_{-6}(\alpha)^{+73}_{-75}\,.$$

$$m_{t,\mathrm{PV}}(163\mathrm{MeV}) = 173033(\mathrm{th})^{+25}_{-28}(\alpha)^{+119}_{-123}\ \mathrm{MeV}\,,$$

$$[\frac{m_{t,\mathrm{PV}}}{\overline{m}_t} - 1] \times 10^5 = 6155\,(\mathrm{th})^{+15}_{-17}\,(\alpha)^{+73}_{-75}\,,$$

Conclusions

- We have organized the OPE along an hyperasymptotic expansion. We use the PV prescription of the Borel integral for reference (scheme/scale independence)
- We have checked the method in the case of the static potential in the large β_0 approximation for two terms of the hyperasymptotic expansion.
- Applied to the top and bottom mass (error determination)

Not discussed:

- Preliminary discussion of the gluon condensate
- It is possible to connect truncated sums with $\mathrm{RS}^{(n)}$ schemes
- It is possible to connect truncated sums with the MRS scheme
- Alternative? $\mu \to \infty$
- Renormalization group analysis of RS masses.
- Any summation method preserves the structure of the OPE?

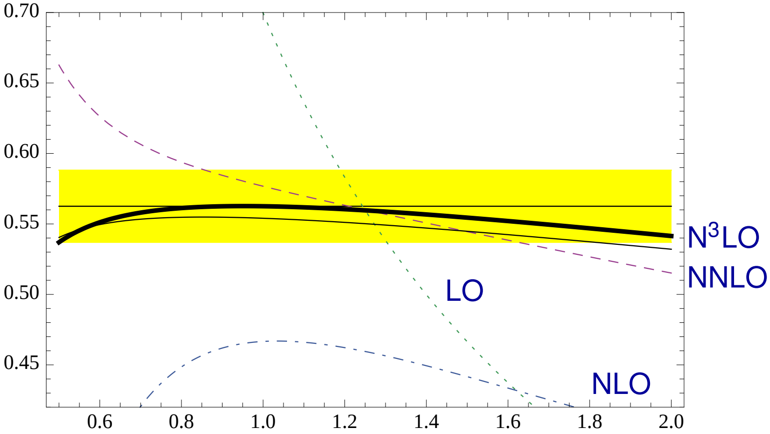

Figure: $2m + V$ renormalon free. $-N_V/2 = N_m$ for $n_l = 3$, as a function of $x \equiv \nu r$, obtained from $-(N_V/2)v_n/v_n^{asym}$. v_n^{asym} is truncated at $\mathcal{O}(1/n^3)$.

$$\boxed{N_m(n_l = 0) = 0.600(29)}\,, \qquad \boxed{N_m(n_l = 3) = 0.563(26)}\,.$$