Hyperasymptotic approximation to the Operator Product Expansion (OPE)

Based on
Ayala, Lobregat, Pineda: 1902.07736; 1909.01370; Montpellier proceedings

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Observable\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = S_{\text{pert}}(\alpha_X(Q)) + \sum_d C_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}.

What is $S_{\text{pert}}(\alpha_X(Q))$?
What is $C_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}$?
OPE

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What is \( C_{O,d}(\alpha X(Q)) \frac{\langle O_d \rangle}{Q^d} \)?

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\]

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= \sum_{n=0}^{\infty} p_n^{(X)} \alpha_x^{n+1}(Q) + \left( K' + \sum_{n=0}^{\infty} p_n^{(X,d)} \alpha_x^{n+1}(Q) \right) \alpha_x^{-db}(Q) e^{-d \frac{2\pi}{\beta_0 \alpha_x(Q)}} + \cdots
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**OPE**

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\]
\[
\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q) \text{ is divergent: } p_n \sim n!
\]

One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations.

It allows for a parametric control of the error.

\[
\text{Observable}(\frac{Q}{\Lambda_{\text{QCD}}}) - \sum_{n=0}^{N} p_n^{(X)} \left( \frac{\mu}{Q} \right) \alpha_X^{n+1}(\mu) \sim O(\alpha_X^{N+2})
\]

but with large coefficient!!

Truncate the sum at the minimal term \( \rightarrow \) superasymptotic approximation:

1) \( N \) and \( \mu \sim Q \) large but finite:

\[
N = N_P \equiv |d| \frac{2\pi}{\beta_0 \alpha_X(\mu)}(1 - c \alpha_X(\mu)),
\]

2) \( N \rightarrow \infty \) and \( \mu \rightarrow \infty \) in a correlated way.

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\]
How to go beyond that?

1. Predict observables with $e^{-A_{\beta_0}^{2\pi}}$ precision ($A > |d|$).
2. Avoid spurious renormalon problems.

Borel transform

$$\sum_{n=0}^{\infty} p_n^{(X)}(\mu)^{\alpha_{\chi}^{n+1}}(\mu) \to B[O](t) = \sum_{n=0}^{\infty} \frac{p_n^{(X)}(\mu)}{n!} t^n$$

Inverse Borel transform

$$\int_{0}^{\infty} dt e^{-t/\alpha_{\chi}(Q)} B[O](t)$$

PV regularization

$$S_{PV}(\alpha(Q)) \equiv \int_{0, PV}^{\infty} dt e^{-t/\alpha_{\chi}(Q)} B[O](t)$$

Scale and Scheme independent
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$$S_{PV} (\alpha(Q)) \equiv \int_{0, PV}^{\infty} dte^{-t/\alpha_x(Q)} B[O](t)$$

Scale and Scheme independent
Assumption

\[
\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = S_{\text{PV}}(\alpha(Q))
\]

\[+ K^{(\text{PV})}_\chi(\alpha_\chi(Q)) \frac{\Lambda^d_X}{Q^d} (1 + O(\alpha_\chi(Q))) + O(\frac{\Lambda'^d_X}{Q'^d}) ,\]

\(S_{\text{PV}}\) can only be computed in an approximated way

\(B[O](t) \to \) analytic function in the complex plane plus cuts.

\[
\delta B[O](t) = Z^{O}_d \frac{\mu^d}{Q^d} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} (1 + b_1(1 - 2u/d) + \cdots )
\]

where \(u \equiv \beta_0 t/(4\pi)\).

These singularities are linked to the asymptotic behavior of perturbation theory

\[I(db) \equiv \int_{0,\text{PV}}^{\infty} dt e^{-t/\alpha} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} = \alpha D_{db-\gamma}(-(2\pi d)/(\beta_0 \alpha))\]

\[\sim \sum_{n=0}^{\infty} \frac{\Gamma(1 + db - \gamma + n)}{\Gamma(1 + db - \gamma)} \left( \frac{\beta_0}{2\pi d} \right)^n \alpha^{n+1}(\mu),\]
\[ \delta S_{PV} = Z_{Od}^{X} \frac{\mu^{d}}{Q^{d}} [l(db) + b_{1} l(db - 1) + \cdots] \]

\[ = Z_{Od}^{X} \frac{\mu^{d}}{Q^{d}} \sum_{n=0}^{N} \frac{\Gamma(1 + db - \gamma + n)}{\Gamma(1 + db - \gamma)} \left[ 1 + b_{1} \frac{db - \gamma}{db - \gamma + n} \right. \]

\[ + b_{2} \frac{(db - \gamma)^{2}}{(n + db - \gamma)(n + db - \gamma - 1)} + \cdots \left. \right] \left( \frac{\beta_{0}}{2\pi d} \right)^{n} \alpha_{\chi}^{n+1}(\mu) + \Omega, \]

The finite sum stands for the contribution to

\[ S_{P} \equiv \sum_{n=0}^{N_{P}(|d_{\min}|)} \rho_{n} \alpha^{n+1}(\mu) \]

associated with the leading renormalon.
\( \Omega_d \) is the terminant (Dingle) of the asymptotic series when we truncate at \( \alpha^{N+1} \):

\[
\Omega_d = \Delta \Omega(db) + b_1 \Delta \Omega(db - 1) + w_2 \Delta \Omega(db - 2) + \cdots
\]

where

\[
w_2 = \frac{b_2 (db - \gamma)}{db - \gamma - 1}
\]

and \( \Delta \Omega \) admits the following integral (but not a Borel integral) representation

\[
\Delta \Omega(db) \equiv Z_{Od}^X \mu^d \frac{1}{Q^d \Gamma(1 + db - \gamma)} \left( \frac{\beta_0}{2\pi d} \right)^{N+1} \alpha_{X}^{N+2}(\mu) \int_{0, PV}^{\infty} dx \frac{x^{db - \gamma + N+1} e^{-x}}{1 - x \frac{\beta_0 \alpha X(\mu)}{2\pi d}}.
\]
\[ \Omega_d = \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\mu^d}{Q^d} e^{-\frac{d^2 \pi}{\beta_0 \alpha_X(\mu)}} \left( \frac{\beta_0 \alpha_X(\mu)}{4 \pi} \right)^{-db} \alpha_X^\gamma(\mu) \]

\[ \times \left( 1 + K_{X,1}^{(P)} \alpha_X(\mu) + K_{X,2}^{(P)} \alpha_X^2(\mu) + O(\alpha_X^3(\mu)) \right), \]

\[ \Omega_d = \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\Lambda_X^d}{Q^d} \alpha_X^\gamma(\mu) \left( 1 + K_{X,1}^{(P)} \alpha_X(\mu) + K_{X,2}^{(P)} \alpha_X^2(\mu) + O(\alpha_X^3(\mu)) \right), \]

\[ K_X^{(P)} = \frac{-Z_{\Omega_d}^X}{\Gamma(1 + bd - \gamma)} \left( \frac{2 \pi d}{\beta_0} \right)^{bd-\gamma+1} \left( \frac{\beta_0}{4 \pi} \right)^{bd} \left( \frac{\beta_0}{d} \right)^{1/2} \left[ - \eta_c + \frac{1}{3} \right] \]

\[ \tilde{K}_{X,1}^{(P)} = \frac{\beta_0/(\pi d)}{-\eta_c + \frac{1}{3}} \left[ - b_1 (bd - \gamma) \left( \frac{1}{2} \eta_c + \frac{1}{3} \right) - \frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c^3 - \frac{1}{1080} \right] \]

\[ K_{X,1}^{(P)} = \tilde{K}_{X,1}^{(P)} - \frac{b \beta_0 ds_1}{2 \pi} \]

\[ \ldots \]

where \( \eta_c \equiv -db + \gamma + \frac{2 \pi |d|c}{\beta_0} - 1 \)

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Hyperasymptotic approximation to the Operator Product Expansion (OPE)

Antonio Pineda
Ultraviolet renormalon $d < 0$

$$p_n^{(as)} = Z_{O_d}^{X} \frac{\mu^d}{Q^d} \frac{\Gamma(n + b' + 1)}{\Gamma(b' + 1)} \left( \frac{\beta_0}{2\pi d} \right)^n \left\{ 1 + c_1 \frac{b'}{n + b'} + c_2 \frac{b'^2}{(n + b')(n + b' - 1)} + \ldots \right\},$$

$\Omega_{d<0}$ reads

$$\Omega_{d<0} = \Delta \Omega_{UV}(db) + c_1 \Delta \Omega_{UV}(db - 1) + \cdots,$$

where $b' = db - \gamma$

$$\Delta \Omega_{UV}(db) = Z_{O_d}^{X} \frac{\mu^d}{Q^d} \frac{1}{\Gamma(b' + 1)} \left( \frac{\beta_0}{2\pi d} \right)^{N_p + 1} \alpha^{N_p + 2} \int_0^\infty dx \frac{e^{-x} x^{N_p + 1 + b'}}{1 + \frac{x \beta_0 \alpha}{2\pi |d|}},$$

$$= Z_{O_d}^{X} \frac{\mu^d}{Q^d} \frac{(-1)^{N_p + 1}}{\Gamma(b' + 1)} \left( \frac{\beta_0}{|d|} \right)^{-b' - 1/2} \alpha^{1/2 - b'} e^{\frac{-2\pi |d|}{\beta_0 \alpha(\mu)}} \left\{ \right.$$  

$$+ \frac{\alpha(\mu)}{\pi} \frac{\beta_0}{12|d|} \left[ -1 + 3\eta_c^2 \right]$$  

$$+ \frac{\alpha^2(\mu)}{\pi^2} \frac{\beta_0^2}{1152|d|^2} \left[ 13 - 48\eta_c - 60\eta_c^2 + 48\eta_c^3 + 36\eta_c^4 \right] + \mathcal{O}(\alpha^3) \right\}.$$
Joining all terms together we have

$$\Omega_{d<0} = \sqrt{\alpha(\mu)} K_X^{(P)} Q^{\mid d\mid} \frac{Q^{\mid d\mid}}{\mu^{\mid d\mid}} e^{-\frac{2\pi \mid d\mid}{\beta_0^\alpha(\mu)}} \left(\frac{\beta_0 \alpha(\mu)}{4\pi}\right)^{-b'}$$

$$\times \left\{1 + \bar{K}_{X,1}^{(P)} \alpha(\mu) + \bar{K}_{X,2}^{(P)} \alpha^2(\mu) + O(\alpha^3(\mu))\right\},$$

where

$$K_X^{(P)} \equiv Z_X^{\diamond}(1)^{N_p+1} \left(\frac{\beta_0}{\pi^2 \mid d\mid}\right)^{-1/2} \frac{1}{\Gamma(b' + 1)} \left(\frac{2}{\mid d\mid}\right)^{-b'},$$

$$\bar{K}_{X,1}^{(P)} \equiv \left(\frac{2}{\pi}\right)^{1/2} \left(c_1 \frac{\beta_0 b'}{2\sqrt{2}\pi \mid d\mid} + \frac{\beta_0}{12\mid d\mid \sqrt{2}\pi} (-1 + 3\eta_c^2)\right),$$

$$\bar{K}_{X,2}^{(P)} \equiv \left(\frac{2}{\pi}\right)^{1/2} \left(c_2 \frac{b'^2 \beta_0^2}{4\sqrt{2} \mid d\mid^2 \pi^{3/2}} + c_1 \frac{b' \beta_0^2 (-1 + 3(\eta_c + 1)^2)}{24\sqrt{2} \mid d\mid^2 \pi^{3/2}} + \frac{\beta_0^2}{1152 \mid d\mid^2 2^{1/2} \pi^{3/2}} \left[13 - 48\eta_c - 60\eta_c^2 + 48\eta_c^3 + 36\eta_c^4\right]\right).$$
$S_{PV}$ will be computed truncating the hyperasymptotic expansion in a systematic way. This means truncating as follows (note that we always define $D$ to be positive):

$$S_{PV}(Q) = S_P(Q; \mu) + \Omega(\mu) + \sum_{n=N_P+1}^{N'_P} (p_n - p_n^{(as)}) \alpha^{n+1}(\mu) + \Omega'(\mu) + \cdots,$$

$$S^{(D,N)}_{PV}(Q) = \sum_{|d|<D} S_{|d|<D} + \sum_{|d|\leq D} \Omega_d + \sum_{n=N_P(D)+1}^{N_P(D)+N} (p_n - p_n^{(as)}) \alpha^{n+1}(\mu).$$

where

$$S_P \equiv \sum_{n=0}^{N_P(|d_{min}|)} p_n \alpha^{n+1}(\mu) \equiv S_{|d|=0},$$

and ($|d| > 0$)

$$S_{|d|} = \sum_{n=N_P(|d'|)+1}^{N_P(|d'|)} (p_n - p_n^{(as)}) \alpha^{n+1}(\mu),$$

$$S_{PV}(Q) = S_P + \sum_{|d|} S_{|d|} + \sum_{d>0} \Omega_d + \sum_{d<0} \Omega_d,$$
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$$S_{PV}^{(D,N)}(Q) = \sum_{\{|d|\} < D} S_{|d| < D} + \sum_{\{|d| \leq D\}} \Omega_d + \sum_{n=N_P(D)+1}^{N_P(D)+N} (p_n - p_n^{(as)}) \alpha^{n+1}(\mu).$$

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and ($|d| > 0$)

$$S_{|d|} = \sum_{n=N_P(|d'|)+1}^{N_P(|d'|)} (p_n - p_n^{(as)}) \alpha^{n+1}(\mu),$$

$$S_{PV}(Q) = S_P + \sum_{\{|d|\}} S_{|d|} + \sum_{\{d > 0\}} \Omega_d + \sum_{\{d < 0\}} \Omega_d,$$
For each value of the couple \((D, N)\) we can state the parametric accuracy of 
\(S_{PV}^{(D,N)}(Q)\). For \(S^{(0,N_P)}\) the error would be (up to a numerical and a \(\sqrt{\alpha_X}\) factor)

\[
\delta S^{(0,N_P)} \sim \mathcal{O} \left( e^{-|d_{min}| \frac{2\pi}{\beta_0 \alpha_X(Q)} } \right),
\]

and for \(S^{(|d_{min}|,0)}\) (up to a numerical and a possible \(\alpha_X^{3/2}\) factor):

\[
\delta S^{(|d_{min}|,0)} \sim \mathcal{O} \left( \left( \frac{d}{d_{min}} \right)^{-|d_{min}| \frac{2\pi}{\beta_0 \alpha_X(Q)}} \right) = e^{-|d_{min}| \frac{2\pi}{\beta_0 \alpha_X(Q)} (1 + \ln(|d/d_{min}|))},
\]

where \(d\) is the location of the next renormalon closest to the origin. This corresponds to the first term in the hyperasymptotic approximation. The expression for the error in the general case \(S_{PV}^{(D,N)}(Q)\) reads \((N \neq N_P\) but large)

\[
\delta S^{(D,N)} \sim \mathcal{O} \left( e^{-D \frac{2\pi}{\beta_0 \alpha_X(Q)} (1 + \ln(|d/D|)) \frac{N}{\alpha_X} } \right),
\]

where \(d\) is the location of the next renormalon closest to the origin after \(D\).
Static potential in the large $\beta_0$ approximation

\[
V(r) = -\frac{2C_F}{\pi} \int_0^{\infty} dq \frac{\sin qr}{qr} \alpha_V(q).
\]

This defines $\alpha_V(q)$ in the V-scheme. In the large-$\beta_0$ approximation

\[
\alpha_V(q) = \alpha_X \sum_{n=0}^{\infty} L^n = \alpha_X \frac{1}{1 - L},
\]

where $L = \frac{\beta_0 \alpha_X}{2\pi} \ln\left(\frac{\mu e^{-c_X/2}}{q}\right)$. $c_{MS} = -5/3$; $c_V = 0$. If $X = \text{latt}$, we take the $n_f = 0$ number for a Wilson action of $d_1 = 5.88359$ Hasenfratz:1980 and use $c_{\text{latt}} = -2\left(\frac{5}{6} + \frac{2\pi d_1}{\beta_0}\right)$: $c_{\text{latt}}(n_f = 0) = -8.38807$ and $c_{\text{latt}}(n_f = 0) = -9.88171$. $	ilde{\Lambda} = \Lambda_X e^{-c_X/2}$ and $\rho = \tilde{\Lambda}r$. Note that $\tilde{\Lambda}$ is scheme independent.
\( V(r) \) is ill defined but not its Borel transform. It reads Aglietti:1995

\[
B[V](t(u)) = B(t(u)) = \frac{-C_F}{\pi^{1/2}} \frac{1}{r} e^{-\alpha x u} \left( \frac{\mu^2 r^2}{4} \right)^u \frac{\Gamma(1/2 - u)}{\Gamma(1 + u)},
\]

We then define (where the single poles of the Borel transform are regulated using the PV prescription)

\[
V_{PV}(r) = \int_{0, PV}^{\infty} dte^{-t/\alpha(\mu)} B[V](t(u)).
\]

We define

\[
V_P \equiv \sum_{n=0}^{N_P} V_n \alpha^{n+1}.
\]

Hyperasymptotic yields

\[
V_{PV} = V_P + \frac{1}{r} \Omega_V + \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1} + \frac{1}{r} \Omega_V' + \mathcal{O}(\Lambda_{QCD}^3 r^2),
\]
where $\Omega_V$ reads for this case

$$\Omega_V = \sqrt{\alpha_X(\mu)K^{(P)}_X} r \mu e^{-\frac{2\pi}{\beta_0 \alpha_X(\mu)}}$$

$$\times \left(1 + K^{(P)}_{X,1} \alpha_X(\mu) + K^{(P)}_{X,2} \alpha_X^2(\mu) + \mathcal{O}(\alpha_X^3(\mu))\right),$$

and $K^{(P)}_X$ and $\bar{K}^{(P)}_{X,i}$ read

$$K^{(P)}_X = -2\pi Z^X_V \beta_0^{-1/2} \left[-\eta_c + \frac{1}{3}\right],$$

$$\bar{K}^{(P)}_{X,1} = \frac{\beta_0/(\pi)}{-\eta_c + \frac{1}{3}} \left[-\frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080}\right],$$

$$\bar{K}^{(P)}_{X,2} = \frac{\beta_0^2/\pi^2}{-\eta_c + \frac{1}{3}} \left[-\frac{1}{160} \eta_c^5 + \frac{1}{96} \eta_c^4 + \frac{1}{144} \eta_c^3 + \frac{1}{96} \eta_c^2 - \frac{1}{640} \eta_c - \frac{25}{24192}\right],$$

where

$$Z^X_V = -2 \frac{C_F}{\pi} e^{-\frac{c_X}{2}}$$

and

$$\eta_c = \frac{2\pi c}{\beta_0} - 1.$$
Figure: We plot $V_{PV}$ (black line) and the differences: (a) $V_{PV} - V_P$ (cyan), (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_{n}^{(as)}) \alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_{n}^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega'_V$ (blue) in the lattice scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. 

Hyperasymptotic approximation to the Operator Product Expansion (OPE)
**Figure:** We plot $V_{PV}$ (black line) and the differences: (a) $V_{PV} - V_P$ (cyan), (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_{n}^{(as)}) \alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_{n}^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega'_V$ (blue) in the $\overline{\text{MS}}$ scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. 

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Figure: Comparison of lattice and $\overline{\text{MS}}$ scheme results for $n_f = 0$. We plot $V_{PV}$ and the differences: (a) $V_{PV} - V_P$, and (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ in the lattice and $\overline{\text{MS}}$ scheme with $n_f = 0$ light flavours.
Figure: (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega'_V$ (blue) in the lattice scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. 

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Figure: (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega_V'$ (blue) in the $\overline{\text{MS}}$ scheme with $n_f = 0$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. 

Hyperasymptotic approximation to the Operator Product Expansion (OPE)
Figure: Comparison of lattice and $\overline{\text{MS}}$ scheme results for $n_f = 0.$
Figure: We plot $V_{PV}$ (black line) and the differences: (a) $V_{PV} - V_P$ (cyan), (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r} \Omega'_V$ (blue) in the lattice scheme with $n_f = 3$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. 

Hyperasymptotic approximation to the Operator Product Expansion (OPE)
Figure: We plot $V_{PV}$ (black line) and the differences: (a) $V_{PV} - V_P$ (cyan), (b) $V_{PV} - V_P - \frac{1}{r} \Omega V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega_V'$ (blue) in the \textbf{MS} scheme with $n_f = 3$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. Ayala, Lobregat, Pineda (Preliminary)
Figure: Comparison of lattice and \( \overline{\text{MS}} \) scheme results for \( n_f = 3 \). We plot \( V_{PV} \) and the differences: (a) \( V_{PV} - V_P \), and (b) \( V_{PV} - V_P - \frac{1}{r} \Omega_V \) in the lattice and \( \overline{\text{MS}} \) scheme with \( n_f = 3 \) light flavours.
Figure: (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega'_V$ (blue) in the lattice scheme with $n_f = 3$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. 
Figure: (b) $V_{PV} - V_P - \frac{1}{r} \Omega_V$ (orange), (c) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}$ (green), and (d) $V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega'_V$ (blue) in the $\overline{\text{MS}}$ scheme with $n_f = 3$ light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of $c$ that yields integer values for $N_P$. 

Hyperasymptotic approximation to the Operator Product Expansion (OPE)
Figure: Comparison of lattice and $\overline{\text{MS}}$ scheme results for $n_f = 3$. 

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Figure: $\mathcal{O}(\Lambda_{QCD}^3)$ accuracy. Comparison of lattice and $\overline{\text{MS}}$ scheme results for $n_f = 3$. Vertical axis in MeV.
Figure: $|V_{PV} - V_{PV}^{\text{Hyperasymptotic}}|$ for $r = 0.1 \, \text{GeV}^{-1}$. Points above the horizontal dotted line are $|V_{PV} - V_N|$. Points between the horizontal dotted and horizontal dashed lines are $|V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{N} (V_n - V_n^{(as)}) \alpha^{n+1}|$. Points below the horizontal dashed lines are $|V_{PV} - V_P - \frac{1}{r} \Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1} - \frac{1}{r} \Omega_V' - \sum_{n=3N_P+1}^{N} (V_n - V_n^{(as)}) \alpha^{n+1}|$. 

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Figure: Blue points are $|r(V_{PV} - V_P)|$. Orange points are $|r(V_{PV} - V_P) - \Omega_V|$, Green
$|r(V_{PV} - V_P) - \Omega_V - r \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}|$. They are plotted as functions of $1/r$. The continuous lines are $e^{-\frac{2\pi}{\beta_0 \alpha}}$ (blue), $e^{-(1+\ln 3)\frac{2\pi}{\beta_0 \alpha}}$ (orange), $e^{-3\frac{2\pi}{\beta_0 \alpha}}$ (green). In the notation $(D, N)$ of they correspond to $(0, N_P)$, $(1, 0)$ and $(1, 2N_P)$ respectively. The computation has been done with $n_f = 3$, in the lattice scheme, and taking the smallest positive value possible of $c$ that yields integer values for $N_P$. 

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Figure: Blue points are $|r(V_{PV} - V_P)|$. Orange points are $|r(V_{PV} - V_P) - \Omega_V|$, Green

$$|r(V_{PV} - V_P) - \Omega_V - r \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^n + 1|.$$

They are plotted as functions of $1/r$. Full points are computed in the $\overline{\text{MS}}$ scheme and empty points in the lattice scheme. The continuous lines are $e^{-\frac{2\pi}{\beta_0 \alpha_{\text{MS}}}} = e^{-\frac{5}{6} e^{-c_{\text{latt}}/2} e^{-\frac{2\pi}{\beta_0 \alpha_{\text{latt}}}}}$ (blue), $e^{-(1+\ln 3)\frac{2\pi}{\beta_0 \alpha_{\text{MS}}}} = e^{-(1+\ln 3)(5/6+c_{\text{latt}}/2)} e^{-(1+\ln 3)\frac{2\pi}{\beta_0 \alpha_{\text{latt}}}}$ (orange),

$$e^{-3\frac{2\pi}{\beta_0 \alpha_{\text{MS}}}} = e^{-5/2 e^{-3c_{\text{latt}}/2} e^{-3\frac{2\pi}{\beta_0 \alpha_{\text{latt}}}}}$$ (green). In the notation $(D, N)$ of they correspond to $(0, N_P)$, $(1, 0)$ and $(1, 2N_P)$ respectively. The dashed orange line corresponds to $4 \times e^{-(1+\ln 3)\frac{2\pi}{\beta_0 \alpha_{\text{MS}}}}$. The computation has been done with $n_f = 3$ and taking the smallest positive value possible of $c$ that yields integer values for $N_P$. 

Hyperasymptotic approximation to the Operator Product Expansion (OPE)

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Bottom pole mass

\[ m_{B(D)} = m_{PV} + \bar{\Lambda}_{PV} + O\left(\frac{1}{m_{PV}}\right). \]

\[ \bar{\Lambda}_{PV} = M_{B/D} - m_P(\bar{m}_b/c) - \bar{m}_b/c \Omega_m - \sum_{N_p+1}^{N' = 2N_P} [r_n - r_n^{(as)}] \alpha^{n+1} + \ldots. \]

\[ m_{b, PV} = 4836(\mu)_+^{8}(Z_m)^{-11}(\alpha)_+^{8} \text{ MeV}. \]

\[ \bar{\Lambda}_{PV} = 477(\mu)_+^{8}(Z_m)^{11}(\alpha)_+^{-8} \text{ MeV}. \]

\[ \mu \in (\bar{m}_b/2, 2\bar{m}_b). \] For \( Z_m \) we take \( \bar{Z}_m^{MS}(n_f = 3) = 0.5626(260). \)
Top pole mass

\[ m_{PV}(\bar{m}) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r^{(n_f)}_n(\bar{m}; \nu = \bar{m}) \alpha^{n+1}_{(n_f)}(\bar{m}) + \delta m^{(n_f)}_b(\bar{m}) + \delta m^{(n_f)}_c(\bar{m}) + \delta m^{(n_f)}_{bc}(\bar{m}), \]

\[ m_{PV}(\bar{m}_t) = \bar{m}_t + \int_{\mu_b}^{\bar{m}_t} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) \right. \right.
\[ \left. + \frac{d}{d\bar{m}} \left( \delta m^{(5)}_b(\bar{m}) + \delta m^{(5)}_c(\bar{m}) + \delta m^{(5)}_{bc}(\bar{m}) \right) \right)
\[ + \int_{\mu_c}^{\mu_b} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) \right. \right.
\[ \left. + \frac{d}{d\bar{m}} \left( \delta m^{(4)}_b(\bar{m}) + \delta m^{(4)}_c(\bar{m}) + \delta m^{(4)}_{bc}(\bar{m}) \right) \right)
\[ + m_{PV}(\mu_c) - \mu_c. \]

\[ \mathcal{F}(\bar{m}, n_f) \equiv \frac{d}{d\bar{m}}(m_{PV}(\bar{m}) - \bar{m}) \sim \frac{d}{d\bar{m}} \sum_{n=0}^{N+1} r^{(n_f)}_n(\bar{m}; \nu = \bar{m}) \alpha^{n+1}_{(n_f)}(\bar{m}) \]
\[ = \sum_{n=1}^{N+1} f_n(\bar{m}) \left( \frac{\alpha(n_f)(\bar{m})}{\pi} \right)^n. \]

\[ m_{PV}(\mu_c) = m_P(\mu_c) + \mu_c \Omega + \delta m^{(3)}_b(\mu_c) + \delta m^{(3)}_c(\mu_c) + \delta m^{(3)}_{bc}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0(\alpha_0(\mu_c))} (1 + \ln(2))}) \]
Top pole mass

\[ m_{PV}(\bar{m}) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha^{n+1}_{(n_f)}(\bar{m}) + \delta m_b^{(n_f)}(\bar{m}) + \delta m_c^{(n_f)}(\bar{m}) + \delta m_{bc}^{(n_f)}(\bar{m}), \]

\[ m_{PV}(\bar{m}_t) = \bar{m}_t + \int_{\mu_b}^{\bar{m}_t} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) + \frac{d}{d\bar{m}} (\delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{bc}^{(5)}(\bar{m})) \right) \]

\[ + \int_{\mu_c}^{\mu_b} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) + \frac{d}{d\bar{m}} (\delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{bc}^{(4)}(\bar{m})) \right) \]

\[ + m_{PV}(\mu_c) - \mu_c. \]

\[ \mathcal{F}(\bar{m}, n_f) \equiv \frac{d}{dm}(m_{PV}(\bar{m}) - \bar{m}) \sim \frac{d}{dm} \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha^{n+1}_{(n_f)}(\bar{m}) \]

\[ \equiv \sum_{n=1}^{N+1} f_n(\bar{m}) \left( \frac{\alpha(n_f)(\bar{m})}{\pi} \right)^n. \]

\[ m_{PV}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{bc}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0^2\alpha(\mu_c)(1+\ln(2))}}) \]

Hyperasymptotic approximation to the Operator Product Expansion (OPE)
Top pole mass

\[ m_{PV}(m) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) + \delta m_b^{(n_f)}(\bar{m}) + \delta m_c^{(n_f)}(\bar{m}) + \delta m_{bc}^{(n_f)}(\bar{m}), \]

\[ m_{PV}(\bar{m}_t) = \bar{m}_t + \int_{\mu_b}^{\bar{m}_t} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) + \frac{d}{d\bar{m}} (\delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{(bc)}^{(5)}(\bar{m})) \right) \]

\[ + \int_{\mu_c}^{\mu_b} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) + \frac{d}{d\bar{m}} (\delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{(bc)}^{(4)}(\bar{m})) \right) \]

\[ + m_{PV}(\mu_c) - \mu_c. \]

\[ \mathcal{F}(\bar{m}, n_f) \equiv \frac{d}{d\bar{m}} (m_{PV}(\bar{m}) - \bar{m}) \simeq \frac{d}{d\bar{m}} \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) \]

\[ \equiv \sum_{n=1}^{N_{\text{max}}} f_n(\bar{m}) \left( \frac{\alpha_{(n_f)}(\bar{m})}{\pi} \right)^n. \]

\[ m_{PV}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0 \alpha(\mu_c)(1+\ln(2))}}). \]

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Top pole mass

\[ m_{PV}(\bar{m}) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r_{n}^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) + \delta m_{b}^{(n_f)}(\bar{m}) + \delta m_{c}^{(n_f)}(\bar{m}) + \delta m_{bc}^{(n_f)}(\bar{m}), \]

\[ m_{PV}(\bar{m}_{t}) = \bar{m}_{t} + \int_{\mu_{b}}^{\bar{m}_{t}} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) + \frac{d}{d\bar{m}}(\delta m_{b}^{(5)}(\bar{m}) + \delta m_{c}^{(5)}(\bar{m}) + \delta m_{bc}^{(5)}(\bar{m})) \right) \]

\[ + \int_{\mu_{c}}^{\mu_{b}} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) + \frac{d}{d\bar{m}}(\delta m_{b}^{(4)}(\bar{m}) + \delta m_{c}^{(4)}(\bar{m}) + \delta m_{bc}^{(4)}(\bar{m})) \right) \]

\[ + m_{PV}(\mu_{c}) - \mu_{c}. \]

\[ \mathcal{F}(\bar{m}, n_f) \equiv \frac{d}{d\bar{m}}(m_{PV}(\bar{m}) - \bar{m}) \approx \frac{d}{d\bar{m}} \sum_{n=0}^{N_{\text{max}}} r_{n}^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) \]

\[ \equiv \sum_{n=1}^{N+1} f_{n}(\bar{m}) \left( \frac{\alpha_{(n_f)}(\bar{m})}{\pi} \right)^{n}. \]

\[ m_{PV}(\mu_{c}) = m_{P}(\mu_{c}) + \mu_{c} \Omega_{m} + \delta m_{b}^{(3)}(\mu_{c}) + \delta m_{c}^{(3)}(\mu_{c}) + \delta m_{bc}^{(3)}(\mu_{c}) + \mathcal{O}(\mu_{c} e^{-\frac{2\pi}{\beta_{0}(\mu_{c})} (1 + \ln(2))}). \]
Figure: **Upper panel**: Plot of the correction to the PV mass of a top mass with varying $\bar{m}_t$ mass due to a heavy quark with $\overline{\text{MS}}$ mass equal to 4.185 GeV (bottom) with and without decoupling (assuming a single heavy quark).
Figure: Plot of the correction to the PV mass of a top mass with varying $\overline{m}_t$ mass due to a heavy quark with $\overline{MS}$ mass equal to 1.223 GeV (charm).
\[
\int_{\mu_b}^{\bar{m}_t} \frac{d \bar{m}}{d \bar{m}} \left( \delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{(bc)}^{(5)}(\bar{m}) \right) \\
+ \int_{\mu_c}^{\mu_b} \frac{d \bar{m}}{d \bar{m}} \left( \delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{(bc)}^{(4)}(\bar{m}) \right) \\
+ \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) = -2.5 \left| O(\alpha^2) \right| + 0.8 \left| O(\alpha^3) \right| = -1.7 \text{ MeV}.
\]

\[
\int_{\mu_b}^{\bar{m}_t} \frac{d \bar{m}}{d \bar{m}} F(\bar{m}, 5) + \int_{\mu_c}^{\mu_b} \frac{d \bar{m}}{d \bar{m}} F(\bar{m}, 4) = 8445 + 837 + 53 - 43 = 9291(22) \text{ MeV}.
\]

\[
(m_P(\mu_c) + \mu_c \Omega_m) \bigg|_{\mu_c = 5 \text{ GeV}} = 5744(\mu)^{+7}_{-15}(Z_m)^{+9}_{-9} \text{ MeV}.
\]
Finally, we also include the error associated to $\alpha$. Combining all errors we obtain

$$m_{t,PV}(163\text{MeV}) = 173033(\text{h.o.})^{+22}_{-22}(\mu)^{+7}_{-15}(Z_m)^{+9}_{-9}(\alpha)^{+119}_{-123} \text{ MeV}.$$ 

$$\left[ \frac{m_{t,PV}}{m_t} - 1 \right] \times 10^5 = 6155(\text{h.o.})^{+13}_{-13}(\mu)^{+4}_{-9}(Z_m)^{+6}_{-6}(\alpha)^{+73}_{-75}.$$ 

$$m_{t,PV}(163\text{MeV}) = 173033(\text{th})^{+25}_{-28}(\alpha)^{+119}_{-123} \text{ MeV},$$

$$\left[ \frac{m_{t,PV}}{m_t} - 1 \right] \times 10^5 = 6155(\text{th})^{+15}_{-17}(\alpha)^{+73}_{-75},$$
Conclusions

- We have organized the OPE along an hyperasymptotic expansion. We use the PV prescription of the Borel integral for reference (scheme/scale independence).
- We have checked the method in the case of the static potential in the large $\beta_0$ approximation for two terms of the hyperasymptotic expansion.
- Applied to the top and bottom mass (error determination)

Not discussed:
- Preliminary discussion of the gluon condensate
- It is possible to connect truncated sums with $\text{RS}^{(n)}$ schemes
- It is possible to connect truncated sums with the MRS scheme
- Alternative? $\mu \to \infty$
- Renormalization group analysis of RS masses.
- Any summation method preserves the structure of the OPE?
Figure: $2m + V$ renormalon free. $-N_V/2 = N_m$ for $n_l = 3$, as a function of $x \equiv \nu r$, obtained from $-(N_V/2)v_n/v_n^{asym}$. $v_n^{asym}$ is truncated at $O(1/n^3)$.

\[
N_m(n_l = 0) = 0.600(29) , \quad N_m(n_l = 3) = 0.563(26)
\]