Diffractive single and di-hadron production at NLO in a saturation framework

Michael Fucilla

Université Paris-Saclay, CNRS/IN2P3, IJCLab

in collaboration with

Andrey V. Grabovsky, Emilie Li, Lech Szymanowski, Samuel Wallon

based on

JHEP **03** (2023) 159 ArXiv: 2310.11066

Resummation, Evolution, Factorization (2023), Madrid, 23-27 Octorber 2023

Outline

Introduction

Saturation physics

- Saturation in the Shockwave formalism
- Balitsky-JIMWLK evolution equations
- Large-N_c limit

Di ractive di-hadron production

- Diffractive di-hadron production at LO
- NLO computation
- Cancellation of divergences

Outline

Introduction

Saturation physics

Saturation in the Shockwave formalism

Balitsky-JIMWLK evolution equations

Large-N_c limit

Diffractive di-hadron production

Diffractive di-hadron production at LO

NLO computation

Cancellation of divergences

Introduction

- Record energies in the center-of-mass reachable by modern and future colliders allow us to study Quantum Chromodynamics (QCD) in its least well understood \ nal frontier"
- Semi-hard collision process ! stringent *scale hierarchy*

$$s \qquad Q^2 \qquad {}^2_{QCD}; \qquad Q^2 \text{ a hard scale};$$

%

Regge kinematic region

$${}_s(Q^2) \ \ln \ \frac{s}{Q^2} \qquad 1 \ =) \ \text{all-order resummation needed}$$

- Linear regime of high-energy QCD

  The BFKL (Balitsky-Fadin-Kuraev-Lipatov) approach

  i. Leading-Logarithmic-Approximation (**LLA**): $(\ _s \ln s)^n$

  ii. Next-to-Leading-Logarithmic-Approximation (**NLLA**): $_s(\ _s \ln s)^n$

- Non-linear (saturation) regime

  B-JIMWLK (Balitsky | Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov) evolution equations

Motivation

- It is established that NLL corrections are necessary in order to reach a precision era in small-x physics
- **Evolution kernels** in the saturation regime are known at NLO
  [I. I. Balitsky, G. Chirilli (2007)], [A. Kovner, M. Lublinsky, Y. Mulian (2013)]
- **Non-perturbative models** for the description of the target
  [L. D. McLerran, R. Venugopalan (1994)]
  [K. J. Golec-Biernat, M. Wusthof (1998)]
- Full NLL predictions requires **NLO impact factors** (still challenging to compute) ! complex analytical results
  [I. I. Balitsky, G. Chirilli (2011)] , [G. Beuf (2016)]
  [G. Chirilli, B. W. Xiao, F. Yuan (2012)]
  [R. Boussarie, A. V. Grabovsky, L. Szymanowski, S. Wallon (2016)]
  [R. Boussarie, A. V. Grabovsky, D. Yu. Ivanov, L. Szymanowski, S. Wallon (2017)]
  [K. Roy, R. Venugopalan (2019)], [R. Venugopalan, F. Salazar, P. Caucal (2021)]
  [G. Beuf, T. Lappi, R. Paatelainen (2021)]
  [F. Bergabo, J. J. Marian (2022)]
  [P. Taels (2023)]
- Huge numerical implementations of impact factors and evolution equations

#

Precise observables to reveal without ambiguity the saturation of gluons in nucleons and nuclei, and to study the **Color Glass Condensate (CGC)**

Outline

Introduction

Saturation physics

- Saturation in the Shockwave formalism
- Balitsky-JIMWLK evolution equations
- Large-N_c limit

Di ractive di-hadron production

- Diffractive di-hadron production at LO
- NLO computation
- Cancellation of divergences

Saturation physics

- DIS total cross-section

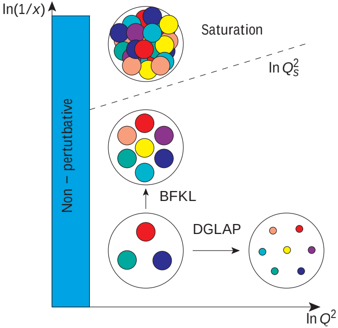
$$_P(x) = \Phi \qquad (k_\perp) \qquad _{k_\perp} F(x; k_\perp)$$

$$\#$$

$$_P(x) \qquad \frac{s}{Q^2} \quad ^{!\,0} \quad = \quad \frac{1}{x} \quad ^{!\,0}$$

- ***Martin-Froissart bound***

$$_{tot} \lesssim c \ln^2 s$$



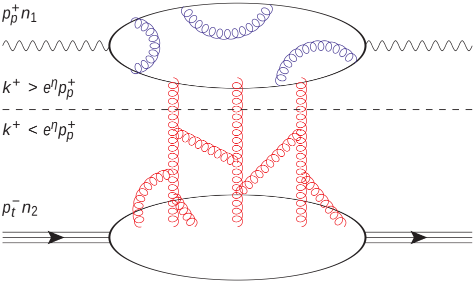
- ***Saturation e ects***
  i. Very dense system $=)$ Recombination effects
  ii. In large nuclei $=)$ Multiple re-scattering ($^2_s A^{1=3}$ resummation)

- Characteristic ***Saturation scale***

$$Q_s^2 \qquad \frac{A}{x} \quad ^{1=3} \quad \Lambda^2_{QCD} \qquad\qquad _s(Q_s^2) \qquad 1 \quad =) \quad \textit{Weakly coupled QCD}$$

Shockwave approach

- High-energy approximation $s = (p_p + p_t)^2 \quad f Q^2 g$
- $n_1 ; n_2$ are light-cone vectors (+/- directions)



- Separation of the gluonic field into “fast” (quantum) part and “slow” (classical) part through a rapidity parameter $\quad < 0$

[I. I. Balitsky (2001)]

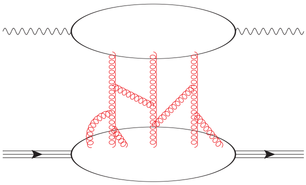
$$A \quad (k^+ ; k \quad ; \not{k}) = A \quad (k^+ \quad > e \quad p_p^+ ; k \quad ; \not{k}) + b \quad (k^+ \quad < e \quad p_p^+ ; k \quad ; \not{k})$$

$e \qquad 1$

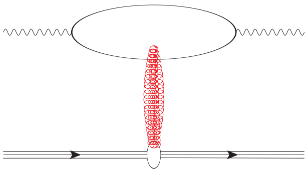
Shockwave approach

- Large longitudinal Boost: $\Lambda = \sqrt{\frac{1+\beta}{1-\beta}} \sim \frac{\sqrt{s}}{m_t}$

$$\begin{cases} b^+(x^+, x^-, \vec{x}) = \Lambda^{-1} b_0^+(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \\ b^-(x^+, x^-, \vec{x}) = \Lambda b_0^-(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \\ b^i(x^+, x^-, \vec{x}) = b_0^i(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \end{cases}$$



$\xrightarrow{\text{boost}}$



$b_0(x)$

$b^-(x^+, x^-, \vec{x}) = \delta(x^+)\mathbf{B}(\vec{x}) n_2 + O(\Lambda^{-1})$

Shockwave approximation

- Light-cone gauge $A \cdot n_2 = 0$

$A \cdot b = 0 \implies$ *Simple effective Lagrangian*

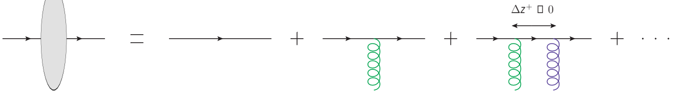
Shockwave approach

- Interactions with the simple shockwave field

  i. Independence on x^- $\Rightarrow$ conservation of p^+ (eikonal approximation)

  ii. $\delta(x^+)$ $\Rightarrow$ interactions at a single transverse coordinate.

- Quark line through the shockwave



$$U_{\vec{z}_i} = 1 + ig\int_{-\infty}^{+\infty} dz_i^+ \, b^-(z_i^+, \vec{z}_i) + (ig)^2 \int_{-\infty}^{+\infty} dz_i^+ \, dz_j^+ \, b^-(z_i^+, \vec{z}_i) \, b^-(z_j^+, \vec{z}_i) \, \theta(z_{ij}^+) + \cdots$$

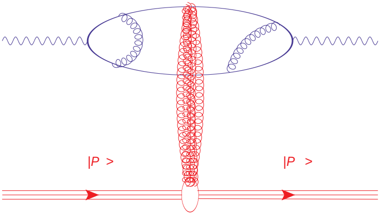
- Multiple interactions with the target $\rightarrow$ *path-ordered Wilson lines*

$$U_{\vec{z}_i} = P \exp\left[ig \int_{-\infty}^{+\infty} dz_i^+ \, b^-(z_i^+, \vec{z}_i) \right]$$

$$\tilde{U}_{\vec{p}} = \int d^d\vec{z} \, e^{-i\vec{p}\cdot\vec{z}} \, U_{\vec{z}_i}(\vec{z})$$

Shockwave approach

- Factorization in the Shockwave approximation



$$M = N_c \int d^d p_1 d^d p_2 \, \Phi(p_1, p_2) \left\langle P' \right| \tilde{U}_{ij}(p_1, p_2) \left| P \right\rangle$$

- *Dipole operator*

$$U_{ij} = \frac{1}{N_c} \mathrm{Tr}\left(U_{z_i} U^{\dagger}_{z_j} \right) - 1$$

$$\tilde{U}_{ij} = \int d^d z_i d^d z_j \, e^{-i p_i \cdot z_i} e^{-i p_j \cdot z_j} U_{ij}$$

Balitsky-JIMWLK evolution equations

- ***Balitsky-JIMWLK evolution equations*** for the dipole
  [Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov]

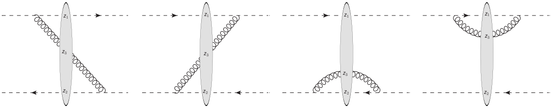
$$\frac{\partial U_{12}}{\partial \eta} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_3 \, \frac{z_{12}^2}{z_{23}^2 z_{31}^2} \Big\{ \underbrace{U_{13} + U_{32} - U_{12}}_{\text{BFKL}} - U_{13}U_{32} \Big\}$$

$$\frac{\partial U_{13}U_{32}}{\partial \eta} = \ldots$$

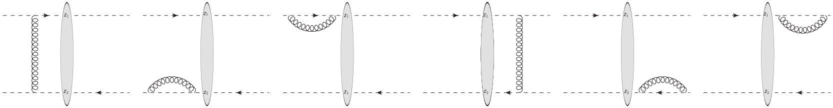
Balitsky hierarchy

$\vdots$

- Double dipole contribution and Dipole contribution

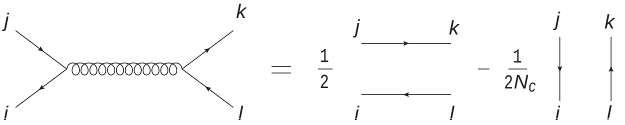


- Dipole contribution



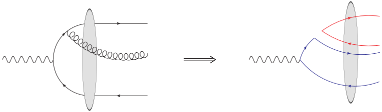
Balitsky-Kovchegov evolution equation

- Large-N_c limit [G. 't Hooft (1974)]



$$t^a_{ij} t^a_{kl} = \frac{1}{2} \quad {}_{il} \; {}_{jk} \qquad \frac{1}{N_c} \quad {}_{ij} \; {}_{kl}$$

- Double dipole ! Dipole dipole



$$U_{13}U_{32} \;\; ! \;\; U_{13} \quad U_{32}$$

- Hierarchy of equations broken ! closed non-linear **BK-equation**
  [I. I. Balitsky (1995)] [Y. V. Kovchegov (1999)]

$$\frac{@ \;\; U_{12}}{@} = \frac{{}_s N_c}{2 \;\; {}^2} \; Z \; d^2 z_3 \; \frac{z_{12}^2}{z_{23}^2 z_{31}^2} \qquad U_{13} + U_{32} \qquad U_{12} \qquad U_{13} \;\; U_{32}$$

with $hU_{12}i \quad hP^0 jU_{12} jP i$

Outline

Introduction

Saturation physics

- Saturation in the Shockwave formalism
- Balitsky-JIMWLK evolution equations
- Large-N_c limit

Di ractive di-hadron production

- Diffractive di-hadron production at LO
- NLO computation
- Cancellation of divergences

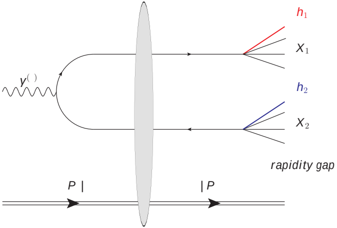
Diffractive di-hadron production

- Precise predictions to detect saturation effects at both the EIC or LHC.
- *Di ractive di-hadron production at NLO*

Diffractive di-hadron production

- Precise predictions to detect saturation effects at both the EIC or LHC.
- *Di ractive di-hadron production at NLO*

$$(\ )(p \ ) + P(p_0) \ ! \ h_1(p_{h1}) + h_2(p_{h2}) + X + P(p_0') \qquad (X = X_1 + X_2)$$



i. General kinematics $(t; Q^2)$ and photon polarization

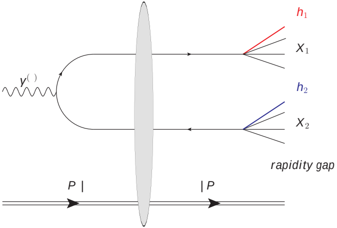
ii. Rapidity gap between $(h_1 h_2 X)$ and P^0

iii. $p_{12}^2 \quad p_{h_1}^2; p_{h_2}^2 \quad {}^2_{QCD}$

Diffractive di-hadron production

- Precise predictions to detect saturation effects at both the EIC or LHC.
- ***Di ractive di-hadron production at NLO***

$$(\ )(p\ \ ) + P(p_0)\ \ !\ \ h_1(p_{h1}) + h_2(p_{h2}) + X + P(p_0^0) \qquad (X = X_1 + X_2)$$



i. General kinematics $(t; Q^2)$ and photon polarization

ii. Rapidity gap between $(h_1 h_2 X)$ and P^0

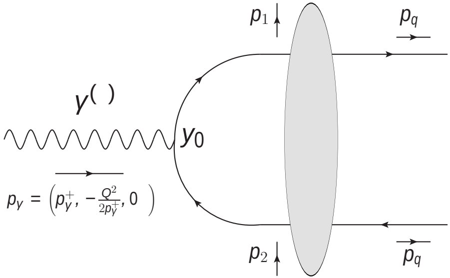
iii. $p_{12}^2 \quad p_{h_1}^2 ; p_{h_2}^2 \quad {}^2_{QCD}$

- Parametrization of the matrix element of the dipole operator

$$P^0 \quad p_0^0 \quad \mathrm{Tr} \quad U \quad \frac{z_?}{2} \quad U^y \quad \frac{z_?}{2} \qquad N_c \quad P\ (p_0) \qquad 2 \qquad p_{00^0} \quad F\ (z_?)$$

$$\int d^d z_?\, e^{i(z_?\ \ p_?)} F\ (z_?) \qquad \mathbf{F}\ (p_?)$$

- Sudakov decomposition for the momenta: $p_i = x_i p^+ n_1 + \frac{p_i^2}{2x_i p^+} n_2 + p_{i?}$



- LO S-matrix element

$$S_0 = \int d^D y_0 \, [\bar{u}(p_q; y_0)]^{nk} \, (\ iee_q) \quad (p^+) \, q \frac{''}{2p^+} e^{\ ip \ y_0} [v(p_{\bar{q}}; y_0)]^{kl} \, p \frac{ln}{N_c}$$

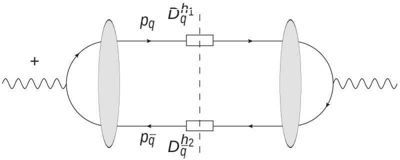
- LO Scattering amplitude

$$M_0 \ / \ N_c \ (p_q^+ + p_{\bar{q}}^+ \quad p^+) \int d^d p_1 d^d p_2 \ (p_{q1} + p_{\bar{q}2}) \, \Phi_0 \, (p_1; p_2) \, \theta_{12}$$

LO cross-section

- Sudakov decomposition for the momenta: $p_i = x_i p^+ n_1 + \frac{p_\perp^2}{2 x_i p^+} n_2 + p_\perp$

$$p_\gamma = \left(p_\gamma^+ \, ; \, -\frac{Q^2}{2p_\gamma^+} \, ; \, \vec{0}_\perp \right)$$



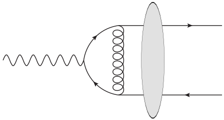
- Collinearity $(p_q^+ ; p_q) = (x_q/x_{h_1})(p_{h_1}^+ ; p_{h_1})$ and $(p_{\bar q}^+ ; p_{\bar q}) = (x_{\bar q}/x_{h_2})(p_{h_2}^+ ; p_{h_2})$

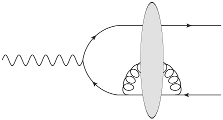
$$\frac{d\sigma_{0JI}^{h_1 h_2}}{dx_{h_1} dx_{h_2} d^d p_{h_1} d^d p_{h_2}} = \sum_q \int_{x_{h_1}}^{1} \frac{dx_q}{x_q} \int_{x_{h_2}}^{1} \frac{dx_{\bar q}}{x_{\bar q}} \left(\frac{x_q}{x_{h1}}\right)^d \left(\frac{x_{\bar q}}{x_{h2}}\right)^d D_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) D_{\bar q}^{h_2}\left(\frac{x_{h_2}}{x_{\bar q}}\right) \frac{d\hat\sigma_{JI}}{dx_q dx_{\bar q} d^d p_q d^d p_{\bar q}} + (h_1 \leftrightarrow h_2)$$

$J, I \rightarrow$ photon polarization for respectively the complex conjugated amplitude and the amplitude.

Treatment of UV and rapidity divergences

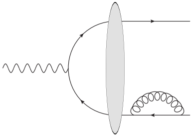
Virtual corrections





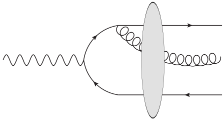
- *UV-divergences* $(p_g^2$! $1)$
- Just dressing of the external quark lines

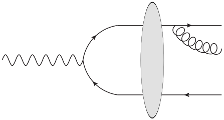
$$\Phi_{\text{dress}} \; / \; \frac{1}{IR} \; \frac{1}{UV}$$



- $_{IR} = \; _{UV}$ mixes **UV** and **IR** divergences

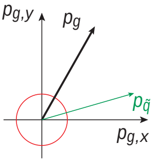
Real corrections

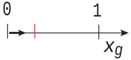




Treatment of UV and rapidity divergences

Rapidity divergences (x_g ! 0)

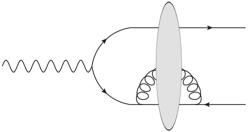




i. x_g ! 0

ii. $p\!\!\!/_g$ generic

- Coming from Φ_{V_2} (double dipole part of the virtual contribution)



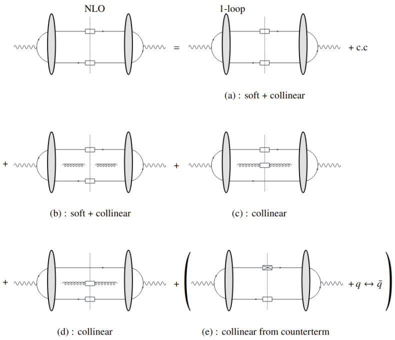
- Regularized by **longitudinal cut-off**: $jp_g^+j = jx_g jp^+ \;>\; p^+ \;=)$ ln term
- **Dim-reg** for transverse momentum integration ($d = D \quad 2 = 2 + 2\;$)
- B-JIMWLK evolution from the non-physical cutoff to the rapidity e

$$\tilde{U}_{12} = \tilde{U}^{e}_{12} + \overset{Z}{}_{e} \; d \quad \frac{@\tilde{U}_{12}}{@}\overset{!}{} \;=) \;\; \Phi_{V_2} \;\; ! \;\; \tilde{\Phi}_{V_2} = \Phi_{V_2} \quad \Phi_0 \quad K_{\mathrm{B}} \;\; \mathrm{JIMWLK}$$

NLO cross-section in a nutshell

- Different *fragmentation mechanisms*

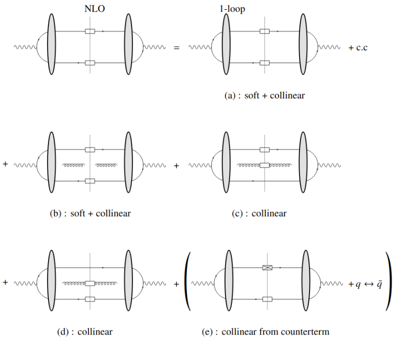
  i. Quark/anti-quark fragmentation
  ii. Quark/gluon fragmentation
  iii. Anti-quark/gluon fragmentation



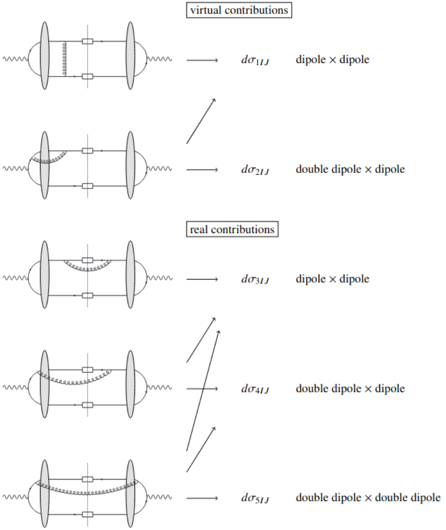
NLO cross-section in a nutshell

- Different *fragmentation mechanisms*

  i. Quark/anti-quark fragmentation
  ii. Quark/gluon fragmentation
  iii. Anti-quark/gluon fragmentation

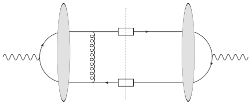


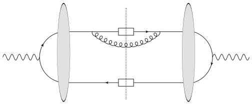
- Operator structure classification

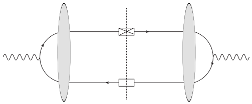


IR singularities: Quark/anti-quark fragmentation

- Divergent contributions

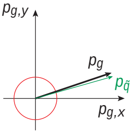


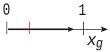




d 1/J d 3/J d counter

- Collinear divergence

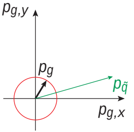


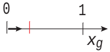


i. p_g ! $p_{\tilde{q}} = \frac{x_g}{x_q} p_q$

ii. x_g generic

- Soft divergence





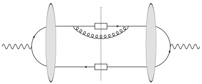
i. p_g $x_g u$

ii. x_g ! 0 and u generic

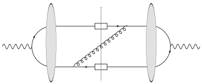
- Soft and collinear divergence (x_g ! 0 and u ! $\frac{p_q}{x_q}$)

IR singularities

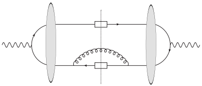
- Divergences: $q\bar{q}$-fragmentation



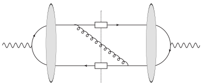
(1): soft + collinear (qg)



(2): soft



(3): soft + collinear ($\bar{q}g$)

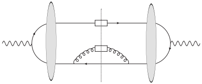


(4): soft

- Treatment of divergences in a nutshell

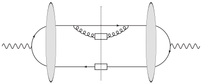
$$d_1 + d_{3;soft} + \underbrace{(d_3^{(1)} \;\; d_{3;soft}^{(1)})}_{d_{3;collinear}^{(1)}} + (d_3^{(2)} \;\; d_{3;soft}^{(2)}) + \underbrace{(d_3^{(3)} \;\; d_{3;soft}^{(3)})}_{d_{3;collinear}^{(3)}} + ((d_3^{(4)} \;\; d_{3;soft}^{(4)})) + d_{counter}$$

- Divergences: qg-fragmentation



(5): collinear ! $d^{qg(5)}_{3;collinear}$

- Divergences: $\bar{q}g$-fragmentation



(6): collinear ! $d^{qg(6)}_{3;collinear}$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h_1} \quad \frac{x_{h_1}}{x_q} \quad = D_q^{h_1} \quad \frac{x_{h_1}}{x_q}; \quad _F \qquad \frac{\ \ _s}{2} \quad \frac{1}{\hat{\ }} + \ln \frac{^2_F}{^2} \qquad P_{qq} \quad D_q^{h_1} \quad \frac{x_{h_1}}{x_q}; \quad _F \quad + \ P_{gq} \quad D_g^{h_1} \quad \frac{x_{h_1}}{x_q}; \quad _F$$

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) = D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\mu_F\right) - \frac{\alpha_s}{2\pi}\left(\frac{1}{\hat{\epsilon}} + \ln\frac{\mu^2}{\mu_F^2}\right)\left[P_{qq}\otimes D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\mu_F\right) + P_{gq}\otimes D_g^{h_1}\left(\frac{x_{h_1}}{x_q};\mu_F\right)\right]$$

$$d\sigma_{LL}^{h_1h_2}\Big|_{ct} = \frac{4\alpha_{em}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q Q_q^2\int_{x_{h_1}}^1 dx_q\int_{x_{h_2}}^1 dx_{\bar{q}}\, x_q x_{\bar{q}}\left(\frac{x_q}{x_{h_1}}\right)^d\left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d\delta(1-x_q-x_{\bar{q}})$$

$$F_{LL}\left(-\frac{\alpha_s}{2\pi}\right)\left(\frac{1}{\hat{\epsilon}} + \ln\frac{\mu^2}{\mu_F^2}\right)\left\{\underbrace{P_{qq}\otimes D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\mu_F\right)}_{(1)} D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}};\mu_F\right)\right.$$

$$\left.+\underbrace{P_{gq}\otimes D_g^{h_1}\left(\frac{x_{h_1}}{x_q};\mu_F\right)}_{(6)} D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}};\mu_F\right) + \left(q;x_q;x_{h_1}\leftrightarrow\bar{q};x_{\bar{q}};x_{h_2}\right)\right\} + (h_1\leftrightarrow h_2)$$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) = D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right) \quad \frac{\ _s}{2} \quad \frac{1}{\hat{\ }} + \ln\frac{\ _F^2}{\ ^2} \quad P_{qq} \quad D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right) + P_{gq} \quad D_g^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right)$$

$$\#$$

$$d\ ^{h_1 h_2}_{LL}\Big|_{ct} = \frac{4\ _{em} Q^2}{(2\ )^{4(d\ \ 1)} N_c} \sum_q Q_q^2 \int_{x_{h_1}}^{1} dx_q \int_{x_{h_2}}^{1} dx_q\, x_q\, x_q \left(\frac{x_q}{x_{h_1}}\right)^{!\,d} \left(\frac{x_q}{x_{h_2}}\right)^{!\,d} (1\ \ x_q\ \ x_q)$$

$$F_{LL} \quad \frac{\ _s}{2} \quad \frac{1}{\hat{\ }} + \ln\frac{\ _F^2}{\ ^2}\ !\ (\underbrace{P_{qq}\quad D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right)}_{(1)}\ !\ D_q^{h_2}\left(\frac{x_{h_2}}{x_q};\ _F\right)\ !$$

$$+ \underbrace{P_{gq}\quad D_g^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right)}_{(6)}\ !\ D_q^{h_2}\left(\frac{x_{h_2}}{x_q};\ _F\right)\ ! + \quad q; x_q; x_{h_1}\ \$\ q; x_q; x_{h_2} \quad) + (h_1\ \$\ h_2)$$

- Finite part of the cross sections

$$d\ _{h_1;h_2} = \sum_{(a;b)} D_a^{h_1} \quad D_b^{h_2} \quad d\hat{\ }_{ab} \qquad (a;b) = f(q;\bar{q});(q;g);(g;\bar{q})g$$

Summary and outlook

Summary

- Computation of the NLO cross-sections of the diffractive production of a pair of hadrons with large p_T.

  [M. F., A. V. Grabovsky, E. Li, L. Szymanowski, S. Wallon (2022)]

- Extension to the single hadron case

  [M. F., A. V. Grabovsky, E. Li, L. Szymanowski, S. Wallon (2023)]

- Full cancellation of divergences has been observed between real, virtual corrections and counterterms from renormalized FFs.
- *General kinematics* $(Q^2; t)$ and *arbitrary photon polarization* means either photo or electro-production.
- Results are also applicable to ultra-peripheral collisions at the LHC.
- Saturation window: $p_T^2 < Q_s^2$

Summary and outlook

Summary

- Computation of the NLO cross-sections of the di ractive production of a pair of hadrons with large p_T.

  [M. F., A. V. Grabovsky, E. Li, L. Szymanowski, S. Wallon (2022)]

- Extension to the single hadron case

  [M. F., A. V. Grabovsky, E. Li, L. Szymanowski, S. Wallon (2023)]

- Full cancellation of divergences has been observed between real, virtual corrections and counterterms from renormalized FFs.
- *General kinematics* $(Q^2; t)$ and *arbitrary photon polarization* means either photo or electro-production.
- Results are also applicable to ultra-peripheral collisions at the LHC.
- Saturation window: $p_T^2 < Q_s^2$

Outlook

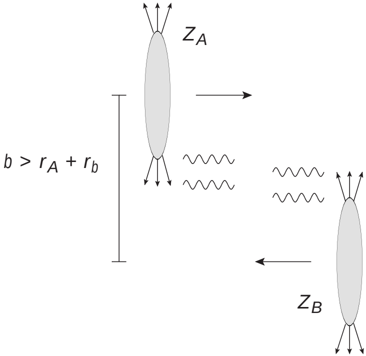
- Phenomenological analyisis

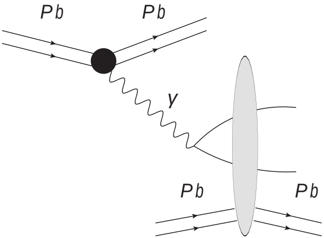
Thanks for your attention

Backup

Ultra-Peripheral collisions

- **Ultra-peripheral collisions (UPCs)** ! two projectiles with radii r_A and r_B interact with an impact parameter $b > R_A + R_B$





- UPCs are mediated by electromagnetic interactions
- Quasi-real photons cloud ! **Equivalent photon approximation (EPA)**

Definition of the non-perturbative functions

- Definition of the matrix element of the dipole operator

$$P^{0} \quad p_0^{0} \quad T \quad \mathrm{Tr} \quad U_{\frac{z_?}{2}} U^{y}{}_{\frac{z_?}{2}} \qquad N_c \quad P\,(p_0) \qquad 2 \qquad p_{00^0} \quad F\,(z_?)$$

Its Fourier transform is

$$\int d^d z_? \, e^{i(z_? \;\; p_?)} F\,(z_?) \qquad \mathbf{F}\,(p_?)$$

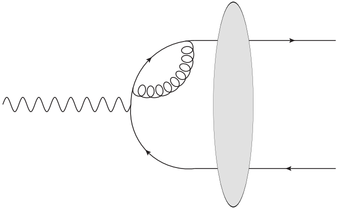
- Definition of the matrix element of the double dipole operator

$$\Big\langle P^{0} \quad p_0^{0} \quad \mathrm{Tr} \quad U_{\frac{z}{2}} U^{y}_{x} \quad \mathrm{Tr} \quad U_x U^{y}{}_{\frac{z}{2}} \qquad N_c\, \mathrm{Tr} \quad U_{\frac{z}{2}} U^{y}{}_{\frac{z}{2}} \qquad P\,(p_0) \Big\rangle \qquad 2 \qquad p_{00^0} \quad \tilde{F}\,(z_?; x_?)$$

with the following Fourier transform

$$\int d^d z_? \, d^d x_? \, e^{i(p_? \;\; x_?) + i(z_? \;\; q_?)} \tilde{F}\,(z_?; x_?) \qquad \tilde{\mathbf{F}}\,(q_?; p_?)$$

Dipole virtual contribution



- Colour factor

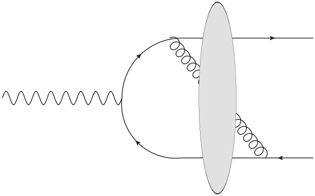
$$C_1 = \ _{ln}V_{nk}(z_1)t^a_{kp}t^a_{pq}V^{y}_{ql}(z_2) \qquad _{ln}\ \ _{nk}t^a_{kp}t^a_{pq}\ \ _{ql}$$

$$= C_F N_c U_{12}$$

- Impact factor

$$M_V^{(1)} \ / \ (p_q^+ + p_q^+ \quad p^+)N_c \int d^d p_1 d^d p_2 \ (p_{q1} + p_{q2})\, C_F U_{12} \quad _{V1}(p_1; p_2)$$

Double dipole virtual contribution



- Colour factor

$$C_2 = {}_{ln} V_{nk}(z_1) t^b_{kp} V^y_{pq}(z_2) t^a_{ql} U^{ab}(z_3) \quad {}_{ln} \; {}_{nk} t^b_{kp} \; {}_{pq} t^a_{ql} \; {}^{ab}$$

$$= \frac{N_c^2}{2}(U_{13} + U_{32} \quad U_{12} \quad U_{13} U_{32}) \quad C_F N_c U_{12}$$

$$= \frac{N_c^2}{2} W_{123} \quad C_F N_c U_{12}$$

- Impact factors

$$M_V^{(2)} \; / \; (p_q^+ + p_{q}^+ \quad p^+) \int d^d p_1 d^d p_2 \; (p_{q1} + p_{q2}) C_F \tilde{U}_{12} \quad {}^{0}_{V1}(p_1; p_2)$$

$$+ \frac{N_c}{2} \int d^d p_1 d^d p_2 \frac{d^d p_3}{(2 \;)^d} \; (p_{q1} + p_{q2} + p_3) \tilde{W}_{123} \quad {}_{V2}(p_1; p_2; p_3)$$

- LL cross-section

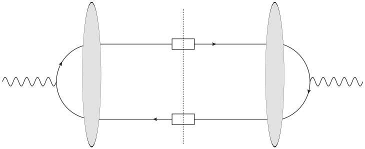
$$\frac{d\ {}^{qq \to h_1 h_2}_{0LL}}{dx_{h_1} dx_{h_2} d^d p_{h_1} d^d p_{h_2}} = \frac{4\ {}_{em} Q^2}{(2\ )^{4(d\ \ 1)} N_c} \sum_q Q_q^2 \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_q\ x_q x_q \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_q}{x_{h_2}}\right)^d$$

$$(1\ \ \ x_q\ \ \ x_q) D_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) D_q^{h_2}\left(\frac{x_{h_2}}{x_q}\right) F_{LL} + (h_1 \leftrightarrow h_2)$$

where

$$F_{LL} = \int d^d p_2 \frac{\mathbf{F}\left(\frac{x_q}{2x_{h_1}} p_{h_1} + \frac{x_q}{2x_{h_2}} p_{h_2}\ \ \ p_2\right)^2}{\left(\frac{x_q}{x_{h_2}} p_{h_2}\ \ \ p_2\right)^2 + x_q x_q Q^2} :$$

Virtual corrections



+ h.c.

1-loop

Cancellation between virtual corrections and soft and with counterterms.

$$\frac{d^{\;qq\,!\,h_1h_2}_{\;1LL}}{dx_{h_1}d_{h_2}d^dp_{h_1\,?}\,d^dp_{h_2\,?}}\;{}_{div} = \frac{4\;{}_{em}Q^2}{(2\;\;)^{4(d\;\;1)}N_c}\sum_q\int_{x_{h_1}}^{1}dx_q\int_{x_{h_2}}^{1}dx_{q}\;x_q x_q\left(\frac{x_q}{x_{h_1}}\right)^{d}\left(\frac{x_q}{x_{h_2}}\right)^{d}$$

$$(1\quad x_q\quad x_q)\,Q_q^2\,D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\;{}_F\right)D_q^{h_2}\left(\frac{x_{h_2}}{x_q};\;{}_F\right)F_{LL}$$

$$\frac{{}_s}{2}C_F\frac{1}{\hat{}}\left[\;4\;\ln(\;\;)\ln\left(\frac{{}^2}{\left(\frac{p_{h_2}}{x_{h_2}}\quad\frac{p_{h_1}}{x_{h_1}}\right)^2}\right)+4\ln(\;\;)\right.$$

$$\left.+4\;\;\ln^2(\;\;)\quad 2\ln(x_qx_q)+3\;\right]+(h_1\;\$\;h_2)$$

Soft divergences

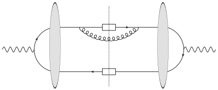
$$\left.\frac{d\sigma^{q\bar q\to h_1 h_2}_{3LL}}{dx_{h_1}\,dx_{h_2}\,d^d p_{h_1\perp}\,dp_{h_2\perp}}\right|_{\text{soft div}} = \frac{4\alpha_{em} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar q}\, x_q x_{\bar q} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar q}}{x_{h_2}}\right)^d$$

$$\times\,\delta(1-x_q-x_{\bar q})\, Q_q^2\, D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right) D_{\bar q}^{h_2}\left(\frac{x_{h_2}}{x_{\bar q}},\mu_F\right) F_{LL}\,\frac{\alpha_s C_F}{2\pi}\,\frac{1}{\hat\epsilon}\left[-4\ln\alpha + 2\ln x_q\right.$$

$$\left. +\,2\ln\left(1-\frac{x_{h_1}}{x_q}\right) - 4\ln^2\alpha - 4\ln\alpha\,\ln\left(\frac{\left(\frac{p_{h_1}}{x_{h_1}}-\frac{p_{h_2}}{x_{h_2}}\right)^2}{\mu^2}\right) + 2\ln x_{\bar q} + 2\ln\left(1-\frac{x_{h_2}}{x_{\bar q}}\right)\right] + (h_1 \leftrightarrow h_2)$$

Cancellation with the residual divergence from the collinear term.

Specific case: diagram (1)



$$\frac{d\ {}^{qq\ !\ h_1 h_1}_{3LL}}{dx_{h_1} d_{h_2} d^d p_{h_1} d^d p_{h_2}}\ \text{coll}\ qg$$

$$= \frac{4\ \ \text{em} Q^2}{(2\ \ )^{4(d\ \ 1)} N_c} \sum_q \int_{x_{h_1}}^{1} \frac{dx_q^0}{x_q^0} \int^{1} \frac{dx_g}{x_g} \int_{x_{h_2}}^{1} \frac{dx_{q^0}}{x_{q^0}} \ (1\ \ x_q^0\ \ x_q^0\ \ x_g)$$

$$\left(\frac{x_q^0}{x_{h_1}}\right)^d \left(\frac{x_q^0}{x_{h_2}}\right)^d Q_q^2 D_q^{h_1}\left(\frac{x_{h_1}}{x_q^0};\ F\right) D_q^{h_2}\left(\frac{x_{h_2}}{x_q^0};\ F\right) \frac{s}{2} C_F \frac{d^d p_{g\,?}}{(2\ \ )^d}$$

$$\int d^d p_{2\,?}\ \mathbf{F}\left(\frac{x_q^0}{2x_{h_1}} p_{h_1\,?} + \frac{x_q^0}{2x_{h_2}} p_{h_2\,?}\ \ p_{2\,?} + \frac{p_{g\,?}}{2}\right)$$

$$\int d^d p_{2^0\,?}\ \mathbf{F}\left(\frac{x_q^0}{2x_{h_1}} p_{h_1\,?} + \frac{x_q^0}{2x_{h_2}} p_{h_2\,?}\ \ p_{2^0\,?} + \frac{p_{g\,?}}{2}\right)$$

$$\frac{(dx_g^2 + 4x_q^0(x_q^0 + x_g))x_q^{02}(1\ \ x_q^0)^2}{\left(x_q^0(1\ \ x_q^0)Q^2 + \left(\frac{x_q^0}{x_{h_2}} p_{h_2}\ \ p_2\right)^2\right)\left(x_q^0(1\ \ x_q^0)Q^2 + \left(\frac{x_q^0}{x_{h_2}} p_{h_2}\ \ p_{2^0}\right)^2\right)\left(x_q^0 p_g\ \ x_g \frac{x_q^0}{x_{h_1}} p_{h_1}\right)^2}$$

$$+ (h_1\ \$\ h_2)$$

Results for diagram (1)

$$
\left.\frac{d\sigma_{3LL}^{q\bar{q}\to h_1h_2}}{dx_{h_1}dx_{h_2}dp_{h_1\perp}d^dp_{h_2\perp}}\right|_{\text{coll. qg div}}
$$

$$
= \frac{4\alpha_{\text{em}}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}}\; x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \delta(1-x_q-x_{\bar{q}})
$$

$$
\times \int d^dp_{2\perp}\int d^dz_{1\perp}\frac{e^{iz_{1\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2\perp}\right)}F(z_{1\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_2\right)^2}
$$

$$
\times \int d^dp_{2'\perp}\int d^dz_{2\perp}\frac{e^{-iz_{2\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2'\perp}\right)}F^*(z_{2\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_{2'}\right)^2}
$$

$$
\times \frac{\alpha_s}{2\pi}\frac{1}{\hat{\epsilon}}Q_q^2\left[\int_{\frac{x_{h_1}}{x_q}}^1 \frac{d\beta_1}{\beta_1}C_F\frac{1+\beta_1^2}{(1-\beta_1)_+}D_q^{h_1}\left(\frac{x_{h_1}}{\beta_1 x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right.
$$

$$
+\int_{\frac{x_{h_1}}{x_q}}^{1-\frac{\alpha}{x_q}} d\beta_1 C_F\frac{2}{1-\beta_1}\left(\frac{c_0^2}{\left(\frac{z_{1\perp}-z_{2\perp}}{2}\right)^2\mu^2}\right)^{\epsilon}D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)
$$

$$
\left.-2C_F\ln\left(1-\frac{x_{h_1}}{x_q}\right)D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right]+(h_1\leftrightarrow h_2)\,.
$$

$$\frac{d\ {}^{h_1 h_2}_{LL}}{dx_{h_1} d_{h_2} d^d p_{h_1 ?}\, d^d p_{h_2 ?}}\Bigg|_{ct} = \frac{4\ \ _{em} Q^2}{(2\ \ )^{4(d\ \ 1)} N_c} \sum_q \int_{x_{h_1}}^{1} dx_q \int_{x_{h_2}}^{1} dx_q\, x_q\, x_q \left(\frac{x_q}{x_{h_1}}\right)^{d} \left(\frac{x_q}{x_{h_2}}\right)^{d} (1\ \ x_q\ \ x_q)$$

$$F_{LL}\ \ \frac{s}{2} \left(\frac{1}{\hat{}} + \ln\frac{{}^2_F}{2}\ \ Q_q^2\right) \left\{ \int_{\frac{x_{h_1}}{x_q}}^{1} \frac{d\ \ _1}{_1} \left[{\color{red} C_F \frac{1 + \ _1^2}{(1\ \ \ _1)_+}} D_q^{h_1}\left(\frac{x_{h_1}}{_1 x_q};\ _F\right) D_q^{h_2}\left(\frac{x_{h_2}}{x_q};\ _F\right) \right.\right.$$

$$\left. + P_{gq}(\ _1) D_g^{h_1}\left(\frac{x_{h_1}}{_1 x_q};\ _F\right) D_q^{h_2}\left(\frac{x_{h_2}}{x_q};\ _F\right) \right] + \int_{\frac{x_{h_2}}{x_q}}^{1} \frac{d\ \ _2}{_2} \left[P_{gq}(\ _2) D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right) D_g^{h_2}\left(\frac{x_{h_2}}{_2 x_q};\ _F\right) \right.$$

$$\left.\left. + C_F \frac{1 + \ _2^2}{(1\ \ \ _2)_+} D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right) D_q^{h_2}\left(\frac{x_{h_2}}{_2 x_q};\ _F\right) \right] + {\color{magenta} 3C_F} D_q^{h_1}\left(\frac{x_{h_1}}{x_q};\ _F\right) D_q^{h_2}\left(\frac{x_{h_2}}{x_q};\ _F\right) \right\}$$

$$+ (h_1 \leftrightarrow h_2) :$$

The divergent part of the partonic cross-section (LL)

$$
\begin{aligned}
d\hat{\sigma}_{3LL}|_{div} = {} & \frac{4\alpha_{em}Q^2}{(2\pi)^{4(d-1)}N_c} Q_q^2 dx'_q dx'_{\bar{q}} \delta(1 - x'_q - x'_{\bar{q}} - x_g) d^d p_{q\perp} d^d p_{\bar{q}\perp} \frac{\alpha_s C_F}{\mu^{2\epsilon}} \frac{dx_g}{x_g} \frac{d^d p_{g\perp}}{(2\pi)^d} \\
& \times \int d^d p_{1\perp} d^d p_{2\perp} \delta(p_{q1\perp} + p_{\bar{q}2\perp} + p_{g\perp}) \mathbf{F}\left(\frac{p_{12\perp}}{2}\right) \\
& \times \int d^d p_{1'\perp} d^d p_{2'\perp} \delta(p_{q1'\perp} + p_{\bar{q}2'\perp} + p_{g\perp}) \mathbf{F}^*\left(\frac{p_{1'2'\perp}}{2}\right) \\
& \times \left\{ \frac{\left(dx_g^2 + 4x'_q(x'_q + x_g)\right)}{\left(Q^2 + \frac{\vec{p}_{\bar{q}2}^{\,2}}{x'_{\bar{q}}(1-x'_{\bar{q}})}\right)\left(Q^2 + \frac{\vec{p}_{\bar{q}2'}^{\,2}}{x'_{\bar{q}}(1-x'_{\bar{q}})}\right)(x'_q \vec{p}_g - x_g \vec{p}_q)^2} \right. \\
& - \frac{\left(2x_g - dx_g^2 + 4x'_q x'_{\bar{q}}\right)\left(x'_q \vec{p}_g - x_g \vec{p}_q\right)\cdot\left(x'_{\bar{q}} \vec{p}_g - x_g \vec{p}_{\bar{q}}\right)}{\left(Q^2 + \frac{\vec{p}_{\bar{q}2'}^{\,2}}{x'_{\bar{q}}(1-x'_{\bar{q}})}\right)\left(Q^2 + \frac{\vec{p}_{q1}^{\,2}}{x'_q(1-x'_q)}\right)\left(x'_q \vec{p}_g - x_g \vec{p}_q\right)^2\left(x'_{\bar{q}} \vec{p}_g - x_g \vec{p}_{\bar{q}}\right)^2} \\
& + \frac{\left(dx_g^2 + 4x'_{\bar{q}}(x'_{\bar{q}} + x_g)\right)}{\left(Q^2 + \frac{\vec{p}_{q1}^{\,2}}{x'_q(1-x'_q)}\right)\left(Q^2 + \frac{\vec{p}_{q1'}^{\,2}}{x'_q(1-x'_q)}\right)(x'_{\bar{q}} \vec{p}_g - x_g \vec{p}_{\bar{q}})^2} \\
& \left. - \frac{\left(2x_g - dx_g^2 + 4x'_q x'_{\bar{q}}\right)\left(x'_q \vec{p}_g - x_g \vec{p}_q\right)\cdot\left(x'_{\bar{q}} \vec{p}_g - x_g \vec{p}_{\bar{q}}\right)}{\left(Q^2 + \frac{\vec{p}_{q1'}^{\,2}}{x'_q(1-x'_q)}\right)\left(Q^2 + \frac{\vec{p}_{\bar{q}2}^{\,2}}{x'_{\bar{q}}(1-x'_{\bar{q}})}\right)\left(x'_q \vec{p}_g - x_g \vec{p}_q\right)^2\left(x'_{\bar{q}} \vec{p}_g - x_g \vec{p}_{\bar{q}}\right)^2} \right\}
\end{aligned}
$$

The rst and third terms are associated with collinear divergences. All of them contribute to the soft divergence.

Finite term from one diagram

$$
\left.\frac{d\sigma_{3LL}^{q\bar{q}\to h_1h_2}}{dx_{h_1}dx_{h_2}dp_{h_1\perp}d^dp_{h_2\perp}}\right|_{\text{coll. qg fin}}
= \frac{4\alpha_{\text{em}}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}}\; x_q x_{\bar{q}}\delta(1-x_q-x_{\bar{q}})\left(\frac{x_q}{x_{h_1}}\right)^d\left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d
$$

$$
\times \int d^dp_{2\perp}\int d^dz_{1\perp}\frac{e^{iz_{1\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2\perp}\right)}F(z_{1\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_2\right)^2}
$$

$$
\times \int d^dp_{2'\perp}\int d^dz_{2\perp}\frac{e^{-iz_{2\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2'\perp}\right)}F^*(z_{2\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_{2'}\right)^2}
$$

$$
\times \frac{\alpha_sC_F}{2\pi}\left\{\int_{\frac{x_{h_1}}{x_q}}^1\frac{d\beta_1}{\beta_1}Q_q^2D_q^{h_1}\left(\frac{x_{h_1}}{\beta_1x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right.
$$

$$
\times\left[\ln\left(\frac{c_0^2}{\left(\frac{z_{1\perp}-z_{2\perp}}{2}\right)^2\mu^2}\right)\frac{1+\beta_1^2}{(1-\beta_1)_+}+\frac{(1-\beta_1)^2+2(1+\beta_1^2)\ln\beta_1}{(1-\beta_1)}\right]
$$

$$
\left.-2\ln\left(1-\frac{x_{h_1}}{x_q}\right)\ln\left(\frac{c_0^2}{\left(\frac{z_{1\perp}-z_{2\perp}}{2}\right)^2\mu^2}\right)D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_q}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right\}+(h_1\leftrightarrow h_2)\,.
$$

with $c_0 = 2e^{-\gamma_E}$