Inhomogeneous Phases in the QCD Phase Diagram

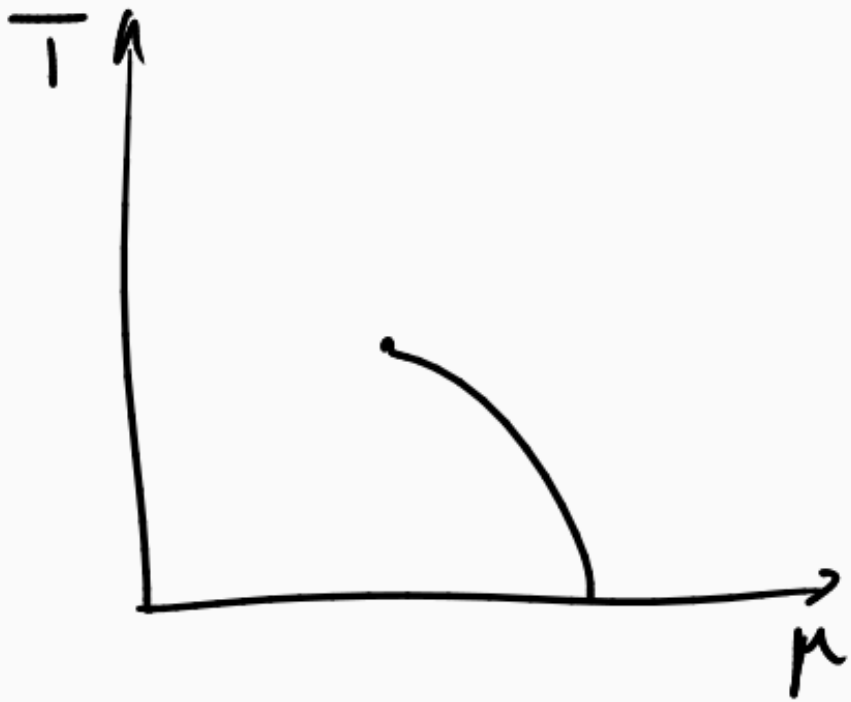
Theo F. Motta (JLU Gießen)

September 22, 2022

Modern equations of state and spectroscopy in neutron-star matter
Universidad de Alcalá

Overview of Inhomogeneous Phases

Inhomogeneous Phases



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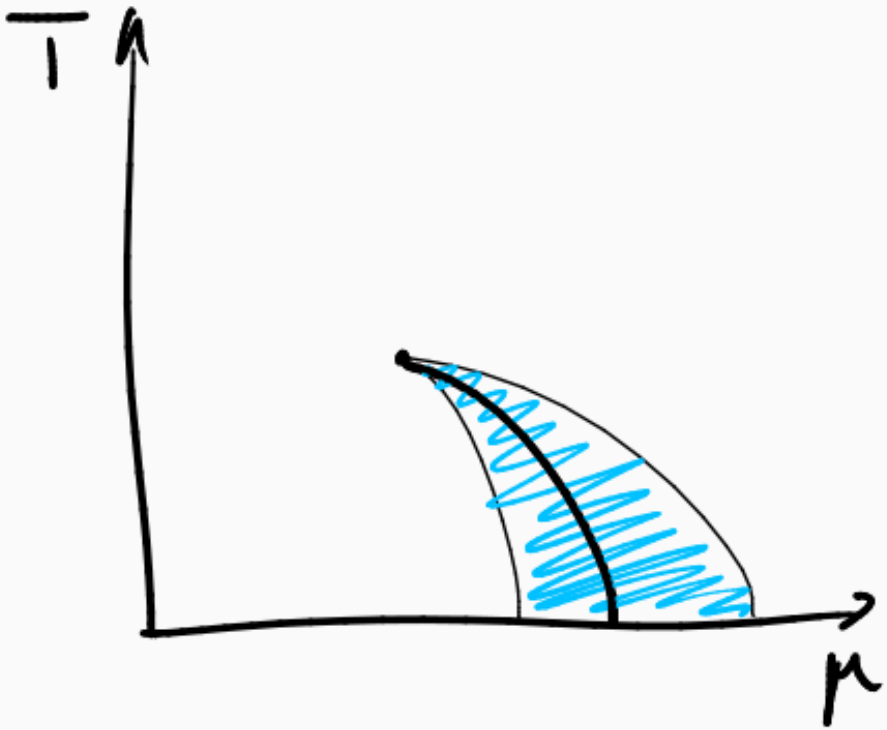

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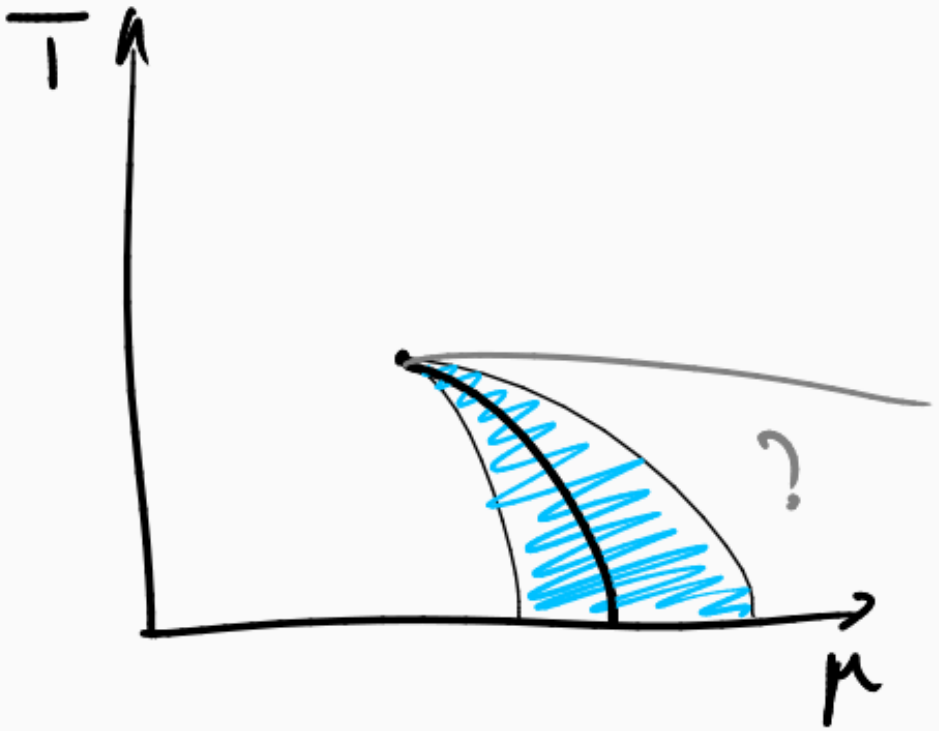

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- With Models of QCD:
  - Gross-Neveu
  - NJL
  - QM

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- By Direct Ansatz
- By Stability Analysis

NJL

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left\{(\bar{\psi}\psi)^2 + \left(\bar{\psi} i\gamma_5 \vec{\tau}\psi\right)^2\right\}$$

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Anzats

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- Chiral Density Wave:

$$\phi_S(\vec{x}) = -\frac{\Delta}{2G_S}\cos(\vec{q}\cdot\vec{x}), \qquad \phi_P(\vec{x}) = -\frac{\Delta}{2G_P}\sin(\vec{q}\cdot\vec{x})$$

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$$M(x) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta x|\nu)$$

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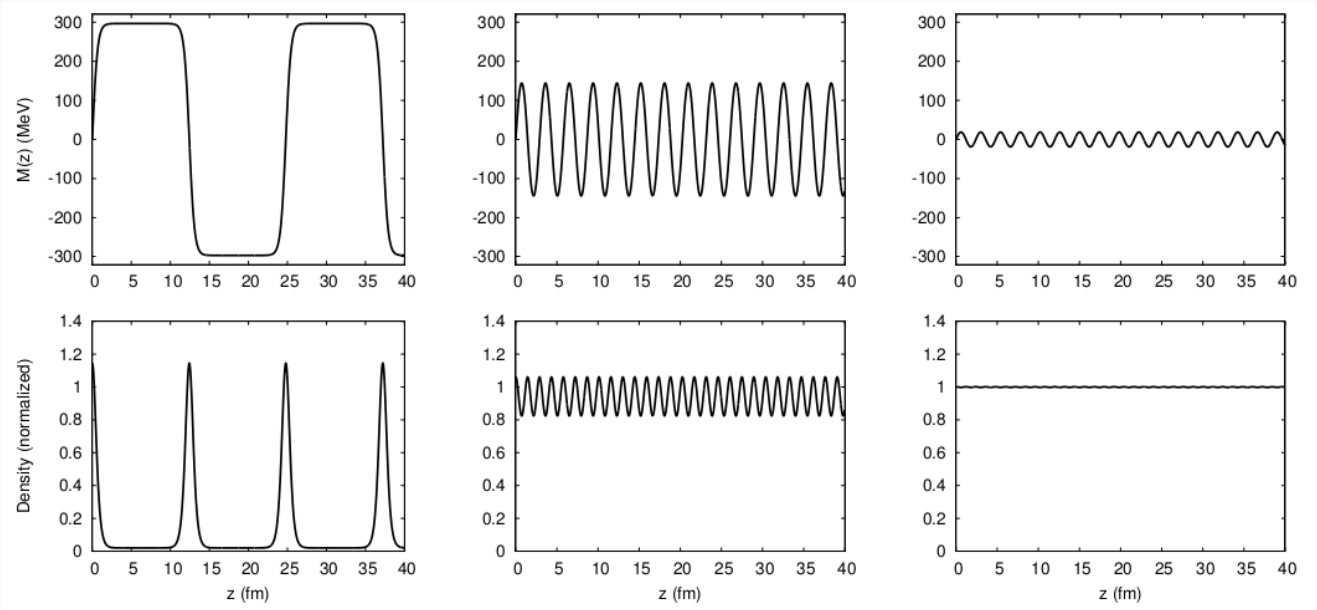

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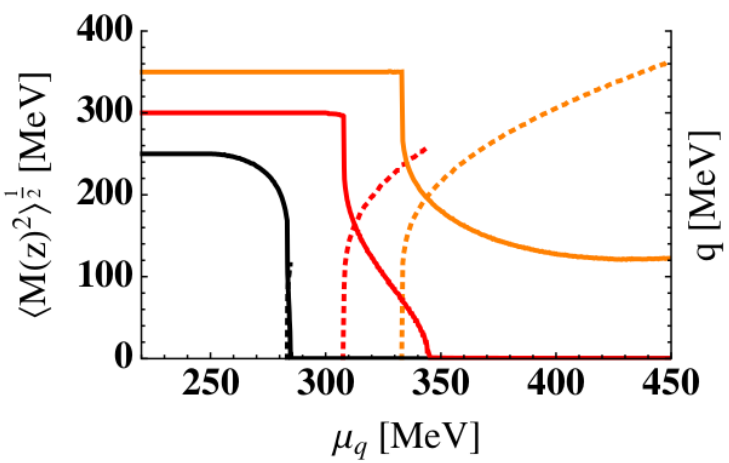
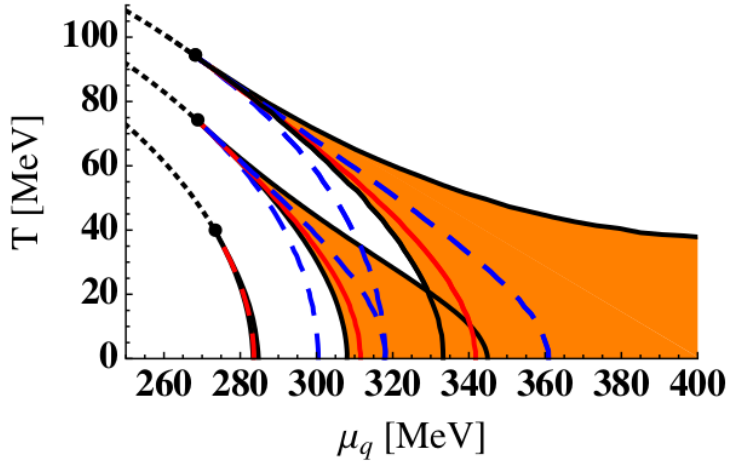
Inhomogeneous phases in the Nambu–Jona-Lasinio and quark-meson model

Dominik Nickel

Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 10 July 2009; published 22 October 2009)





Stability Analysis

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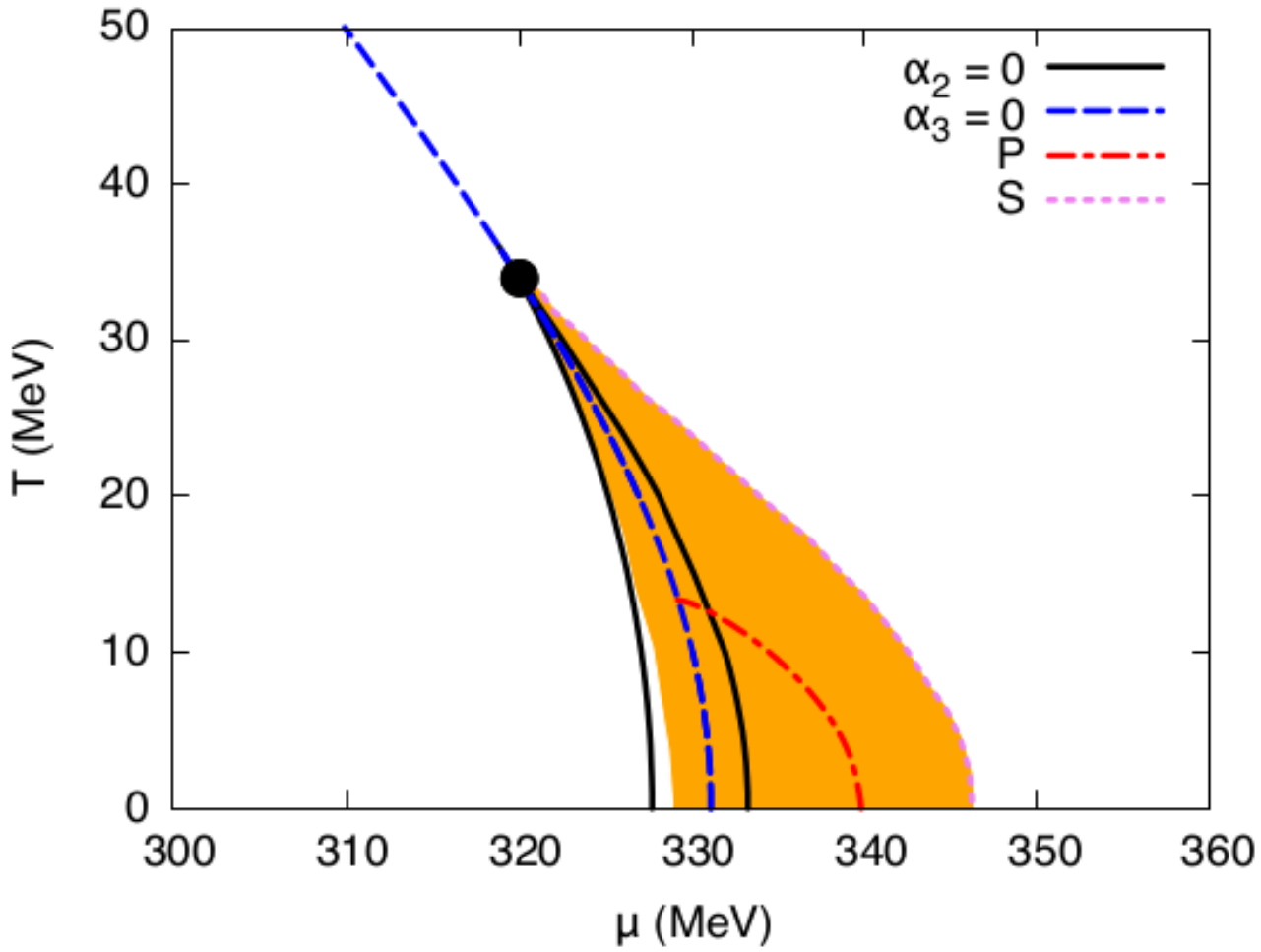
Inhomogeneous chiral phases away from the chiral limit

Michael Buballa[1] and Stefano Carignano[2]

[1] *Theoriezentrum, Institut für Kernphysik, Technische Universität Darmstadt, Schlossgartenstr. 2, D-64289 Darmstadt, Germany*

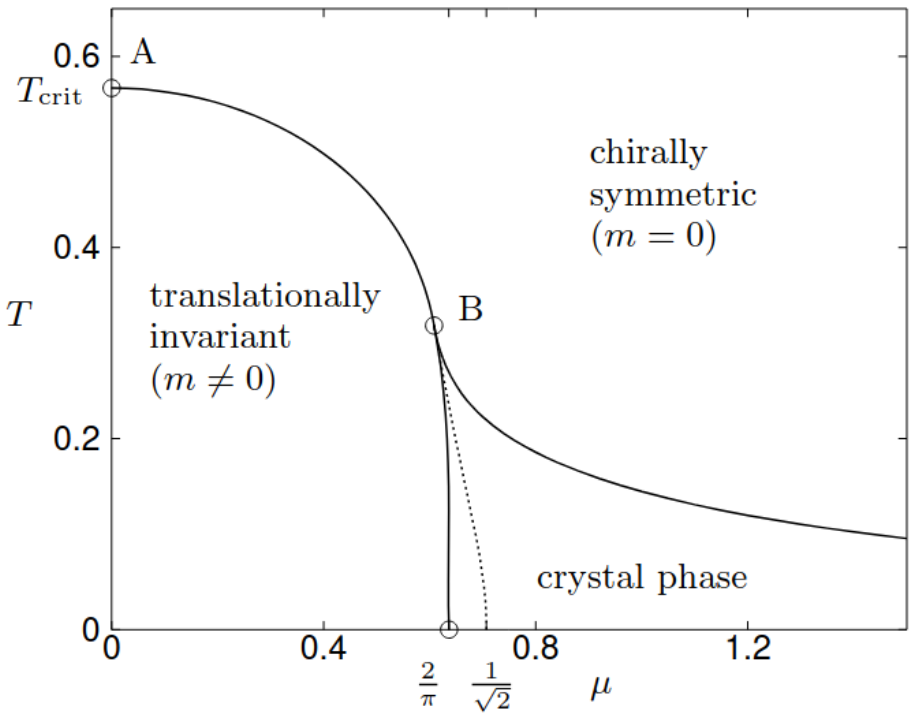
[2] *Departament de Física Quàntica i Astrofísica and Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Catalonia, Spain.*



A plot twist? Gross-Neveu Model!

Revised Phase Diagram of the Gross-Neveu Model

Michael Thies and Konrad Urlichs
Institut für Theoretische Physik III
Universität Erlangen-Nürnberg
Staudtstraße 7
D-91058 Erlangen
Germany
(Dated: October 25, 2018)



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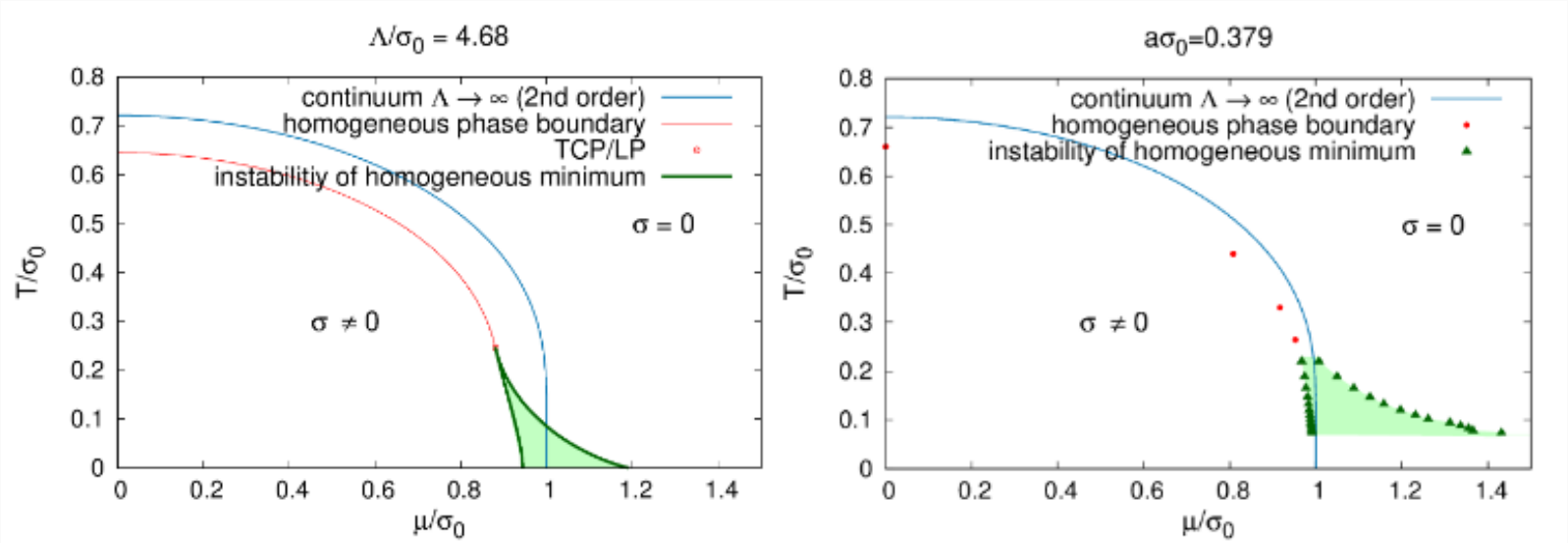

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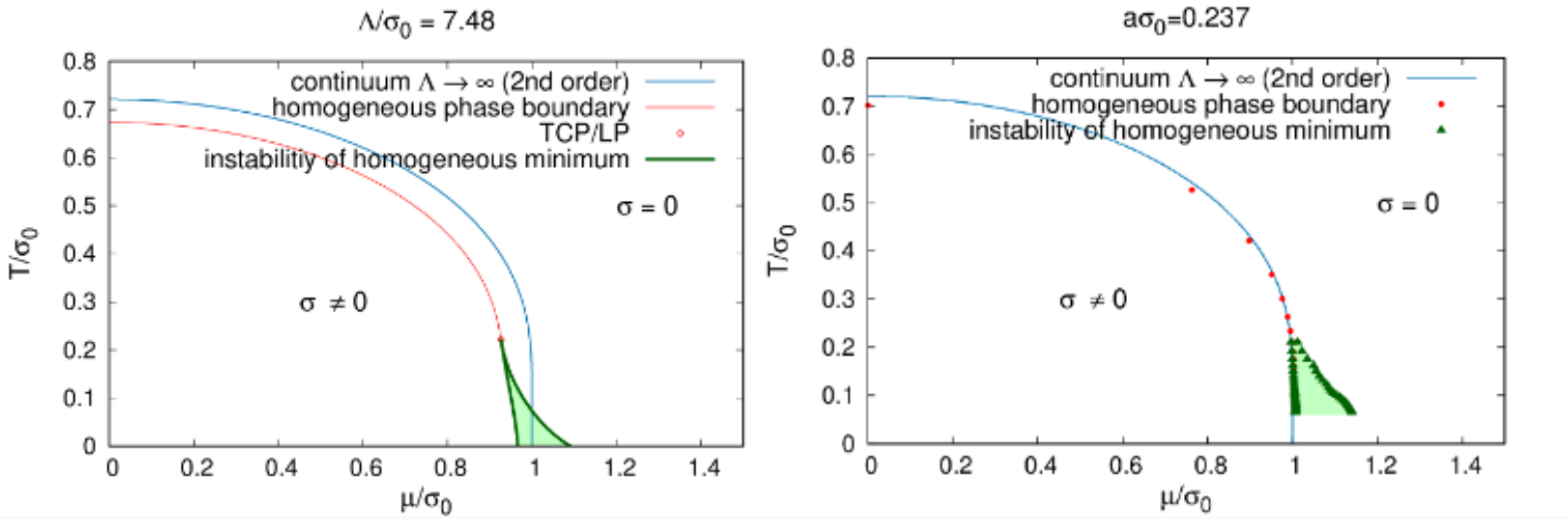

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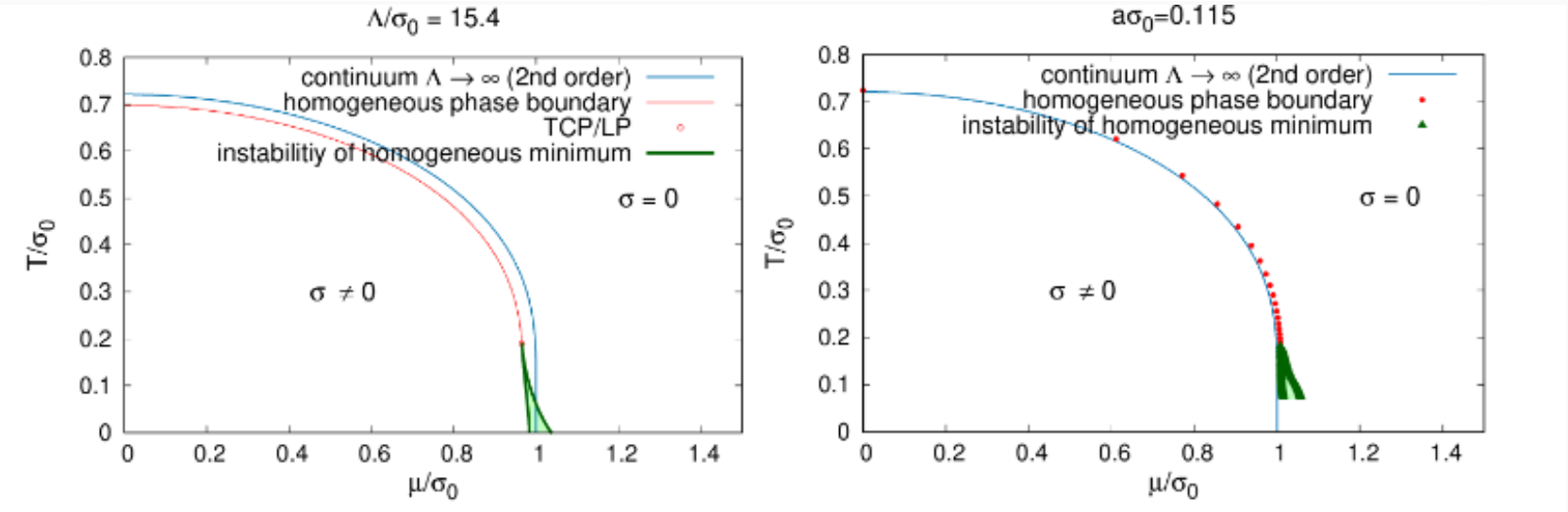

Quantum Chromodynamics

How about QCD?

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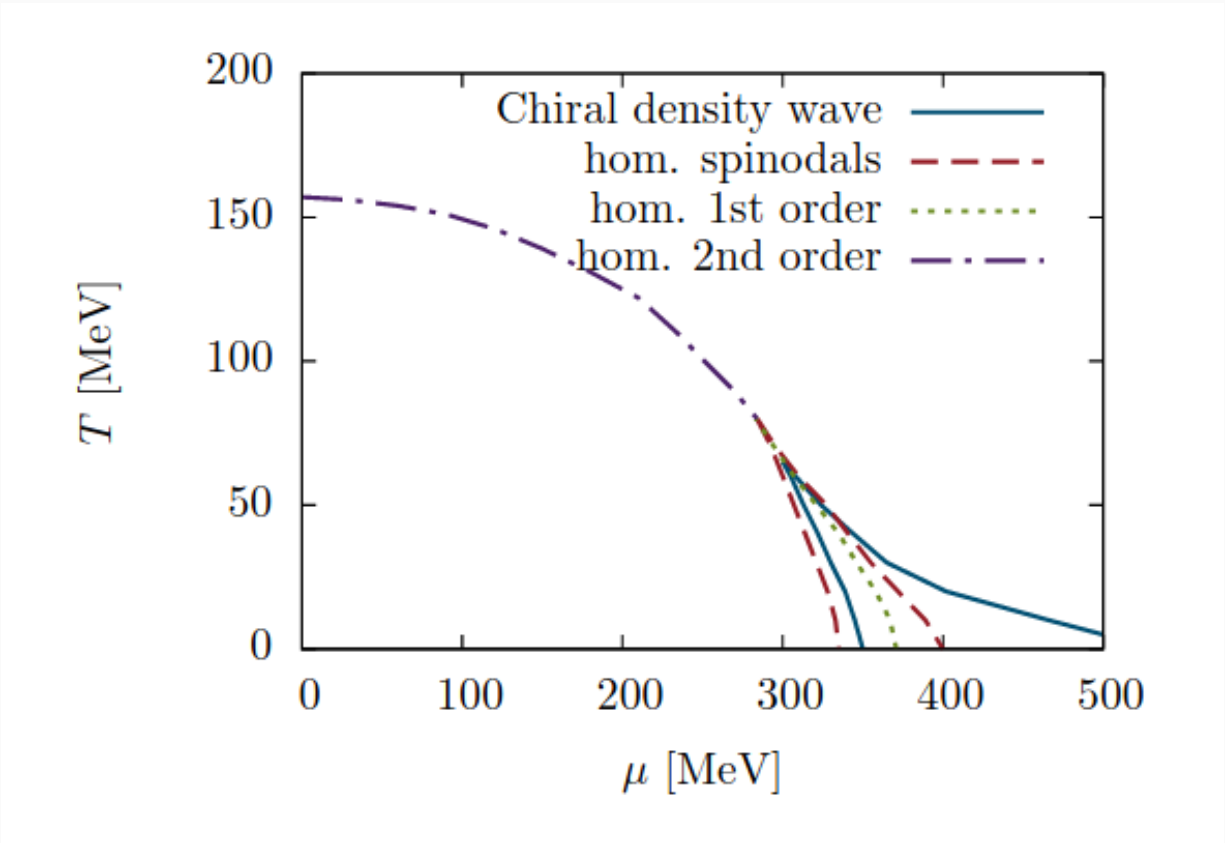
Dyson-Schwinger study of chiral density waves in QCD

D. Müller[a], M. Buballa[a], J. Wambach[a,b]

[a] *Institut für Kernphysik (Theoriezentrum), Technische Universität Darmstadt, Germany*
[b] *GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany*



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& + \left(B\left(p,p'\right) - i\gamma_4\gamma_3 F\left(p,p'\right) - i\gamma_4\frac{\vec{p}_\perp}{|\vec{p}_\perp|}G\left(p,p'\right) - i\gamma_3\frac{\vec{p}_\perp}{|\vec{p}_\perp|}H\left(p,p'\right)\right)\frac{(\mathrm{I}-\gamma_5)}{2}\delta\left(p-p'+Q\right) \\
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- Then you solve the DSE **and**, in theory, you must calculate whether or not this solution is favoured!
- They calculated a necessary condition for stability, namely

$$\frac{d\Gamma}{dQ} > 0.$$

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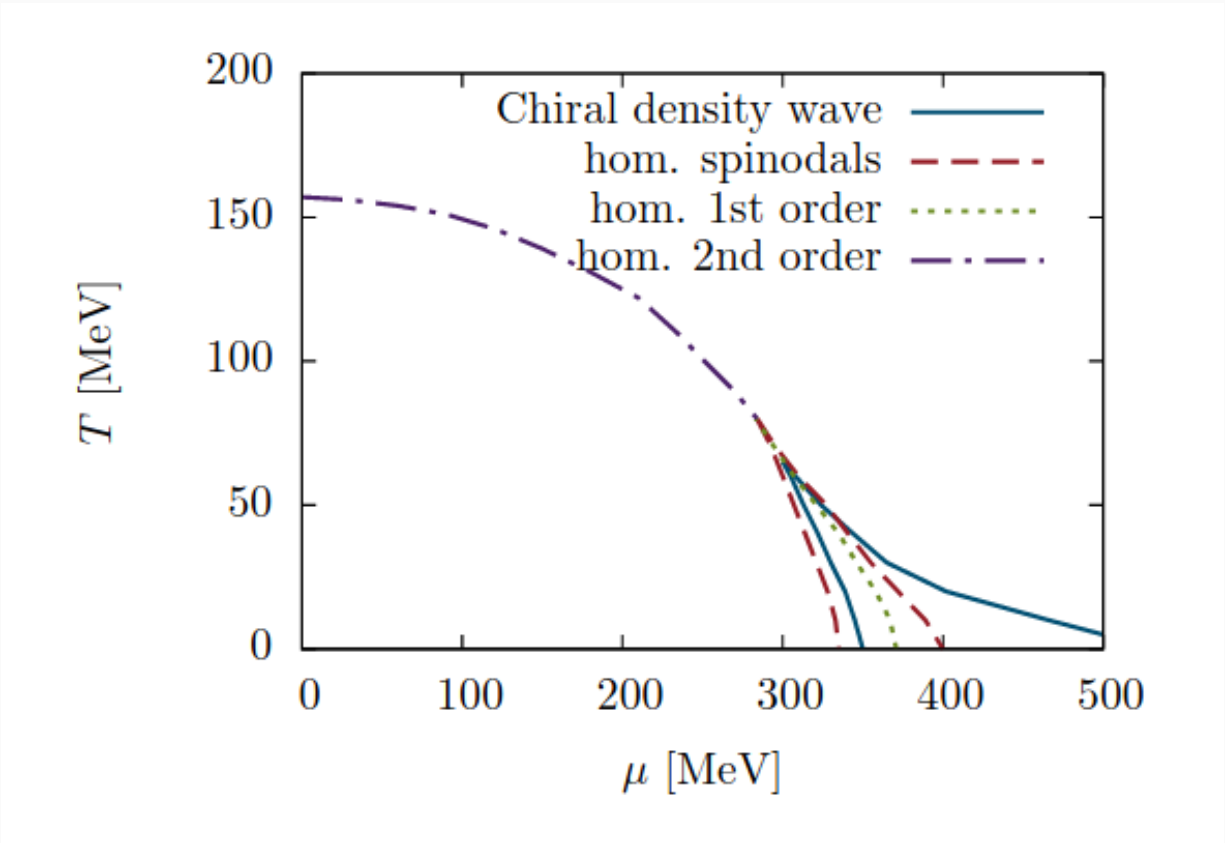

Stability Analysis

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- We chose to start from a 2PI effective action

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- Fundamentally, we want a Taylor expansion

$$\begin{aligned} F[\varphi(u)] =& F[\varphi_0(u)] + \int \left.\frac{\delta F[\varphi(u)]}{\delta\varphi(u')}\right|_{\varphi=\varphi_0} \times (\varphi(u) - \varphi_0(u'))du' \\ &+ \frac{1}{2!}\int\int \left.\frac{\delta^2 F[\varphi(u)]}{\delta\varphi(u')\delta\varphi(u'')}\right|_{\varphi=\varphi_0} \times (\varphi(u) - \varphi_0(u')) \times (\varphi(u) - \varphi_0(u''))du'du'' + \cdots \end{aligned}$$

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- Second order is the leading order

$$\Gamma^{(2)} = \frac{1}{2}\mathrm{Tr}[(\bar{S}^{-1}\delta S)^2] + \frac{1}{2}\mathrm{Tr}\left[\overline{\frac{\delta^2\Gamma_{2\mathrm{PI}}}{\delta S_{12}\delta S_{34}}}\delta S_{12}\delta S_{34}\right]$$

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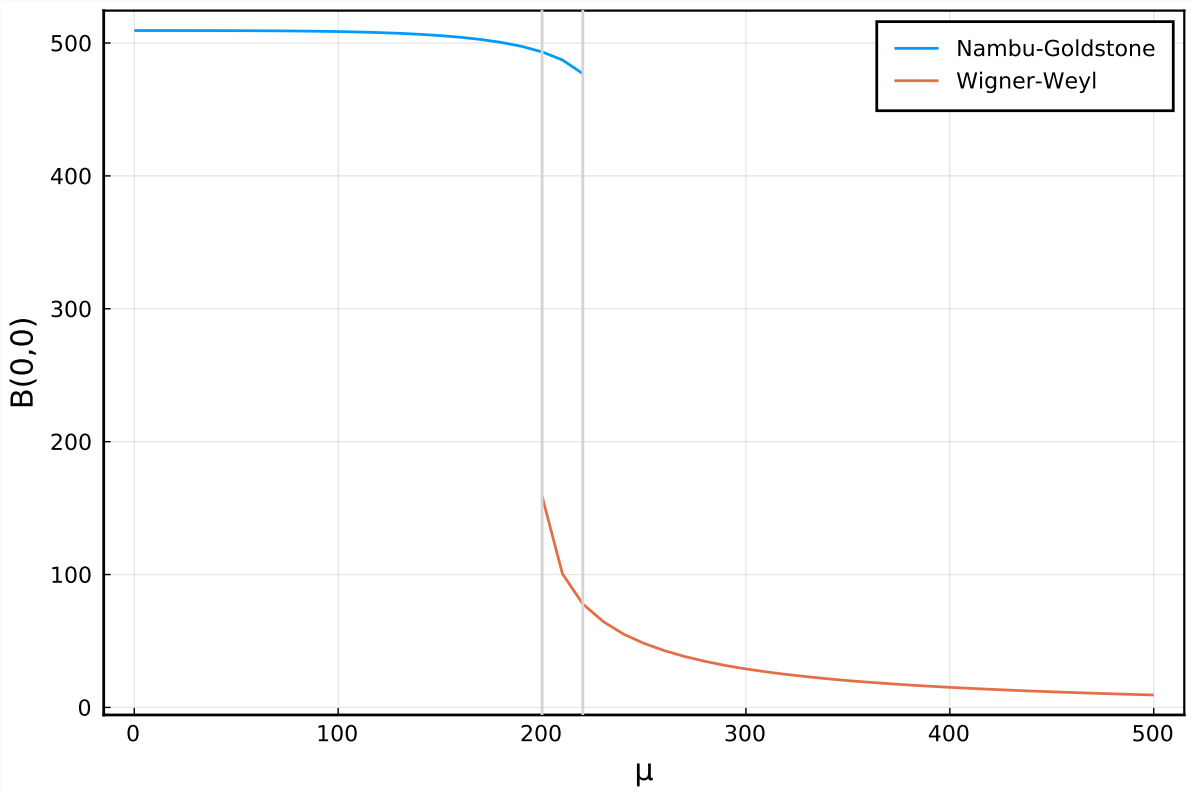

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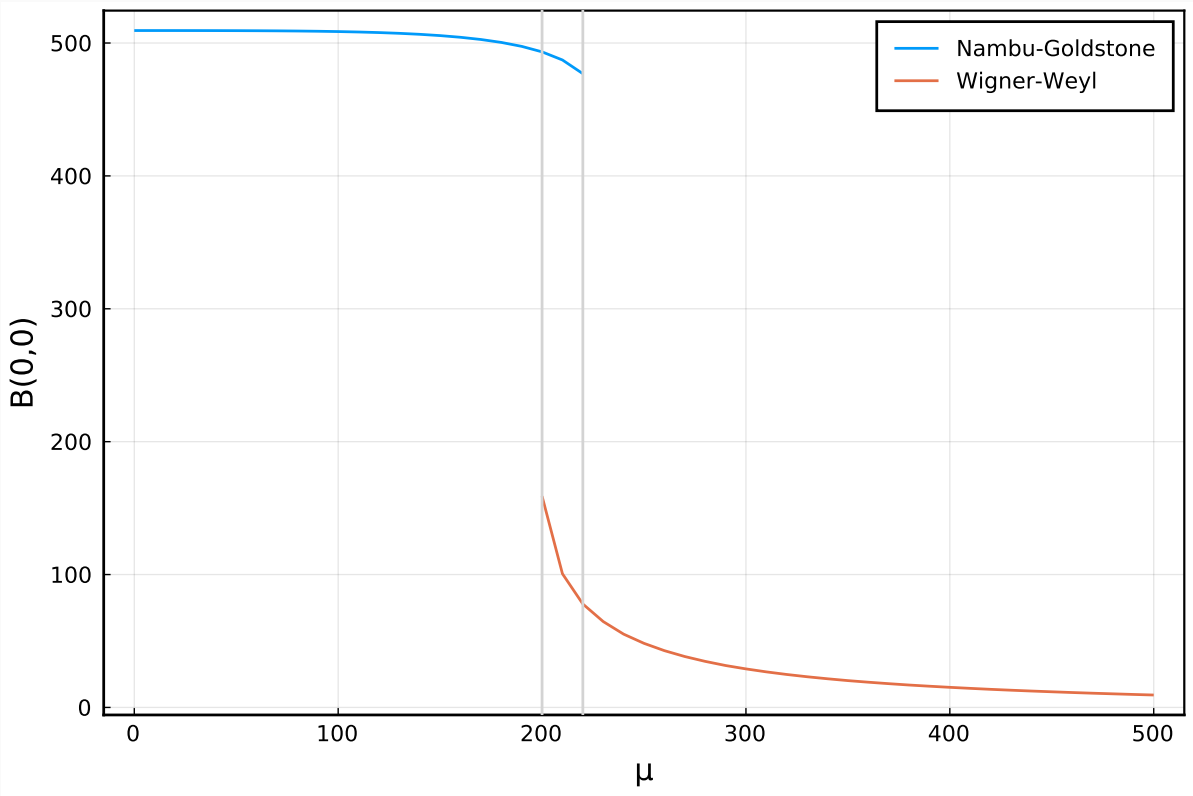
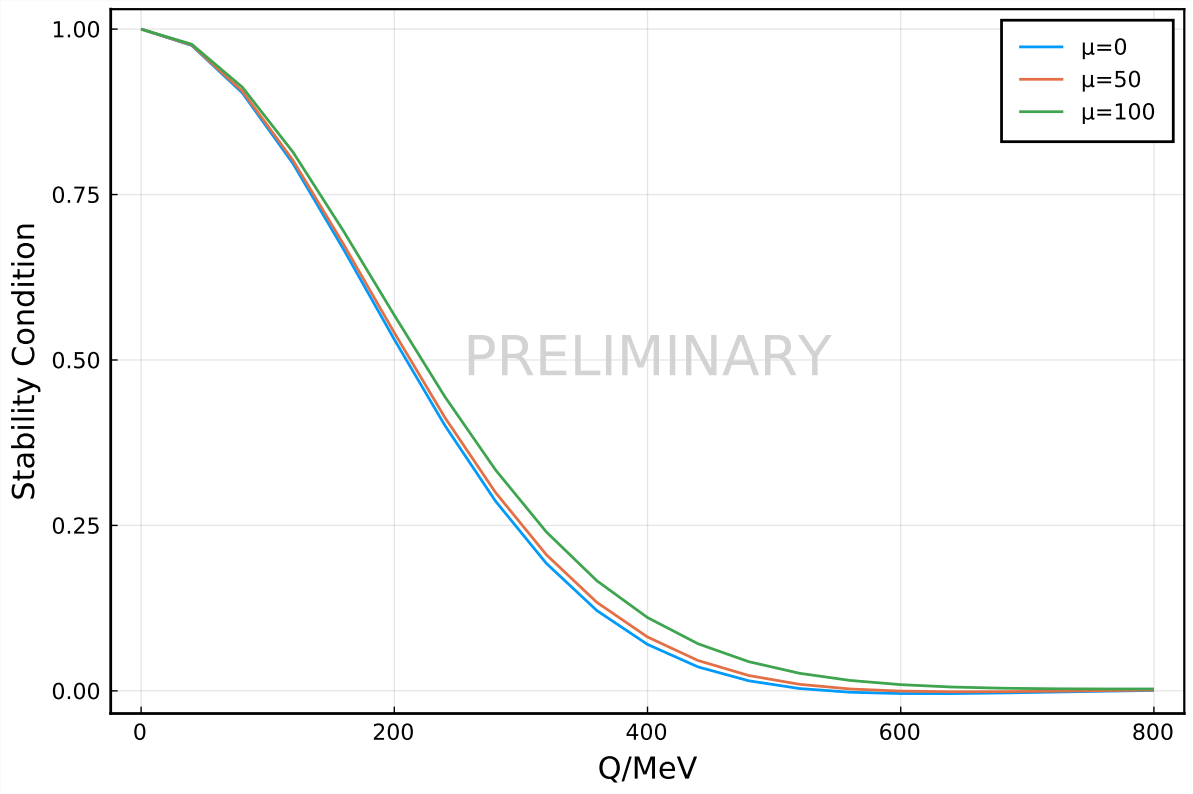


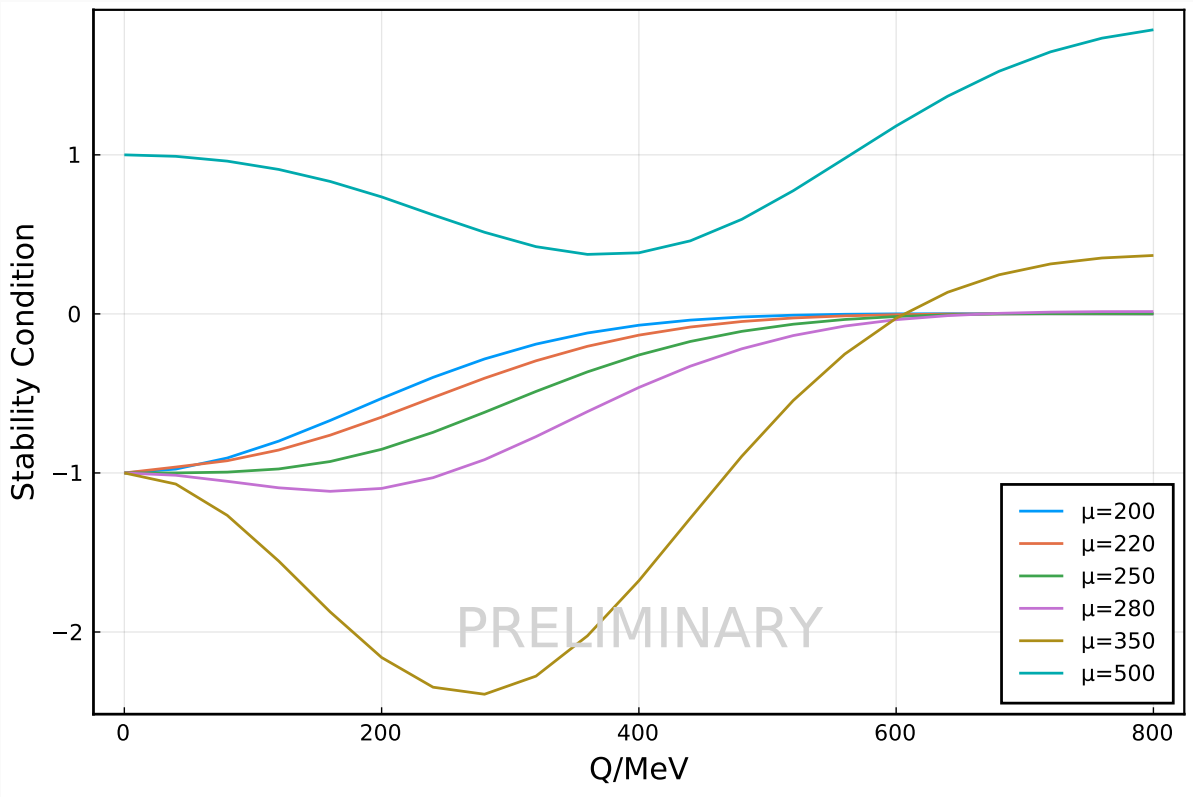
- Take a fixed temperature slice of the phase diagram.
- $\delta S(k_1, k_2) = \bar{S}(k_1) H(k_1, k_2) F(k_1 - k_2) \bar{S}(k_2)$

Stability Analysis



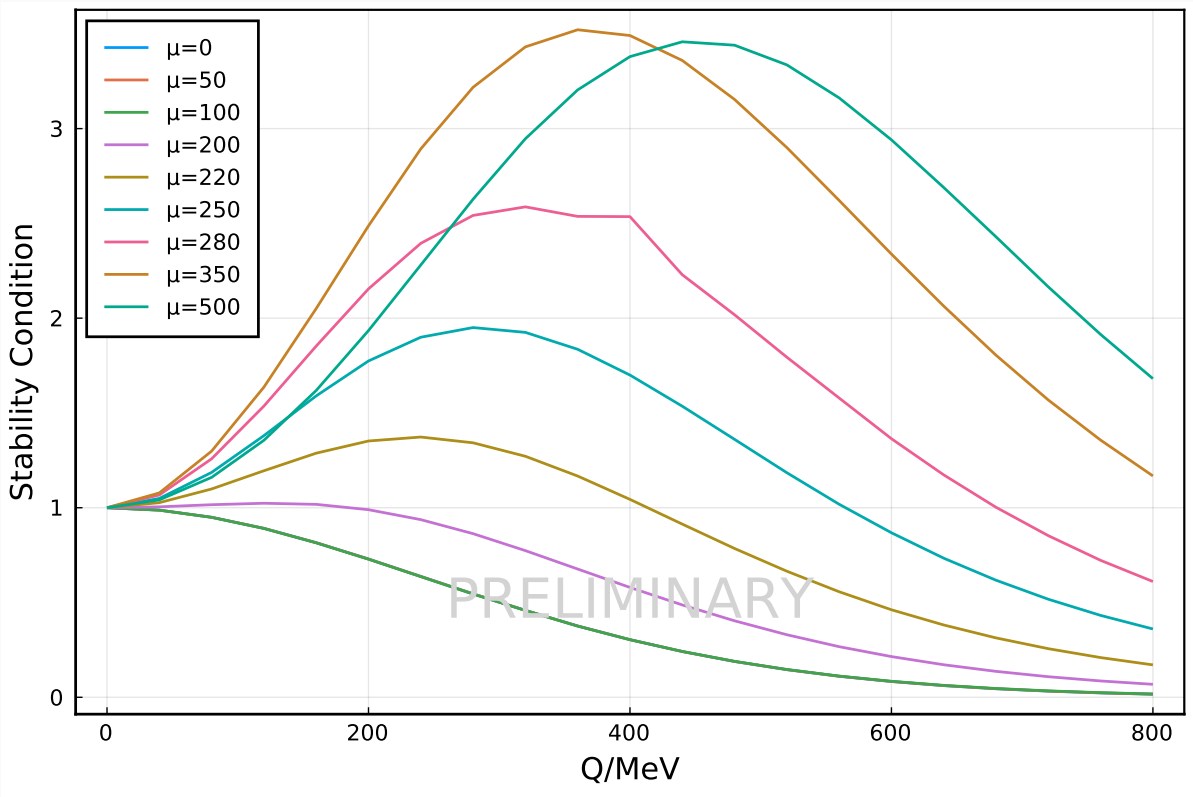
- The normalised stability condition is always positive for $\mu < \mu_c$ (Nambu-Goldstone)

Stability Analysis



- The Wigner-Weyl solution can show instabilities!

Stability Analysis



- However, not for every simplification of δS.

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Thanks!