Proton image and momentum distributions from light-front dynamics

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In hadron physics, one of the most important remaining challenges is to describe the dynamics and structure of the proton in terms of its basic constituents (quarks and gluons).

The proton light-front wave function, defined on the null plane $x^+ = t + z = 0$, gives through the parton probability densities access to various observables.

For example:
- Electromagnetic form factors
- The parton distribution function
- Generalized parton distribution functions

Additionally, the double parton scattering cross section depends on the double parton distribution function (DPDF) [1]:

$$D(x_1, x_2, \vec{\eta}_\perp) = \sum_{n=3}^\infty \mathcal{D}_n(x_1, x_2, \vec{q}_\perp) = \sum_{n=3}^\infty \int \frac{d^2k_{1\perp}}{(2\pi)^2} \frac{d^2k_{2\perp}}{(2\pi)^2} \left\{ \prod_{i\neq 1,2} \int \frac{d^2k_{i\perp}}{(2\pi)^2} \int_0^1 dx_i \right\} 
\times \delta \left( 1 - \sum_{i=1}^n x_i \right) \delta \left( \sum_{i=1}^n \vec{k}_{i\perp} \right) \Psi_n^\dagger(x_1, \vec{k}_{1\perp} + \vec{\eta}_\perp, x_2, \vec{k}_{2\perp} - \vec{\eta}_\perp, \ldots) \Psi_n(x_1, \vec{k}_{1\perp}, x_2, \vec{k}_{2\perp}, \ldots), \right.$$  

The first of Mellin moments of DPDF has recently been calculated within lattice QCD [2].

• The LF wave function is defined on the LF plane, i.e. solely in Minkowski space. In that sense it is not directly available in Euclidean space.

• Therefore, in this work we considered a simple (e.g. spinless quarks) but dynamical three-body model on the light-front in the valence approximation.

• One of the basic building block is the scalar diquark, introduced through a pole in the quark-quark transition amplitude.
Three-body Faddeev-Bethe-Salpeter equation with zero interaction

- Faddeev-Bethe-Salpeter (FBS) equation with zero interaction [1]:
  \[ v(q, p) = 2i \mathcal{F}(M_{12}^2) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p - q - k)^2 - m^2 + i\epsilon} v(k, p) \]  
  \[ (2) \]

- Currently, bare propagators for the quarks.
- \( v(q, p) \) is one of the Faddeev components of the total vertex function.
- Di-quark implemented by assuming a pole in \( \mathcal{F}(M_{12}^2) \), corresponding either to a two-body bound \((a > 0)\) or virtual \((a < 0)\) state, where \( a \) denotes the scattering length.
- \( \mathcal{F}(M_{12}^2) \), where \( M_{12}^2 = (p - q)^2 \), given by
  \[ \mathcal{F}(M_{12}^2) = \frac{\Theta(-M_{12}^2)}{16\pi^2 y} \log \frac{1+y}{1-y} - \frac{1}{16\pi ma} + \frac{\Theta(M_{12}^2) \Theta(4m^2 - M_{12}^2)}{8\pi^2 y'} \arctan y' - \frac{1}{16\pi ma} + \frac{\Theta(M_{12}^2 - 4m^2)}{16\pi^2} \log \frac{1+y''}{1-y''} - \frac{1}{16\pi ma} - \frac{iy''}{16\pi} \]  
  \[ (3) \]

- The FBS equation was recently solved including the infinite number of Fock components in Euclidean [2] and Minkowski [3] space.

After the LF projection, i.e. introducing $k_\pm = k_0 \pm k_z$ and integrating over $k_-$, one obtains the three-body LF equation [1, 2]:

$$\Gamma(x, k_\perp) = \frac{\mathcal{F}(M_{12}^2)}{(2\pi)^3} \int_0^{1-x} \frac{dx'}{x'(1-x-x')} \int_0^\infty \frac{d^2k'_\perp}{M_0^2 - M_N^2} \Gamma(x', k'_\perp)$$

(4)

with the squared free three-body mass

$$M_0^2 = (k'_\perp^2 + m^2)/x' + (k_\perp^2 + m^2)/x + ((k'_\perp + k_\perp)^2 + m^2)/(1-x-x')$$

(5)

The three-body valence LF wave function is given by

$$\Psi_3(x_1, \vec{k}_1\perp, x_2, \vec{k}_2\perp, x_3, \vec{k}_3\perp) = \frac{\Gamma(x_1, \vec{k}_1\perp) + \Gamma(x_2, \vec{k}_2\perp) + \Gamma(x_3, \vec{k}_3\perp)}{\sqrt{x_1x_2x_3(M_N^2 - M_0^2(x_1, \vec{k}_1\perp, x_2, \vec{k}_2\perp, x_3, \vec{k}_3\perp))}},$$

(6)

where due to momentum conservation: $x_3 = 1 - x_2 - x_3$ and $\vec{k}_3\perp = -\vec{k}_1\perp - \vec{k}_2\perp$.

Results for the vertex function

<table>
<thead>
<tr>
<th>Model</th>
<th>( m ) [MeV]</th>
<th>( a ) ([m^{-1}])</th>
<th>( M_2 ) [MeV]</th>
<th>( M_N/m )</th>
<th>( r_{F_1} ) [fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>317</td>
<td>-1.84</td>
<td>-</td>
<td>2.97</td>
<td>0.97</td>
</tr>
<tr>
<td>II</td>
<td>362</td>
<td>3.60</td>
<td>681</td>
<td>2.60</td>
<td>0.72</td>
</tr>
</tbody>
</table>

Two different values of \( a \) considered, with negative and positive \( a \), fitted to reproduce the experimental Dirac form factor (up to \( \sim 1 \text{ GeV}^2 \)). For the model with a bound diquark the obtained value of the di-quark mass same as a recent Lattice QCD calculation.

The proton structure contained in the vertex function \( \Gamma(x,k_{\perp}) \). As seen for the bound diquark case it has a node at roughly \( x = 0.8 \).
The distribution amplitude is defined as

$$
\phi(x_1, x_2) = \int d^2k_1 \perp d^2k_2 \perp \Psi_3(x_1, \vec{k}_1 \perp, x_2, \vec{k}_2 \perp, x_3, \vec{k}_3 \perp).
$$

(7)

It shows the dependence of the wave function on the momentum fractions for the case when the quarks share the same position.

For the two considered cases similar results.
Alternatively, the proton can be studied in the on the null-plane, in terms of the transverse position \( \vec{b}_i \perp \) and the Ioffe-time \( \tilde{x}_i = b_i^+ p^+ \). The image of the proton is then obtained through the Fourier transform of the proton LF wave function.

For simplicity, we consider here the case \( \vec{b}_1 \perp = \vec{b}_2 \perp = \vec{0} \perp \), and then one has

\[
\Phi(\tilde{x}_1, \tilde{x}_2) \equiv \Psi_3(\tilde{x}_1, \vec{0} \perp, \tilde{x}_2, \vec{0} \perp) = \int_0^1 dx_1 e^{ix_1 x_1} \int_0^{1-x_1} dx_2 e^{ix_2 x_2} \phi(x_1, x_2),
\] (8)
For $\tilde{x}_2 = 0$ the two parameter sets give almost identical results.

- For $\tilde{x}_2 = 10$ and $\tilde{x}_1 \geq 10$ a rather dramatic decrease of the amplitude is seen. Similar behavior for the two parameter sets.

- An exponential damping is seen with respect to the relative distance in Ioffe-time between the two quarks. We expect this damping to be even more significant if confinement is incorporated, as its more effective at large distances.
The valence contribution to the Dirac form factor is given by

\[
F_1(Q^2) = \left\{ \prod_{i=1}^{3} \int \frac{d^2k_{i\perp}}{(2\pi)^2} \int_0^1 dx_i \right\} \delta \left( 1 - \sum_{i=1}^{3} x_i \right) \delta \left( \sum_{i=1}^{3} \vec{k}_{f\perp}^i \right) \times \Psi_3^\dagger(x_1, \vec{k}_{1\perp}^f, ...) \Psi_3(x_1, \vec{k}_{1\perp}^i, ...),
\]

where \(Q^2 = \vec{q}_\perp \cdot \vec{q}_\perp\) and the magnitudes of the momenta read

\[
|\vec{k}_{i\perp}^{f(i)}|^2 = |\vec{k}_{i\perp} \pm \frac{\vec{q}_\perp}{2} x_i|^2 = \vec{k}_{i\perp}^2 + \frac{Q^2}{4} x_i^2 \pm \vec{k}_{i\perp} \cdot \vec{q}_\perp x_i \quad (i = 1, 2),
\]

and

\[
|\vec{k}_{3\perp}^{f(i)}|^2 = \left| \pm \frac{\vec{q}_\perp}{2} (x_3 - 1) - \vec{k}_{1\perp} - \vec{k}_{2\perp} \right|^2 =
(1 - x_3)^2 \frac{Q^2}{4} \pm (1 - x_3) \vec{q}_\perp \cdot (\vec{k}_{1\perp} + \vec{k}_{2\perp}) + (\vec{k}_{1\perp} + \vec{k}_{2\perp})^2.
\]
Both parameters give a fair reproduction of experimental data for low $Q^2$, i.e $Q^2 < 1\text{GeV}^2$, where the model should be applicable.

The bound diquark case give also quite good agreement for moderate $Q^2$. But, this should be viewed with caution since the scaling laws of the QCD are not built-in.
We define the single parton distribution function (PDF) as

\[
f_1(x_1) = \frac{1}{(2\pi)^6} \int_0^{1-x_1} dx_2 \int d^2k_{1\perp} d^2k_{2\perp} |\Psi_3(x_1, \vec{k}_{1\perp}, x_2, \vec{k}_{2\perp}, x_3, \vec{k}_{3\perp})|^2 = I_{11} + I_{22} + I_{33} + I_{12} + I_{13} + I_{23}.
\]

(12)

with the Faddeev contributions

\[
I_{ii} = \frac{1}{(2\pi)^6} \int_0^{1-x_1} dx_2 \int d^2k_{1\perp} d^2k_{2\perp} \frac{\Gamma^2(x_i, \vec{k}_{i\perp})}{x_1x_2x_3(M_N^2 - M_0^2(x_1, \vec{k}_{1\perp}, x_2, \vec{k}_{2\perp}, x_3, \vec{k}_{3\perp}))^2}
\]

(13)

\[
I_{ij} = \frac{2}{(2\pi)^6} \int_0^{1-x_1} dx_2 \int d^2k_{1\perp} d^2k_{2\perp} \frac{\Gamma(x_i, \vec{k}_{i\perp})\Gamma(x_j, \vec{k}_{j\perp})}{x_1x_2x_3(M_N^2 - M_0^2(x_1, \vec{k}_{1\perp}, x_2, \vec{k}_{2\perp}, x_3, \vec{k}_{3\perp}))^2}; \quad i \neq j.
\]

Evolution of the PDF will be performed in the near future.
The valence double parton distribution function (DPDF) is given by

\[
D_3(x_1, x_2; \vec{\eta}_\perp) = \frac{1}{(2\pi)^6} \int d^2k_{1\perp} d^2k_{2\perp} \\
\times \Psi_3^\dagger(x_1, \vec{k}_{1\perp} + \vec{\eta}_\perp; x_2, \vec{k}_{2\perp} - \vec{\eta}_\perp; x_3, \vec{k}_{3\perp}) \Psi_3(x_1, \vec{k}_{1\perp}; x_2, \vec{k}_{2\perp}; x_3, \vec{k}_{3\perp}).
\]

(14)

Fourier transform of \( D_3(x_1, x_2, \vec{\eta}_\perp) \) in \( \vec{\eta}_\perp \) gives the probability of finding the quarks 1 and 2 with momentum fractions \( x_1 \) and \( x_2 \) at a relative distance \( \vec{y}_\perp \) within the proton.

In the figure is shown results for \( \eta_\perp = 0 \). For the case of virtual diquark (left panel) a rather narrow distribution is obtained due to the small binding energy.
The single quark transverse momentum density in the forward limit and integrated in the longitudinal momentum is associated with the probability density to find a quark with momentum $k_\perp$.

It can be computed as:

$$L_1(k_\perp) = \frac{k_\perp}{(2\pi)^6} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{2\pi} d\theta_1 \int d^2k_{2\perp} |\psi_3(x_1, \vec{k}_{1\perp}, x_2, \vec{k}_{2\perp}, x_3, \vec{k}_{3\perp})|^2.$$

For model I (solid line) a more narrow distribution is seen compared to model II (dashed line), because the radius is larger.
Conclusions

- We have, in this work, studied the proton in a simple but fully dynamical valence LF model based on a zero-range interaction.
- The model is based on the concept of a strongly interacting diquark, either virtual or bound.
- We have studied the structure of the proton by computing the LF wave function in its Ioffe-time representation and also momentum distributions.
- However, the model is rather crude since e.g. the spin degree of freedom hasn’t been included yet. But is a first step towards studying the proton directly in Minkowski space.
- Future plans:
  - Generalization to the infinite set of Fock components (The Faddeev-Bethe-Salpeter equation solved in PLB 791 (2019) 276)
  - Implementation of a more realistic interaction (gluon exchange)
  - Inclusion of spin degree of freedom